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APPROXIMATION OF THE STOCHASTIC RAYLEIGH-BÉNARD PROBLEM NEAR THE ONSET OF CONVECTION AND RELATED PROBLEMS

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Approximation of the stochastic Rayleigh-Bénard problem near the onset of convection and related problems

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Abstract

It is well known that complicated stochastic evolution systems near a change of stability are often dominated by slow modes. We rigorously approximate the evolution of the system by stochastic ordinary differential equations describing the amplitudes of the dominating modes on a slow time-scale.

In this article we focus on equations with quadratic nonlinearity and give applications to the Rayleigh-Bénard problem and a model from surface growth perturbed by fractional noise. The unique existence of global solutions is not necessary.

Keywords: Amplitude equation, Rayleigh-Bénard convection, multiple scale analysis, approximate centre manifold, surface growth, fractional Brownian motion, systems of SPDEs

Classification: 60H15, 35R60, 60H10, 76E06, 35Q72

1 Introduction

Bifurcation points play a central role in the theory of evolution equations given by deterministic partial differential equations (PDEs). At these points the dynamical behaviour of solutions can change significantly. One celebrated example is a pitchfork-bifurcation, where the number of stable fixed points of the evolution changes from one to two. But for stochastic systems the question of how to describe a bifurcation, is not completely settled even for stochastic ordinary equations (see e.g. [Ar98, Sec. 9] or [CF98]). There are several concepts of bifurcations, which could lead to different results.

Our approach describes the transient dynamics, and it uses the well known approximation via amplitude equations. Here the dynamics of complicated system of stochastic PDEs (SPDEs) is described when the bifurcation parameter is near the deterministic bifurcation point. In that case a natural separation of time-scales is present, which allows to separate finitely many dominating modes from the infinite-dimensional problem. Furthermore the essential dynamics is well described by these finitely many degrees of freedom. The dynamics of these modes are given by a stochastic ordinary differential equation (SDE), the so called amplitude equation. Moreover, there are several results for transient dynamics of SDEs (e.g. [BG02]) that, using this method, directly extend to SPDEs.

On a formal level the approximation of stochastic PDEs in a neighbourhood of a change of stability by a set of simple SDEs or SPDEs is well understood, and is a valuable tool in the physics community. See for example [CH93] or [W97] for a review of models related to pattern formation, or [Ge98] for applications to Rayleigh-Bénard problem. We give more examples later.

Despite of that, the mathematical theory of amplitude equations for SPDEs was rigorously only developed for cubic type nonlinearities (see for example [BMS01, B03, BH04]). The main advantage in this case is a simple separation of the dynamics, as the dominating modes decouple easily from the fast modes. For deterministic equations there are numerous publications, even for equations on unbounded domains. See for example [CE90, KSM92], or for Rayleigh-Bénard problem [Schn94, Schn99], and the references therein.

In contrast quadratic nonlinearities are much more involved, as they tend to mix Fourier modes much stronger. We will see in the formal derivation that the main problem is the fact that the nonlinearity does not act directly on the dominant modes. It only influences them through non-dominant modes.

The method presented could also be referred to as an approximate centre manifold (cf. [BlHa, Bl05b] and the references therein). Nevertheless the vector space given by the dominant modes is only a first order approximation of the true invariant manifold.

A related result is [FvE], where the stochastic flow of a perturbed gradient system is investigated on a slow time-scale in some slow manifold, which is finite-dimensional. A second example is [Ro03] for a single mode amplitude equation for Burgers equation. Moreover in [K01] envelope-equations for noise induced oscillations in delay-equations are derived by multi-scale analysis. But neither result contains a rigorous proof of the reduction.

Let us finally comment on some limitations of our approach. Moments, mean values, or correlation functions of solutions could also be approximated by this method, but we will not treat this here. The power of the presented approach is that it still establishes results even if equation (1) exhibits blow-ups in finite time, and statistical quantities are no longer defined. This can also be the case in both our examples. For surface growth in dimension $d = 1, 2$ and the Rayleigh-Bénard problem for $d = 3$, the uniqueness of global solutions is not settled. Therefore our result relies only on the stability and the global existence of unique solutions for the amplitude equation.

Our approach neglects all the other modes, apart from the dominating ones. Therefore it is limited to bounded domains. The main drawback is the following. Our proofs rely heavily on the existence of a spectral gap between the first non-zero eigenvalue and 0 (ω in Assumption 3.2). In most examples it is well known that this gap shrinks to 0, if the domain size is large, hence leading to a blow up of constants. See e.g. the constant $\int_0^\infty \tau^{-\alpha} \mathrm{e}^{-\tau\omega} d\tau$ in the proof of Theorem 4.1. Another example is the time-scale of attractivity, which is $\frac{1}{\omega} \ln(\varepsilon^{-2})$. Therefore the presented approach is limited to bounded domains only, and the validity of the theory is restricted to $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \to 0$ for domain-size to infinity. For unbounded (or just very large) domains, we need modulation equations like in e.g. [KSM92, S95] or in many other publications by these authors. For a first result for SPDEs see [BHP], which treats the stochastic Swift-Hohenberg equation on large domains.

The paper is organised as follows. The next section is devoted to a formal derivation of the amplitude equation. Section 3 contains a rigorous formulation of our problem, the assumptions that are necessary, and the statement of the main results. Section 4 provides the proof of the attractivity result, and Sections 5 and 6 the proof of the approximation. In Section 7 we present two applications. The final Section 8 establishes technical results used throughout the proofs of the main results.

2 Formal Derivation of the Results

We consider equations of the following type:

$$\partial_t u(t) = Lu(t) + \varepsilon^2 Au(t) + B(u(t), u(t)) + \varepsilon^{2H+1}\xi(t), \quad u(0) = u_0. \tag{1}$$

The equation is considered in a Banach space X, where L and A are some (unbounded) operators, B is a bilinear mapping, and ξ is some Gaussian noise, both in space and time, where the correlation in time is fractional noise with Hurst-parameter $H \in [\frac{1}{2}, 1)$. Hence, for $H > \frac{1}{2}$

$$\mathbb{E}\xi(t,x) = 0 \qquad \text{and} \qquad \mathbb{E}\xi(t,x)\xi(s,y) = c_H|t-s|^{2H-2}q(x,y)\ ,$$

where q is the kernel of the corresponding covariance operator, and c_H some constant depending only on $H > 0$. The case $H = \frac{1}{2}$ corresponds to white noise in time.

We use fractional noise in order to simplify some of the proofs, by using path-wise estimates only available for fractional noise with $H > \frac{1}{2}$. The particular scaling of the noise leads to noise and linear (in)stability in the amplitude equation. We can easily consider different scalings, which would lead to a loss of one of the effects.

In [B03, BMS01] the scaling of the noise strength and the deterministic perturbation $\varepsilon^2 Au$ is of the same order. But for $H > \frac{1}{2}$ the scaling is different, in order to get an interesting stochastic approximation. Different noise strength leads to the loss of either the linear instability or the noise in the approximation.

Our main examples are the Rayleigh-Bénard problem and a model from surface growth (cf. Section 7). Therefore, we omit cubic or higher order terms in (1) for simplicity of presentation. Nevertheless they can be incorporated in a straightforward way. Cubic nonlinearities give a contribution to the amplitude equation (like in [B03]), while higher order nonlinearities give no contribution at all, at least not on the time-scale of interest. It is even possible that (1) exhibits blow-ups in finite time. But if the amplitude equation is stable, then this happens only after a very long time.

The key observation is that the projection of (1) onto the kernel $N(L)$ of L lives on a much slower time-scale, as it is not subject to an exponential decay on a time-scale of order $\mathcal{O}(1)$. In order to derive the slow dynamics, we preform the following formal calculation. Denote the projection onto $N(L)$ by P_c, define $P_s = I - P_c$, and consider the ansatz

$$u(t) = \varepsilon\Phi_c(\varepsilon^2 t) + \varepsilon^2\psi_s(t) \;, \tag{2}$$

where $\Phi_c \in N(L)$ and $\psi_s \in P_sX$. Let us remark that the ansatz $u(t) = \varepsilon\Phi_c(\varepsilon^2 t)$ is not appropriate, as in our examples the nonlinearity maps $N(L)$ into P_sX. Hence, the formal argument just leads to the linearization of the SPDE, which is well known. Here we want to capture also nonlinear effects. Furthermore, the higher order corrections are necessary, although in the final result, they are not visible, as the error of the approximation is of order $\mathcal{O}(\varepsilon^2)$.

Plugging (2) into (1), we obtain first by collecting all terms of order $\mathcal{O}(\varepsilon^2)$ in P_sX:

$$\partial_t\psi_s(t) = L\psi_s(t) + P_sB(\Phi_c(\varepsilon^2 t), \Phi_c(\varepsilon^2 t)) + \varepsilon^{2H-1}P_s\xi(t) \;. \tag{3}$$

Secondly, projecting onto $N(L)$ and collecting all terms of order ε^2 yields

$$P_cB(\Phi_c(T), \Phi_c(T)) = 0 \;. \tag{4}$$

Furthermore, we collect all terms in $N(L)$ of order ε^3 to obtain

$$\partial_T\Phi_c = P_cA\Phi_c + 2P_cB(\Phi_c, \psi_s(\varepsilon^{-2}\cdot)) + P_c\hat{\xi} \;, \tag{5}$$

where $T = \varepsilon^2 t$ is the slow time-scale and $\hat{\xi}(T) = \varepsilon^{2H-2}\xi(\varepsilon^{-2}T)$ is a rescaled version of our noise, i.e. both stochastic processes posses the same distributions.

Equation (4) is an Assumption on P_cB (cf. Assumption 3.5). The other two equations are on the one hand a dominating equation (5) on a slow time-scale coupled to an equation (3) on the fast time-scale.

Equations with a similar structure are treated in [PS03, PS] where tracers in a fast moving velocity field are considered. Or for systems of SDEs see [BG03]. Moreover, the problem is similar to averaging principles. See e.g. [KK01] or [Fr96]. Nevertheless the structure of our problem is slightly different, as we get an effective equation for the slow component completely independent of the fast modes.

In order to simplify (3) and (5), rescale (3) to the slow time-scale $T = \varepsilon^2 t$ by defining $\psi_s(t) = \Phi_s(\varepsilon^2 t)$. Hence ($L_s = P_sL$)

$$\varepsilon^2\partial_T\Phi_s = L_s\Phi_s + P_sB(\Phi_c, \Phi_c) + \varepsilon P_s\hat{\xi} \;.$$

As L_s is invertible on P_sX, we get in lowest order of ε

$$\Phi_s = -L_s^{-1} P_s B(\Phi_c, \Phi_c)\ .$$

Plugging this into (5) we end up with the *amplitude equation*:

$$\partial_T \Phi_c = P_c A \Phi_c - 2 P_c B\Big(\Phi_c, L_s^{-1} P_s B(\Phi_c, \Phi_c)\Big) + P_c \hat{\xi}\ . \qquad (6)$$

Surprisingly, this equation involves a cubic nonlinearity, although the nonlinearity in the original equation was quadratic.

Our main results show that these formal calculations can be made rigorous. First the *attractivity* result (Theorem 4.1) justifies the ansatz (2) for small initial conditions after a short time of order $\mathcal{O}(\ln(\varepsilon^{-1}))$. This relies on the linear damping of the fast modes. The *approximation* result (Theorem 6.1) is more involved. It verifies the formal calculation for a solution u of (1). Provided $u(0) = \varepsilon a(0) \cdot e + \mathcal{O}(\varepsilon^2)$ then $u(t) = \varepsilon a(\varepsilon^2 t) \cdot e + \mathcal{O}(\varepsilon^2)$ on a time-scale of order $\mathcal{O}(\varepsilon^{-2})$, where $e = (e_1, \ldots, e_n)$ is a basis of $N(L)$ and the amplitudes $a \in \mathbb{R}^n$ fulfil an SDE given by (6).

3 Notation and Formulation of the Problem

Let X be some Banach space with norm $\|\cdot\|$. For the linear operator L in (1) we assume the following:

Assumption 3.1 (Linear Operator and Projections) *Suppose L is an unbounded linear operator on X. Denote the kernel (or null-space) of L by $\mathcal{N} := N(L)$, and suppose that $n := \dim(\mathcal{N}) < \infty$. Define by $e = (e_1, \ldots, e_n) \in X^n$ a basis of $\mathcal{N}$, and fix a projection onto $\mathcal{N}$ by P_c. Define $P_s := I - P_c$, and suppose that P_c and hence P_s commute with L.*

As the dimension of $\mathcal{N}$ is finite, it is well known that both P_c and P_s are bounded linear operators on X (cf. [W80]), i.e. $P_c, P_s \in \mathcal{L}(X)$.

The second assumption on L, which induces the separation of time-scales is the following:

Assumption 3.2 (Semigroup and the Space Y) *The operator L from Assumption 3.1 generates an analytic semigroup $\{\mathrm{e}^{tL}\}_{t\geq 0}$ of linear operators on X such that there are constants $\omega > 0$ and $M \geq 1$ with*

$$\|\mathrm{e}^{tL} P_s x\| \leq M \mathrm{e}^{-t\omega} \|x\| \quad \textit{for all } t \geq 0,\ x \in X. \qquad (7)$$

Suppose there is a second Banach space Y such that X is continuously embedded into Y. Assume we can extend e^{tL} to a semigroup on Y. Additionally suppose that for $t > 0$ the operator e^{tL} is a bounded operator from Y to X (i.e., $\mathrm{e}^{tL} \in \mathcal{L}(Y, X)$) such that for some $\alpha \in [0, 1)$

$$\|P_s \mathrm{e}^{tL} y\| \leq M(1 + t^{-\alpha})\mathrm{e}^{-t\omega}\|y\|_Y \quad \textit{for all } t > 0,\ y \in Y. \tag{8}$$

Let us make some comments on the properties of e^{tL}. First by Assumption 3.1 the projections P_c and P_s commute with e^{tL}. As e^{tL} is also a semigroup on Y we can extend L_s^{-1} to a continuous operator from P_sY to P_sX, e.g. by writing $L_s^{-1} = \int_0^\infty \mathrm{e}^{\tau L} P_s d\tau$. We can also extend P_c and hence P_s to projections in the space Y (i.e., $P_c, P_s \in \mathcal{L}(Y, Y)$), as $\mathcal{N} \subset X \subset Y$.

Moreover, $L \equiv 0$ on $\mathcal{N}$ implies $\mathrm{e}^{tL} = Id$ on $\mathcal{N}$ for all $t \geq 0$, and, as $\mathcal{N}$ is finite dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_Y$ are equivalent on $\mathcal{N}$. Therefore, we can assume that M is sufficiently large such that

$$\|\mathrm{e}^{tL} x\| \leq M\|x\| \quad \text{for all } t \geq 0,\ x \in X$$

and

$$\|\mathrm{e}^{tL} P_c y\| \leq M\|P_c y\|_Y \quad \text{for all } t \geq 0,\ y \in Y\ .$$

Remark 3.3 (Spectrum of L) *Assumption 3.2 and all its implications are usually ensured by assumptions on the spectrum $\sigma(L)$ of L. We basically need that $\sigma(L)$ lies in the half-plane with negative real part with the exception of an isolated eigenvalue 0 of finite multiplicity.*

The space Y is for many applications given by a fractional Sobolev space, and bounds like (8) follow then from the existence of a spectral gap between 0 and the rest of the spectrum.

For the stochastic perturbation let the following assumption be true. A detailed discussion of Q-Wiener processes and stochastic convolutions can be found in [dPZ92] for white noise and in [DPM02] for fractional noise.

Assumption 3.4 (Noise) *Suppose that the fractional noise process ξ is the generalised derivative of some fractional Q-Wiener process $\{W(t)\}_{t\geq 0}$ with Hurst parameter $H \in [\frac{1}{2}, 1)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the stochastic convolution*

$$W_L(t) = \int_0^t \mathrm{e}^{(t-\tau)L} dW(\tau) \tag{9}$$

is a well defined stochastic process with continuous paths in X.

To be more precise, we assume that there is a basis of X consisting of eigenfunctions f_k of Q with $Qf_k = \alpha_k^2 f_k$, and $W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k^H(t) f_k$, where β_k^H are i.i.d. real (fractional) Brownian motions with Hurst parameter $H \geq \frac{1}{2}$. We say that the noise (or W) is of trace-class, if $\mathrm{tr}(Q) < \infty$.

The case $H = \frac{1}{2}$ corresponds to white noise. Here we experience a lot of technical difficulties if $\mathrm{tr}(Q) = \infty$. This is the main reason for considering fractional noise. Some results will apply to $H < \frac{1}{2}$, but we do not focus on that.

Assumption 3.5 (Bilinear Operator B) *Suppose $B : X \times X \to Y$ is a bilinear and continuous mapping which is symmetric, i.e. $B(u, v) = B(v, u)$ and there is a constant $C_B > 0$ such that $\|B(u, v)\|_Y \leq C_B \|u\| \|v\|$. Denote $B(u) = B(u, u)$, $B_s = P_s B$, and $B_c = P_c B$. The key assumption is $B_c(\cdot, \cdot) = 0$ on $\mathcal{N} \times \mathcal{N}$, which was already indicated in (4).*

Assumption 3.6 (Linear Operator A) *Suppose $A : X \to Y$ is a continuous linear operator, i.e. there is a constant $C_A > 0$ such that $\|Au\|_Y \leq C_A \|v\|$. Define $A_s = P_s A$ and $A_c = P_c A$, which are both bounded linear operators from X to Y.*

To give a meaning to (1) we consider mild solutions.

Assumption 3.7 (Mild Solutions) *Assume that for all (stochastic) initial conditions $u_0 \in X$ equation (1) has a mild local solution u. This means we have a stopping time $t^* > 0$ and a stochastic process u such that $u : [0, t^*] \to X$ is $\mathbb{P}$-a.s. a solution of*

$$u(t) = \mathrm{e}^{tL} u_0 + \int_0^t \mathrm{e}^{(t-\tau)L} \Big[\varepsilon^2 A u(\tau) + B(u(\tau)) \Big] d\tau + \varepsilon^{2H+1} W_L(t) \qquad \text{for } t \leq t^*. \tag{10}$$

Moreover, either $t^* = \infty$ or $\|u(t)\| \to \infty$ for $t \to t^*$.

This Assumption is mainly for convenience. Under the previous Assumptions it is easy to verify it. The proof of existence of unique local solutions is standard under our Assumptions. See e.g. [dPZ92] for a textbook. For L^p-theory with application to Navier-Stokes eq. see e.g. [BP99, BP00]. Moreover there are results for the Kuramoto-Shivashinsky (see [DJ01]) and the surface growth equation from (45) (see e.g. [BlGu]). Note that most of the above cited articles do not use fractional noise, but as most proofs are done path-wise, the results carry over to this case, too. Only regularity of the stochastic convolution W_L is needed, which is usually well understood (cf. [DPM02]).

As the projections commute with the semigroup, it is straightforward to verify that

$$P_s[W_L(t)] = \int_0^t \mathrm{e}^{(t-\tau)L} dP_s W(\tau) \quad \text{and} \quad P_c[W_L(t)] = P_c W(t).$$

We now split the variation of constants formula (10) into two parts:

$$P_s u(t) = \mathrm{e}^{tL} P_s u_0 + \int_0^t \mathrm{e}^{(t-\tau)L} \Big[\varepsilon^2 A_s u(\tau) + B_s(u(\tau))\Big] d\tau + \varepsilon^{2H+1} \int_0^t \mathrm{e}^{(t-\tau)L} dP_s W(\tau) \tag{11}$$

and

$$P_c u(t) = P_c u_0 + \int_0^t \Big[\varepsilon^2 A_c u(\tau) + B_c(u(\tau))\Big] d\tau + \varepsilon^{2H+1} P_c W(t). \tag{12}$$

Definition 3.8 *We call* $u_s(t) = P_s u(t)$ fast modes*, as they are subject to a deterministic exponential decay on a time-scale of order* $\mathcal{O}(1)$*. Moreover* $u_c(t) = P_c u(t)$ *are the* slow modes*.*

3.1 The Amplitude Equation

The amplitude equation is an SDE (or a system of SDEs) that describes the essential dynamics of mild solutions of (1) near 0. It was formally derived in (6). Here we state the rigorous formulation. First we define a projection onto the amplitudes of the dominating modes in $\mathcal{N}$.

Definition 3.9 *For* $a \in \mathbb{R}^n$ *denote* $a \cdot e = \sum_{k=1}^n a_k e_k$*. Define the projection* $\Pi : X \to \mathbb{R}^n$ *by* $\Pi(a \cdot e + z) = a$ *for all* $a \in \mathbb{R}^n$ *and* $z \in N(P_c)$*.*

As the spaces $\mathcal{N}$ and $\mathbb{R}^n$ are finite dimensional, we easily obtain that Π is continuous, i.e., there is a constant $C_\pi > 0$ such that $|\Pi(x)| \le C_\pi \|x\|$ for all $x \in X$, where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

Definition 3.10 *Define the cubic nonlinearity* $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\begin{aligned} \Gamma[a] &= -2\Pi\{B_c\left(a \cdot e, L_s^{-1} B_s(a \cdot e)\right)\} \\ &= -2 \sum_{i,j,k=1}^n a_i a_j a_k \Pi B_c\left(e_i, L_s^{-1} B_s(e_j, e_k)\right)\} \end{aligned} \tag{13}$$

and the linearity $\nu : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\nu(a) = \Pi\{A_c(a \cdot e)\} = \sum_{i=1}^{n} a_i \Pi\{A_c(e_i)\}. \tag{14}$$

We discussed after Assumption 3.2 that $L_s := P_s L$ is invertible on $P_s X$, as $N(L_s) = \{0\}$. Hence, the mapping $L_s^{-1} B_s(\cdot) : P_c X \to P_s X$ is well defined. Furthermore, under our Assumptions, it is easy to verify that Γ and ν are continuous.

Definition 3.11 *Define the* amplitude equation *by*

$$a(T) = a(0) + \int_0^T \nu(a(s))ds + \int_0^T \Gamma[a(s)]ds + \beta(T) \tag{15}$$

where $\{\beta(T)\}_{T\geq 0}$ *is a (fractional) Wiener process in* $\mathbb{R}^n$ *given by* $\beta(T) = \varepsilon^{2H}\Pi(W(\varepsilon^{-2}T))$.

Remark 3.12 *The distribution of* β*, and therefore also of solutions of (15), is actually independent of* ε *due to the scaling properties of a fractional Wiener process (see e.g. [DÜ99]). Note also that due to scaling properties* $\varepsilon^{2H}W(\varepsilon^{-2}\cdot)$ *is a version of* W *for all* $\varepsilon > 0$*. The paths depend on* ε*, but the distribution not. Therefore, we do not use an index* ε *for* a*.*

The last assumption deals with the nonlinear stability of the amplitude equation (15). For instance, if $\dim(\mathcal{N}) = 1$, then it ensures that the cubic nonlinearity $\Gamma(a) = ca^3$ has the right sign.

Assumption 3.13 *For* Γ *from (13) we assume* $\Gamma(a) \cdot a \leq 0$.

This assumption ensures large deviation estimates for a solution of (15). It mainly prevents blow ups in finite time such that exponential moments of solutions of (15) are well defined.

3.2 Main Results

Our main results are the attractivity (see Theorems 3.14 and 4.1) and the approximation (see Theorems 3.16 and 6.1). A simplified version of Theorem 4.1 is the following theorem. It ensures that the ansatz (2) of our formal computation is justified with high probability, at least for small initial conditions. The main ingredient for proving the attractivity is the existence of a spectral gap ω (see Assumption 3.2) leading to an exponential decay in $P_s X$.

Theorem 3.14 (Attractivity) *Suppose all assumptions of Section 3 are true. Fix $\delta > 0$ and the time $t_\varepsilon = \frac{1}{\omega}\ln(\varepsilon^{-2})$ with ω from (7). We can write all mild solutions of (1) with (random) initial conditions $u(0) = u_0$ as*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon \cdot e + \varepsilon^2 R_\varepsilon$$

with $a_\varepsilon \in \mathbb{R}^n$ and $R_\varepsilon \in P_s X$ such that

$$\mathbb{P}\Big\{|a_\varepsilon|_{\mathbb{R}^n} \leq C_\pi(M\delta + 2), \quad \|R_\varepsilon\| \leq \ln(\varepsilon^{-1})\Big\} \geq \mathbb{P}\Big\{\|u_0\| \leq \delta\varepsilon\Big\} - o_\varepsilon(1)\ . \quad (16)$$

Here $o_\varepsilon(1)$ denotes a term that converges to 0 for $\varepsilon \to 0$. For simplicity, we do not focus on precise convergence rates. These are quite technical, as we need explicit bounds on various probabilities combined with detailed knowledge on how our constants depend on other constants.

For a given solution u of (10) define the approximation $\varepsilon\psi$ by

$$\varepsilon\psi(t) := \varepsilon \underbrace{a(\varepsilon^2 t) \cdot e}_{=:\psi_c(t)} + \varepsilon^2 \psi_s(t)\ , \quad (17)$$

where a is a solution of the amplitude equation (15) with initial condition $a(0) = \varepsilon^{-1}\Pi u_0$. Hence, $\psi_c(0) = \varepsilon^{-1} P_c u_0$ Furthermore, ψ_s is the correction on the fast modes satisfying $\psi_s(0) = \varepsilon^{-2} P_s u_0$ and

$$\psi_s(t) = \mathrm{e}^{tL}\psi_s(0) + \varepsilon^{2H-1} P_s W_L(t) + \int_0^t \mathrm{e}^{(t-\tau)L} B_s(\psi_c(\tau)) d\tau\ . \quad (18)$$

As discussed previously, the term ψ_s is a higher order correction, in order to deal with the coupling of modes in $\mathcal{N}$ to modes in $P_s X$ due to the nonlinearity.

The residual of $\varepsilon\psi$ is given by

$$\mathrm{Res}(\varepsilon\psi)(t) = -\varepsilon\psi(t) + \mathrm{e}^{tL}\varepsilon\psi(0) + \int_0^t \mathrm{e}^{(t-\tau)L}[\varepsilon^2 A\varepsilon\psi(\tau) + B(\varepsilon\psi(\tau))]d\tau + \varepsilon^{2H+1} W_L(t). \quad (19)$$

In order to show that $\varepsilon\psi$ is a good approximation of a solution u of (10), we have to control the residual. This is done in two steps. First we discuss $P_c\mathrm{Res}(\varepsilon\psi)$ relying on the amplitude equation (15). The second step bounds $P_s\mathrm{Res}(\varepsilon\psi)$. The main tool is the equation for the second order correction (18). It is crucial for our result that the latter bound is much better than the first one.

Theorem 3.15 (Residual) *Suppose all assumptions of Section 3 are true. Fix* $\delta > 0$*, some small* $1 \gg \kappa \geq 0$*, and some time* $T_0 > 0$*. Furthermore fix* $\gamma^* = 2H > 1$ *for* $H > \frac{1}{2}$ *or* $\gamma^* = 2 - 2\kappa$ *for trace-class noise.*

Let a *and* ψ_s *be as in (17), then for all solutions* u *of (10)*

$$\begin{aligned}\mathbb{P}\Big\{ \sup_{t\in[0,T_0\varepsilon^{-2}]} &\|P_c \mathrm{Res}(\varepsilon\psi(t))\| \leq \ln(\varepsilon^{-1})(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa}),\\ &\sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s \mathrm{Res}(\varepsilon\psi(t))\| \leq \ln(\varepsilon^{-1})\varepsilon^{3-\kappa}\Big\}\\ \geq\ & 1 - \mathbb{P}\Big\{\|u(0)\| > \delta\varepsilon\Big\} - \mathbb{P}\Big\{\|P_s u(0)\| > \delta\varepsilon^2\Big\} - o_\varepsilon(1)\ .\end{aligned}$$

A simplified version of the main approximation result (cf. Theorem 6.1) is:

Theorem 3.16 (Approximation) *Suppose all assumptions of Section 3 are true. Fix* $\delta > 0$*, some small* $1 \gg \kappa \geq 0$*, and some time* $T_0 > 0$*.*

Let a *and* ψ_s *be as in (17), then for all solutions* u *of (10)*

$$\mathbb{P}\Big\{ \sup_{t\in[0,T_0\varepsilon^{-2}]} \|u(t) - \varepsilon\psi(t)\| \leq \ln(\varepsilon^{-1})(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa})\Big\}$$

$$\geq 1 - \mathbb{P}\Big\{\|u(0)\| > \delta\varepsilon\Big\} - \mathbb{P}\Big\{\|P_s u(0)\| > \delta\varepsilon^2\Big\} - o_\varepsilon(1)\ ,$$

where the constant $\gamma^*(H)$ *was defined in Theorem 3.15.*

Note that we have to be careful, when using a time-shift, as our processes are not Markovian for $H > \frac{1}{2}$. Nevertheless, we can apply the attractivity (cf. Theorem 4.1) on $[0, t_\varepsilon]$ and the approximation (cf. Theorem 6.1) on $[t_\varepsilon, T_0\varepsilon^{-2}]$. Although obviously we cannot combine the simplified versions of this section directly, we derive

$$\mathbb{P}\Big\{ \sup_{t\in[t_\varepsilon,T_0\varepsilon^{-2}]} \|u(t) - \varepsilon\psi(t)\| \leq \ln(\varepsilon^{-1})(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa})\Big\}$$

$$\geq \mathbb{P}\Big\{\|u(0)\| \leq \delta\varepsilon\Big\} - o_\varepsilon(1)\ .$$

A sketch of the typical dynamics is given in Figure 1.

Let us finally remark that we can give estimates for the stopping time t^* from Assumption 3.7, as $t^* \geq T_e\varepsilon^{-2}$ with high probability. This is especially important, as it is not yet settled, if our examples exhibit unique global

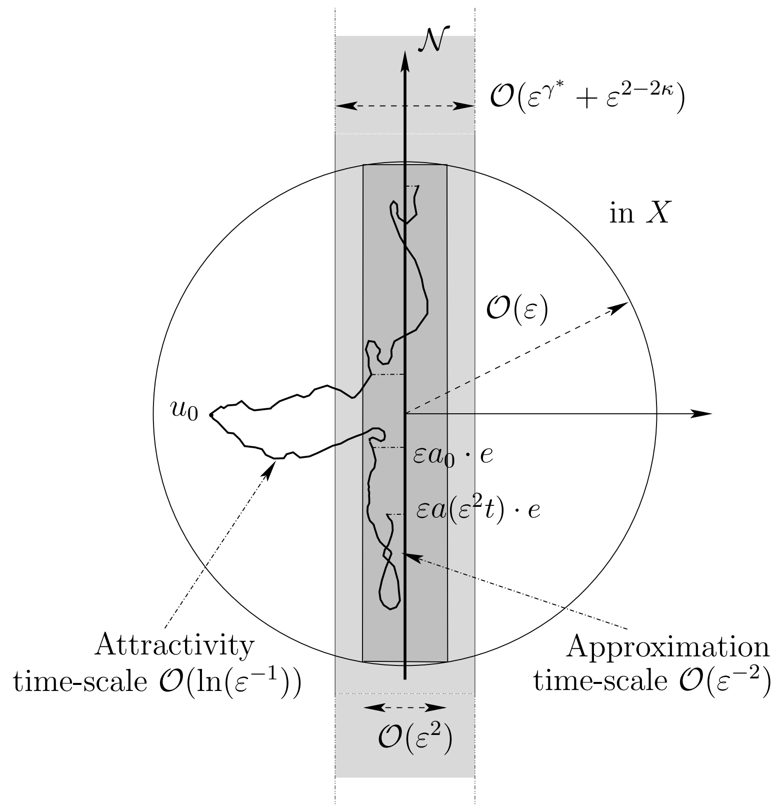

Figure 1: Two typical trajectories of mild solutions of (1)

solutions, or not. Here we obtain that the unique local solution exists for a very long time with very high probability.

The $\mathcal{O}$-notation is used in the following way. A term $G_\varepsilon = \mathcal{O}(g_\varepsilon)$ if and only if there are positive constants ε_0 and C depending only on other constants (and not on $\varepsilon > 0$) such that $|G_\varepsilon| \leq C g_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

4 Attractivity

This section contains the proof of the following theorem.

Theorem 4.1 (Attractivity) *Suppose all assumptions of Section 3 are true. Fix $\delta > 0$, a constant $C_w > 0$, and the time $t_\varepsilon = \frac{1}{\omega}\ln(\varepsilon^{-2})$ with ω from (7). We can write all mild solutions u of (1) with (random) initial conditions u_0 as*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon \cdot e + \varepsilon^2 R_\varepsilon$$

with $a_\varepsilon \in \mathbb{R}^n$ and $R_\varepsilon \in P_s X$ such that for sufficiently small $\varepsilon \in (0,1]$

$$\|u_0\| \leq \delta\varepsilon, \quad \sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| \leq \varepsilon^{-1}, \text{ and } \|P_s W_L(t_\varepsilon)\| \leq C_w \varepsilon^{1-2H} \tag{20}$$

$$\Rightarrow \quad |a_\varepsilon|_{\mathbb{R}^n} \leq C_\pi(M\delta+2) \quad \text{and} \quad \|R_\varepsilon\| \leq K+1\,, \tag{21}$$

where $K = M(1+C_B D)D\int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{-\tau\omega}d\tau + C_w$, $D = M\delta+2$, and C_π was defined after Definition 3.9.

Note that both the constant K and the time t_ε defined in the previous theorem blow up for $\omega \to 0$.

Before we turn to the proof of Theorem 4.1 let us first discuss how this theorem implies the simplified version stated in Theorem 3.14.

Proof of Theorem 3.14: For the result that (21) is true with high probability, we have to bound the probability of (20) from below.

First $\mathbb{P}(\|P_s W_L(t_\varepsilon)\| > C_w)$ is arbitrarily small for large C_w independent of ε, as the law of $\|P_s W_L(t_\varepsilon)\|$ converges to a unique measure for $t_\varepsilon \to \infty$ (see [DPM02, Prop. 3.4]). Of course this bound is for $H > \frac{1}{2}$ much better than the one necessary for (20).

Secondly, for $H > \frac{1}{2}$ it is straightforward to use the factorisation method for $\mathbb{P}(\sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| > \varepsilon^{-1})$ similar to [DPM02, Prop. 3.2], in order to establish an analog to the well known case $H = \frac{1}{2}$. Here it is easy to show that the probability is bounded by $\mathcal{O}(\varepsilon^n)$ for all $n \in \mathbb{N}$ (cf. e.g. Theorem 5.1 of [BMS01]).

Summarising these estimates, we immediately derive from Theorem 4.1 that for every given probability $p > 0$ there is a constant C_p such that

$$\mathbb{P}\Big\{|a_\varepsilon|_{\mathbb{R}^n} \leq C_\pi(M\delta+2), \quad \|R_\varepsilon\| \leq C_p\Big\} \geq \mathbb{P}\Big\{\|u_0\| \leq \delta\varepsilon\Big\} - p$$

This easily implies Theorem 3.14. □

For the proof of Theorem 4.1 we follow basically the proof of analogous results in [B03]. First we establish a bound on solutions of (10). Using straightforward estimates, we show in Lemma 4.2 that solutions with initial conditions of order $\mathcal{O}(\varepsilon)$ stay of order $\mathcal{O}(\varepsilon)$ on a large time-scale of order $\mathcal{O}(\varepsilon^{-1})$. It is not trivial to extend this result to time-scales of order $\mathcal{O}(\varepsilon^{-2})$. We need the approximation result for this.

Lemma 4.2 *Suppose all assumptions of Section 3 are true. Let u be a solution of (10) with (random) initial condition u_0. Then for all times $t_\varepsilon \leq \varepsilon^{-1}$*

and all constants $\delta > 0$ and $C_w > 0$ we obtain with $D := M\delta + 2$ and $\varepsilon \in (0,1]$ sufficiently small that

$$\sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| \le \varepsilon^{-1} \quad \text{and} \quad \|u_0\| \le \delta\varepsilon \quad \Rightarrow \quad \sup_{t\in[0,t_\varepsilon]} \|u(t)\| \le D\varepsilon. \tag{22}$$

Proof: By Assumptions 3.5 and 3.6 we easily show

$$\|B(v) + \varepsilon^2 A(v)\|_Y \le \varepsilon^2 C_A \|v\| + C_B \|v\|^2. \tag{23}$$

Define the exit time $\tau_\varepsilon^* := \inf\{\tau > 0 \ : \ \|u(\tau)\| > D\varepsilon\}$. Hence, as long as $\tau < \tau_\varepsilon^*$ we obtain

$$\|B(u(\tau)) + \varepsilon^2 A(u(\tau))\|_Y \le \varepsilon^2 (C_A\varepsilon + C_B D) D. \tag{24}$$

Now we derive from (10) for all $t \le \min\{t_\varepsilon, \tau_\varepsilon^*\}$

$$\begin{aligned}
\|u(t)\| &\le M\|u_0\| + M\int_0^t (1 + (t-\tau)^{-\alpha})\|B(v) + \varepsilon^2 A(v)\|_Y d\tau + \varepsilon^{2H+1}\|W_L(t)\| \\
&\le [M\delta + \varepsilon^{2H-1}]\varepsilon + M\varepsilon^2 (C_A\varepsilon + C_B D) D \int_0^{t_\varepsilon} (1 + \tau^{-\alpha}) d\tau \\
&\le [M\delta + 1]\varepsilon + M(C_A\varepsilon + C_B D) D \frac{2-\alpha}{1-\alpha} \cdot \varepsilon^2 \\
&< D\varepsilon
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. This yields immediately $\tau_\varepsilon^* \ge t_\varepsilon$ on the set of interest, which finishes the proof. □

Proof of Theorem 4.1: Define $a_\varepsilon = \varepsilon^{-1}\Pi(u(t_\varepsilon))$ and $R_\varepsilon = \varepsilon^{-2} P_s u(t_\varepsilon)$. By Lemma 4.2 all we need to show is a bound on $P_s u$, as $|\varepsilon a_\varepsilon| = |\Pi(u(t_\varepsilon))| \le C_\pi D\varepsilon$ with C_π from Definition 3.9.

Using (11) and then (7) we obtain

$$\begin{aligned}
\|P_s u(t_\varepsilon)\| \le\ & M e^{-\omega t_\varepsilon}\|u_0\| + \varepsilon^{2H+1}\|P_s W_L(t_\varepsilon)\| \\
& + M\int_0^{t_\varepsilon} (1 + (t_\varepsilon - \tau)^{-\alpha}) e^{-(t_\varepsilon-\tau)\omega} \|\varepsilon^2 A(u(\tau)) + B(u(\tau))\|_Y d\tau.
\end{aligned}$$

As $\tau \le t_\varepsilon \le \varepsilon^{-1}$ and $\|u(\tau)\| \le D\varepsilon$ (by Lemma 4.2) we use (24) to finally end up with

$$\|P_s u(t_\varepsilon)\| \le M\delta\varepsilon^3 + M(C_A\varepsilon + C_B D)\varepsilon^2 D \int_0^\infty (1 + \tau^{-\alpha}) e^{-\tau\omega} d\tau + C_w \varepsilon^2 \ .$$

This implies the result. □

5 Residual

Let us first define a proper subset of the probability space Ω, which has sufficiently high probability.

Definition 5.1 *For W_L as in Assumption 3.4, and a, ψ_s as in (17) fix some $1 \gg \kappa \geq 0$. For $H = \frac{1}{2}$ we need $\kappa > 0$, but for $H > \frac{1}{2}$, we can simply choose $\kappa = 0$. Given C_w, δ, $C_a > 0$, and a time $T_0 > 0$ define the event $\mathcal{B}_\varepsilon \subset \Omega$ as*

$$\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(C_w, \delta, C_a, T_0) := \Big\{ \tilde{\omega} \in \Omega \Big| \sup_{t \in [0, T_0 \varepsilon^{-2}]} \|P_s W_L(t)\| \leq C_w \varepsilon^{1-2H-\kappa}, \\ \int_0^{T_0} \|P_s W_L(t\varepsilon^{-2})\| dt \leq C_w, \quad \|\psi_s(0)\| \leq \delta, \quad \sup_{T \in [0,T_0]} |a(T)| \leq C_a \Big\} . \tag{25}$$

As usual, we suppress the dependence of a, ψ_s, etc. on the randomness $\tilde{\omega} \in \Omega$ in the notation.

For fixed $0 < q < H$ define furthermore the following set:

$$\mathcal{D}_\varepsilon := \mathcal{B}_\varepsilon \cap \Big\{ \|a\|_{C^q([0,T_0])} \leq C_a \Big\} \tag{26}$$
$$\cap \quad \begin{cases} \Big\{ \|P_s W(\varepsilon^{-2} \cdot)\|_{C^q([0,T_0],X)} \leq C_w \varepsilon^{-2H} \Big\} & : \ \mathrm{tr}(Q) < \infty \\ \Omega & : \ \text{otherwise} \end{cases}$$

Here $C^q([0,T_0])$ denotes the space of Hölder-continuous functions from $[0,T_0]$ to $\mathbb{R}^n$ with Hölder exponent q.

Let us first show that the probability of $\mathcal{B}_\varepsilon$ and $\mathcal{D}_\varepsilon$ is large.

Lemma 5.2 *Consider the sets $\mathcal{B}_\varepsilon$ and $\mathcal{D}_\varepsilon$, as in Definition 5.1. For all $p > 0$ we can choose C_a and C_w sufficiently large, such that*

$$\mathbb{P}(\mathcal{B}_\varepsilon) \geq 1 - \mathbb{P}\{\|\psi_s(0)\| > \delta\} - \mathbb{P}\{|a(0)| > \delta\} - p \, . \tag{27}$$

Furthermore, for trace-class noise

$$\mathbb{P}(\mathcal{D}_\varepsilon) \geq 1 - \mathbb{P}\{\|\psi_s(0)\| > \delta\} - \mathbb{P}\{|a(0)| > \delta\} - p \, , \tag{28}$$

with $\varepsilon > 0$ sufficiently small.

Proof: The probability involving $\sup_{t\in[0,T_0\varepsilon^{-2}]}\|P_sW_L(t)\|$ is easily discussed with the methods of Section 4 in the proof of Theorem 3.14. Note that for $H=\frac{1}{2}$ and $\kappa=0$ we have $\mathbb{P}(\sup_{t\in[0,T_0\varepsilon^{-2}]}\|P_sW_L(t)\|\le C_w)\to 0$ for $\varepsilon\to 0$. Hence, in this case $\kappa>0$ is essential for $\mathbb{P}(\mathcal{B}_\varepsilon)$ to be large.

Large deviation results for $\|a\|_{L^\infty([0,T_0])}$ are well known. A very simple one was verified in [B03] for white noise (i.e., $H=\frac{1}{2}$). Nevertheless, that result relies on the continuity of the solution map and did not use $H=\frac{1}{2}$. It carries immediately over to amplitude equations driven by fractional Brownian motion, using large deviation results for the latter. We obtain for all $p>0$ that

$$\mathbb{P}\Big\{\|a\|_{L^\infty([0,T_0])}>C_a\Big\}\le\mathbb{P}\Big\{|a(0)|>\delta\Big\}+p\;,$$

provided C_a is sufficiently large, where C_a depends on δ and p.

The probability of the integral being bounded by C_w is obviously high in C_w/T_0. To verify this we can take the expectation and use the fact that the distribution of $P_sW_L(t)$ converges for $t\to\infty$ to a unique invariant measure. For the probability we use Chebychev's inequality.

Summarising all results, we derive (27).

Large deviation results for $\|a\|_{C^q([0,T_0])}$ are slightly more involved than the ones in $L^\infty([0,T_0])$, but it is straightforward to carry them over from large deviation results for the fractional Brownian motion. It is well known that due to the regularity of a and β we can only use $q\in(\frac{1}{2},H)$ for fractional noise with $H>\frac{1}{2}$, but for white noise we need $q<\frac{1}{2}$. Nevertheless, in this case we can at least choose q arbitrarily close to $\frac{1}{2}$

For the last term we use $\|W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)}=\varepsilon^{-2H}\|W\|_{C^q([0,T],X)}$ in law, due to the scaling invariance of the noise. Hence, it is obvious how to bound $\mathbb{P}\{\|W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)}>C_w\varepsilon^{-2H}\}$ using Chebychev's inequality. It is small, if C_w is large. Note that we need trace-class noise, because otherwise $\mathbb{P}$-almost surely $W(t)\notin X$.

Summarising all bounds, we derive (28). □

The main result of this section is the following theorem, which is proved in Lemma 5.6 and Lemma 5.5 with the help of a bound from Lemma 5.4, which are all stated below.

Theorem 5.3 (Residual) *Suppose all assumptions of Section 3 are true. For constants $\delta,C_a,C_w>0$, $q\in(0,H)$, some small $1\gg\kappa\ge 0$, and some time $T_0>0$ consider the set $\mathcal{D}_\varepsilon$ defined in Definition 5.1.*

Then there exist positive constants $C_{\rm res}^{(c)}$ and $C_{\rm res}^{(s)}$ such that for sufficiently small $\varepsilon > 0$ we obtain

$$\mathcal{D}_\varepsilon \quad\subset\quad \Big\{\tilde\omega\in\Omega\Big|\sup_{t\in[0,T_0\varepsilon^{-2}]}\|P_c\mathrm{Res}(\varepsilon\psi(t))\|\le C_{\rm res}^{(c)}(\varepsilon^{\gamma^*}+\varepsilon^{2-2\kappa}),\\ \sup_{t\in[0,T_0\varepsilon^{-2}]}\|P_s\mathrm{Res}(\varepsilon\psi(t))\|\le C_{\rm res}^{(s)}\varepsilon^{3-\kappa}\Big\}\;,$$

where $\gamma^* = 2H > 1$ for $H > \frac{1}{2}$, and $\gamma^* = \max\{3-2H, 2H\} \ge \frac{3}{2}$ for trace-class noise. For $q < \frac{1}{2}$ and trace-class noise $\gamma^* = 2-\eta(q)$ with $\eta(q)\to 0$ for $q\to\frac{1}{2}$. To be more precise $\eta(q)\sim\frac{2}{3}\sqrt{\frac{1}{2}-q}$ for $q\to\frac{1}{2}$.

Now the simplified Theorem 3.15 is a direct consequence of Theorem 5.3 and Lemma 5.2.

Note that for $\mathrm{tr}(Q) = \infty$ and $H = \frac{1}{2}$ we do not have a useful result. Nevertheless, it is possible to improve γ^* for noise not of trace-class. In that case we need certain bounds on the eigenvalues of Q. This is much more involved, as we need more knowledge on the space, in which W is defined (e.g. some fractional power spaces of L). We do not treat this here.

Let us now turn to the proof of Theorem 5.3. For realisations in the set $\mathcal{B}_\varepsilon$ from (25), we readily obtain bounds for ψ_s and ψ_c from (17). The proof of the following lemma is straightforward.

Lemma 5.4 *Suppose all assertions of Section 3 are true, and let ψ_c and ψ_s be as in (17). Then for all constants C_w, δ, $C_a > 0$, $1 \gg \kappa \ge 0$, and all times $T_0 > 0$, there are positive constants C_c and C_s such that for sufficiently small $\varepsilon > 0$*

$$\mathcal{B}_\varepsilon \quad\subset\quad \Big\{\tilde\omega\in\Omega\Big|\sup_{t\in[0,\frac{T_0}{\varepsilon^2}]}\|\psi_s(t)\|\le C_s\varepsilon^{-\kappa},\qquad \sup_{t\in[0,\frac{T_0}{\varepsilon^2}]}\|\psi_c(t)\|\le C_c,\\ \int_0^{\frac{T_0}{\varepsilon^2}}\|\psi_s(t)\|dt\le C_s\Big\}\;.$$

Let us now turn to the residual of $\varepsilon\psi$ from (19).

Lemma 5.5 *Under the assumptions of Lemma 5.4 define*

$$K_{\alpha,\omega} = \int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{-\omega\tau}d\tau\;. \qquad (29)$$

Then if $\varepsilon(C_A C_s + C_B C_s^2 \varepsilon^{-\kappa}) \le 1$

$$\mathcal{B}_\varepsilon \subset \Big\{\tilde\omega \in \Omega \Big| \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s \mathrm{Res}(\varepsilon\psi(t))\| \le C_{\mathrm{res}}^{(s)} \varepsilon^{3-\kappa}\Big\}$$

with $C_{\mathrm{res}}^{(s)} = K_{\alpha,\omega} M \|P_s\|_{\mathcal{L}(X)}[2C_s C_c C_B + 1 + C_A C_c]$.

Proof: Split $\mathrm{Res}(\varepsilon\psi)(t) = P_c\mathrm{Res}(\varepsilon\psi)(t) + P_s\mathrm{Res}(\varepsilon\psi)(t)$. Projecting (19) with P_s we obtain

$$\begin{aligned}
&P_s\mathrm{Res}(\varepsilon\psi)(t)\\
&= -\varepsilon^2\psi_s(t) + \mathrm{e}^{tL}\varepsilon^2\psi_s(0) + \varepsilon^{2H+1}P_s W_L(t)\\
&\quad + \int_0^t \mathrm{e}^{(t-\tau)L}\Big[\varepsilon^2 A_s\Big(\varepsilon\psi_c(\tau)+\varepsilon^2\psi_s(\tau)\Big) + B_s\Big(\varepsilon\psi_c(\tau)+\varepsilon^2\psi_s(\tau)\Big)\Big]d\tau\\
&= \varepsilon^2\Big[-\psi_s(t) + \mathrm{e}^{tL}\psi_s(0) + \varepsilon^{2H-1}P_s W_L(t) + \int_0^t \mathrm{e}^{(t-\tau)L}B_s(\psi_c(\tau))]d\tau\Big]\\
&\quad +2\varepsilon^3\int_0^t \mathrm{e}^{(t-\tau)L}B_s(\psi_c(\tau),\psi_s(\tau))d\tau + \varepsilon^3\int_0^t \mathrm{e}^{(t-\tau)L}A_s\psi_c(\tau)d\tau\\
&\quad +\varepsilon^4\int_0^t \mathrm{e}^{(t-\tau)L}\Big[A_s\psi_s(\tau) + B_s(\psi_s(\tau))\Big]d\tau.
\end{aligned}$$

Using (8), Assumptions 3.5 and 3.6, Lemma 5.4 and (18) to cancel terms of order ε^2, it is straightforward to finish the proof of Lemma 5.5. □

The bound for $P_c\mathrm{Res}(\varepsilon\psi(t))$ is a little bit more involved.

Lemma 5.6 *Under the assumptions of Theorem 5.3 define*

$$\mathcal{D}_\varepsilon \subset \Big\{\tilde\omega \in \Omega \Big| \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_c \mathrm{Res}(\varepsilon\psi(t))\| \le C_{\mathrm{res}}^{(c)} (\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa})\Big\}\,, \tag{30}$$

where $C_{\mathrm{res}}^{(c)}$ *is some sufficiently large constant. For* $H \in (\frac{1}{2}, 1)$ *the exponent is either* $\gamma^* = 2H > 1$ *for the general case or* $\gamma^* = \max\{3-2H, 2H\} \ge \frac{3}{2}$ *for trace-class noise. For* $H = \frac{1}{2}$ *and* $\mathrm{tr}(Q) < \infty$ *we have* $\gamma^* = 2 - \eta(q)$*, where* $\eta(q) \to 0$ *for* $q \to \frac{1}{2}$.

Proof: First we obtain from (19) that

$$\begin{aligned}
P_c\mathrm{Res}(\varepsilon\psi)(t) &= -\varepsilon\psi_c(t) + \varepsilon\psi_c(0) + \varepsilon^{2H+1}P_c W(t)\\
&\quad + \int_0^t \Big[\varepsilon^2 A_c\Big(\varepsilon\psi_c(\tau)+\varepsilon^2\psi_s(\tau)\Big) + B_c\Big(\varepsilon\psi_c(\tau)+\varepsilon^2\psi_s(\tau)\Big)\Big]d\tau.
\end{aligned}$$

Using $B_c(\psi_c) = 0$ from Assumption 3.5 we derive

$$
\begin{aligned}
&P_c\mathrm{Res}(\varepsilon\psi)(t)\\
&= \varepsilon \cdot \Big(\psi_c(0) - \psi_c(t) + \varepsilon^{2H} P_c W(t) + \varepsilon^2 \int_0^t \Big[A_c\psi_c(\tau) + 2B_c(\psi_c(\tau), \psi_s(\tau))\Big] d\tau\Big)\\
&\quad + \varepsilon^4 \cdot \int_0^t \Big[B_c(\psi_s(\tau)) + A_c\psi_s(\tau)\Big] d\tau. \qquad (31)
\end{aligned}
$$

For $t \leq T_0\varepsilon^{-2}$ the last term in (31) is bounded by $\mathcal{O}(\varepsilon^{2-2\kappa})$ due to Lemma 5.4. Using the substitution $T = \varepsilon^2 t$ to the slow time-scale and (15) to cancel out most $\mathcal{O}(\varepsilon)$-terms, we derive

$$
P_c\mathrm{Res}(\varepsilon\psi)(t) = 2\varepsilon\int_0^T\Big[B_c\Big(a(s)\cdot e, \psi_s(s\varepsilon^{-2})\Big) + B_c\Big(a(s)\cdot e, L^{-1}B_s(a(s)\cdot e)\Big)\Big] ds + \mathcal{O}(\varepsilon^{2-2\kappa})
$$

Plugging in the definition of ψ_s from (18), we obtain

$$
\begin{aligned}
&\int_0^T B_c\Big(a(s)\cdot e, \Big[\psi_s(s\varepsilon^{-2}) + L^{-1}B_s(a(s)\cdot e)\Big]\Big) ds \qquad (32)\\
&= \int_0^T B_c\Big(a(s)\cdot e, \mathrm{e}^{s\varepsilon^{-2}L}\psi_s(0))\Big) ds \qquad (33)\\
&\quad + \int_0^T B_c\Big(a(s)\cdot e, \varepsilon^{2H-1} P_s W_L(s\varepsilon^{-2}))\Big) ds \qquad (34)\\
&\quad + \int_0^T B_c\Big(a(s)\cdot e, \int_0^{\frac{s}{\varepsilon^2}} \mathrm{e}^{(\frac{s}{\varepsilon^2}-\tau)L} B_s(\psi_c(\tau)) d\tau + L^{-1}B_s(\psi_c(s))\Big) ds. \qquad (35)
\end{aligned}
$$

Now we have to bound the terms in (33)–(35) separately. The main idea is that formally $\varepsilon^{-2}L\mathrm{e}^{t\varepsilon^{-2}L} \to \delta_0 I$ for $\varepsilon \to 0$, where δ_0 is the Delta-distribution in time. Unfortunately, we do not have enough regularity, to preform this limit in a simple way.

For simplicity of presentation, we postpone now technical details to the forthcoming Lemmas 5.7, 5.8, and 5.9. Here Lemma 5.7 is a straightforward estimate to bound (33). Lemma 5.8 uses fractional integration by parts, and gives a bound for (34) in terms of Hölder norms of W and a. While the third Lemma 5.9 bounds (35) in terms of Hölder norms of a. We now first finish the proof of Lemma 5.6 .

First by Lemma 5.7 we obtain for $T \leq T_0$ and $\tilde{\omega} \in \mathcal{B}_\varepsilon$

$$
\|(33)\| \leq C\varepsilon^2\|a\|_{L^\infty([0,T])} \cdot \|\psi_s(0)\| \leq C\varepsilon^2 C_a\delta = \mathcal{O}(\varepsilon^2)\ .
$$

For the third term we derive for $q > \frac{1}{2}$ and by Lemma 5.9 for $\tilde{\omega} \in \mathcal{D}_\varepsilon$ and $T \le T_0$

$$\|(35)\| \le C\varepsilon^2 \|a\|_{C^q([0,T])} = \mathcal{O}(\varepsilon^2) \ .$$

For $q < \frac{1}{2}$ there is, again by Lemma 5.9, some $\eta(q)$ with $\eta(q) \to 0$ for $q \to \frac{1}{2}$, such that

$$\|(35)\| \le C\varepsilon^{2-\eta(q)} \|a\|_{C^q([0,T])} = \mathcal{O}(\varepsilon^{2-\eta(q)}) \ .$$

For (34) we establish two bounds. The direct estimate gives

$$\|(34)\| \le C\varepsilon^{2H-1} \|a\|_{L^\infty([0,T])} \cdot \int_0^T \|P_s W_L(s\varepsilon^{-2})\| ds = \mathcal{O}(\varepsilon^{2H-1}).$$

But for small H and trace-class noise, we get a better bound by Lemma 5.8. As long as $q > \frac{1}{2}$ and $T \le T_0$

$$\|(34)\| \le C\varepsilon^2 \|a\|_{C^q([0,T])} \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)} = \mathcal{O}(\varepsilon^{2-2H}) \ .$$

Moreover, for $q < \frac{1}{2}$ and $T \le T_0$ we derive, again by Lemma 5.8,

$$\|(34)\| \le C\varepsilon^{2-\eta(q)} \|a\|_{C^q([0,T])} \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)} = \mathcal{O}(\varepsilon^{2-2H-\eta(q)})$$

with $\eta(q)$ as above.

Hence, we can bound (32) for $q > \frac{1}{2}$ by $\mathcal{O}(\varepsilon^{2H-1})$. If additionally $\mathrm{tr}(Q) < \infty$, then we derive the bound $\mathcal{O}(\varepsilon^{2-2H})$. For $q < \frac{1}{2}$ we get roughly the same result, only our exponent is slightly smaller by some $\eta(q)$. This finishes the proof of Lemma 5.6. □

Let us finally prove the lemmas used in the proof of Lemma 5.6.

Lemma 5.7 *Let $a : [0,T] \to \mathbb{R}^n$ be a measurable function such that $\|a\|_{L^\infty} := \sup_{t\in[0,T]} |a(t)| < \infty$. Let Assumptions 3.2 and 3.5 be true. Furthermore fix $e = (e_1, \ldots, e_n) \in X^n$ and define $\|e\|^2 := \sum_{i=1}^n \|e_i\|^2$. Then for all $w \in X$*

$$\| \int_0^T B_c\Big(a(\tau) \cdot e, \mathrm{e}^{\tau\varepsilon^{-2}L} w)\Big) d\tau \| \le \varepsilon^2 M C_B \int_0^\infty \mathrm{e}^{-s\omega} ds \|a\|_{L^\infty} \|e\| \|w\|.$$

Proof: This is a straightforward estimate. Note that the left hand side is obviously bounded by $MC_B \int_0^T \|a(\tau)\cdot e\| \mathrm{e}^{-\tau\varepsilon^{-2}\omega} d\tau \|w\|$. Now Cauchy-Schwarz inequality and the substitution $\tau' = \tau\varepsilon^{-2}$ gives the claim. □

Lemma 5.8 *Suppose that L as in Assumption 3.1 generates an analytic semigroup, as in Assumption 3.2, and W is a fractional Wiener-process as in Assumption 3.4 with $H \geq \frac{1}{2}$ and $\operatorname{tr}(Q) < \infty$. Then for $q > \frac{1}{2}$ and all Hölder continuous $a : [0,T] \to \mathbb{R}^n$*

$$\Big\| \underbrace{\int_0^T B_c\Big(a(\tau)\cdot e, P_s W_L(\tau\varepsilon^{-2})\Big)d\tau}_{=:I_w} \Big\| \leq C\varepsilon^2 \|a\|_{C^q([0,T],\mathbb{R}^n)} \cdot \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)}$$

Moreover, for all $r > 1$ we obtain for some constant $C_r > 0$ that

$$\|I_w\| \leq C_r \varepsilon^{2-\frac{3r+1}{2r(r+1)}} \cdot \|a\|_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T],\mathbb{R}^n)} \cdot \|P_s W(\varepsilon^{-2}\cdot)\|_{C^{\frac{1}{2}-\frac{r+1}{4r(r+2)}}([0,T],X)}.$$

Proof: First we establish a straightforward estimate using Assumption 3.5.

$$\begin{aligned}
&\Big\| \int_0^T B_c\Big(a(\tau)\cdot e, P_s W_L(\tau\varepsilon^{-2})\Big)d\tau \Big\| \\
&= \Big\| \sum_{i=1}^n B_c\Big(e_i, \int_0^T a_i(\tau) P_s W_L(\tau\varepsilon^{-2}) d\tau\Big) \Big\| \\
&\leq C_B \|P_c\|_{\mathcal{L}(X)} \cdot \sum_{i=1}^n \|e_i\| \cdot \Big\| \int_0^T a_i(\tau) \frac{\partial}{\partial\tau} \int_0^\tau P_s W_L(\sigma\varepsilon^{-2}) d\sigma d\tau \Big\| \\
&\leq C\|a\|_{C^q([0,T])} \cdot \Big\| t \mapsto \int_0^t P_s W_L(\tau\varepsilon^{-2}) d\tau \Big\|_{C^q([0,T],X)} \qquad (36)
\end{aligned}$$

for $q > \frac{1}{2}$, where we used the fractional integration by parts formula (cf. Lemma 8.4).

For trace-class noise it is well known that $W(t)$ exhibits Hölder continuous paths in X. Hence, we obtain using integration by parts

$$\begin{aligned}
&\int_0^t P_s W_L(\tau\varepsilon^{-2}) d\tau \\
&= \int_0^t \int_0^{\tau\varepsilon^{-2}} \mathrm{e}^{(\tau\varepsilon^{-2}-\sigma)L} dP_s W(\sigma) d\tau \\
&= \int_0^t P_s W(\tau\varepsilon^{-2}) d\tau + L_s \int_0^t \int_0^{\tau\varepsilon^{-2}} \mathrm{e}^{(\tau\varepsilon^{-2}-\sigma)L} P_s W(\sigma) d\sigma d\tau \\
&= \int_0^t P_s W(\tau\varepsilon^{-2}) d\tau + \varepsilon^{-2} L_s \int_0^t \int_0^\tau \mathrm{e}^{(\tau-\sigma)L\varepsilon^{-2}} P_s W(\sigma\varepsilon^{-2}) d\sigma d\tau \\
&= \int_0^t \mathrm{e}^{(t-\sigma)L\varepsilon^{-2}} P_s W(\sigma\varepsilon^{-2}) d\sigma\ , \qquad (37)
\end{aligned}$$

where we used Fubini in the last step. The previous equation (37) follows also from integrating the equation $d(P_sW_L) = L_sP_sW_Ldt + d(P_sW)$.

Now the bound on the convolution operator with the semigroup in terms of Hölder norms (cf. (55) of Lemma 8.2 with $h = P_sW(\varepsilon^{-2}\cdot)$) together with (36) and (37) yields the first result.

For the second result choose some small $\alpha > 0$ fixed later and some large $r > 0$. Then by applying the fractional integration by parts formula (cf. Lemma 8.4 with $p_1 = \frac{1}{2} - \alpha/(r+1)$ and $p_2 = \frac{1}{2} + 2\alpha/(r+1)$) we change (36) to

$$\|I_w\| \leq C\|a\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \Big\| t \mapsto \int_0^t P_sW_L(\tau\varepsilon^{-2})d\tau \Big\|_{C^{\frac{1}{2}+\frac{2\alpha}{r+1}}([0,T],X)} .$$

Finally (37) together with the bound for the convolution operator from Corollary 8.3 implies

$$\|I_w\| \leq C\varepsilon^{2-\frac{1}{r+1}-2\alpha\frac{r+2}{r+1}} \cdot \|a\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \|P_sW(\varepsilon^{-2}\cdot)\|_{C^{\frac{1}{2}-\alpha}([0,T],X)} .$$

Choosing $\alpha = \frac{r+1}{4r(r+2)}$ gives the assertion. An easy calculation shows that the condition $r > 1$ is obviously sufficient to apply Corollary 8.3. □

Lemma 5.9 *Let Assumptions 3.1, 3.2, and 3.5 be true. For $a \in C^q([0,T],\mathbb{R}^n)$ define*

$$J := \int_0^T B_c\Big(a(t)\cdot e, \int_0^{t\varepsilon^{-2}} \mathrm{e}^{(t\varepsilon^{-2}-\tau)L} B_s(a(\varepsilon^2\tau)\cdot e)d\tau + L_s^{-1}B_s(a(t)\cdot e)\Big)dt .$$

Then for $q > \frac{1}{2}$ we obtain

$$\|J\| \leq C\varepsilon^2\|a\|^3_{C^q([0,T])}.$$

Moreover, for $r > 1$ and $q = \frac{1}{2} - 1/(4r(r+2))$ we derive

$$\|J\| \leq C\varepsilon^{2-\frac{3r+1}{2r(r+1)}}\|a\|^3_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T])} .$$

Proof: Inserting $a\cdot e = \sum_{j=1}^n a_je_j$ we obtain

$$\begin{aligned} J &= \int_0^T B_c\Big(a(t)\cdot e, \varepsilon^{-2}\int_0^t \mathrm{e}^{(t-\tau)L\varepsilon^{-2}} B_s(a(\tau)\cdot e)d\tau + L_s^{-1}B_s(a(t)\cdot e)\Big)dt \\ &= \sum_{i,j,k=1}^n B_c\Big(e_i, L_s^{-1}\int_0^T a_i(t)\Big[\varepsilon^{-2}L_s\int_0^t \mathrm{e}^{(t-\tau)L\varepsilon^{-2}} B_s(e_j,e_k)a_j(\tau)a_k(\tau)d\tau \\ &\qquad + B_s(e_j,e_k)a_j(t)a_k(t)\Big]dt\Big) . \end{aligned}$$

Hence,

$$\begin{aligned}\|J\| &\le C\sum_{i,j,k=1}^{n}\Big\|\int_0^T a_i(s)L_s^{-1}\Big[\varepsilon^{-2}L\int_0^s \mathrm{e}^{(s-\tau)L\varepsilon^{-2}}B_s(e_j,e_k)a_j(\tau)a_k(\tau)d\tau \\ &\qquad\qquad +B_s(e_j,e_k)a_j(s)a_k(s)\Big]ds\Big\| \\ &\le C\varepsilon^2\sum_{i,j,k=1}^{n}\|a_i\|_{C^q([0,T])}\|a_j\|_{C^p([0,T])}\|a_k\|_{C^p([0,T])}\end{aligned}$$

by Lemma 8.5, where $p\in(1-q,1)$ arbitrary. The main idea of Lemma 8.5 is to rewrite the sum of the two terms as a derivative of a convolution integral, and use fractional integration by parts. Note that constants depend on p and q. Choosing $p\in(1-q,q]$, we derive

$$\|J\|\le C\varepsilon^2\|a\|^3_{C^q([0,T])}.$$

Moreover the second result of Lemma 8.5 immediately yields the second assertion of this lemma. □

6 Approximation

This section provides the following theorem. Its simplified form was stated in Theorem 3.16, which is now a direct consequence of Lemma 5.2 and Theorem 6.1.

Theorem 6.1 (Approximation) *Suppose all assumptions of Section 3 are true. Fix positive constants δ, C_a, C_w, $q\in(0,H)$, some small $1\gg\kappa>0$, and some time $T_0>0$. Consider the set $\mathcal{D}_\varepsilon$ from Definition 5.1.*

Then there is a constant $C_{\text{att}}>0$ such that for sufficiently small $\varepsilon>0$ we obtain for all solutions u of (10), where ψ is defined in (17)

$$\mathcal{D}_\varepsilon\subset\Big\{\tilde\omega\in\Omega\Big|\sup_{t\in[0,T_0\varepsilon^{-2}]}\|u(t)-\varepsilon\psi(t)\|\le C_{\text{att}}(\varepsilon^{\gamma^*}+\varepsilon^{2-2\kappa})\Big\},$$

where the constant $\gamma^*(H,q)$ was defined in Theorem 5.3.

Proof: Define R via

$$u-\varepsilon\psi=\varepsilon^2R=\varepsilon^2R_c+\varepsilon^3R_s, \tag{38}$$

where as usual $R_c \in \mathcal{N}$ and $R_s \in P_sX$. Note that by definition $R(0) = 0$. Our aim is to prove that R_c, R_s are of order $\mathcal{O}(1)$. However, our result is slightly worse, but nevertheless sufficient for the proof.

Taking the difference of (10) and (19) yields

$$\begin{aligned} R(t) &= \varepsilon^2 \int_0^t \mathrm{e}^{(t-\tau)L} A(R(\tau)) d\tau - \varepsilon^{-2} \mathrm{Res}(\varepsilon\psi)(t) \\ &\quad + \varepsilon^{-2} \int_0^t \mathrm{e}^{(t-\tau)L} [B(u(\tau)) - B(\varepsilon\psi(\tau))] d\tau. \end{aligned} \tag{39}$$

As $u = \varepsilon\psi + \varepsilon^2 R$,

$$\begin{aligned} B(u) - B(\varepsilon\psi) &= 2B(\varepsilon\psi_c + \varepsilon^2\psi_s, \varepsilon^2 R_c + \varepsilon^3 R_s) + B(\varepsilon^2 R_c + \varepsilon^3 R_s) \\ &= 2\varepsilon^3 B(\psi_c + \varepsilon\psi_s, R_c + \varepsilon R_s) + \varepsilon^4 B(R_c + \varepsilon R_s) \; . \end{aligned}$$

Projecting (39) to P_cX and expanding the quadratic terms we obtain

$$\begin{aligned} R_c(t) &= \varepsilon^2 \int_0^t A_c(R_c + \varepsilon R_s) d\tau - \varepsilon^{-2} P_c \mathrm{Res}(\varepsilon\psi)(t) \\ &\quad + 2\varepsilon^2 \int_0^t [B_c(\psi_c, R_s) + B_c(\psi_s, R_c)] d\tau + \varepsilon^4 \int_0^t B_c(R_s) d\tau \\ &\quad + \varepsilon^3 \int_0^t [B_c(\psi_s, R_s) + 2B_c(R_c, R_s)] d\tau \; , \end{aligned} \tag{40}$$

where we used that $B_c = 0$ on P_cX by Assumption 3.5. Analogously,

$$\begin{aligned} R_s(t) &= \varepsilon \int_0^t \mathrm{e}^{(t-\tau)L} A_s(R_c + \varepsilon R_s) d\tau \\ &\quad + 2 \int_0^t \mathrm{e}^{(t-\tau)L} B_s(\psi_c + \varepsilon\psi_s, R_c + \varepsilon R_s) d\tau \\ &\quad + \varepsilon \int_0^t \mathrm{e}^{(t-\tau)L} B_s(R_c + \varepsilon R_s) d\tau - \varepsilon^{-3} P_s \mathrm{Res}(\varepsilon\psi)(t) \; . \end{aligned} \tag{41}$$

Define the stopping time

$$\tau_R^* := \inf\{t > 0 \; : \; \|R_s(t)\| > \varepsilon^{-1} \quad \text{or} \quad \|R_c(t)\| > \varepsilon^{-1}\} \; . \tag{42}$$

Now we use Lemma 5.4 and Theorem 5.3 to obtain for $t \leq \min\{\tau_R^*, \; T_0\varepsilon^{-2}\}$ and $\tilde{\omega} \in \mathcal{D}_\varepsilon$

$$\begin{aligned} \|R_c(t)\| &= \mathcal{O}(\varepsilon^{-2\kappa} + \varepsilon^{\gamma^*-2}) \\ &\quad + C\varepsilon^2 \int_0^t \Big[\|R_c(\tau)\| (1 + \|\psi_s(\tau)\|) + \|R_s(\tau)\| \Big] d\tau \; , \end{aligned} \tag{43}$$

where we used Assumptions 3.5 and 3.6. Note that for $\kappa > 0$ we do not have a uniform $\mathcal{O}(1)$ bound on ψ_s. Therefore, we have to leave this term in (43), in order to get the right estimate by Gronwall's inequality.

Analogously we obtain,

$$\begin{aligned} \|R_s(t)\| &= \mathcal{O}(\varepsilon^{-\kappa}) \\ &\quad + C\int_0^t (1+(t-\tau)^{-\alpha})\mathrm{e}^{-(t-\tau)\omega}\Big[\|R_c(\tau)\| + \varepsilon\|R_s(\tau)\|\Big]d\tau \ , \end{aligned} \tag{44}$$

where we additionally used (7).

Now the claim follows from the generalised Gronwall's inequality of Lemma 8.1 with $\gamma(t) = 1+\|\psi_s(\tau)\|$ and Lemma 5.4 to bound $\varepsilon^2 \int_0^{T_0\varepsilon^{-2}} \gamma(\tau)d\tau = \mathcal{O}(1)$ appearing in the exponent. □

7 Applications

There are numerous examples of equations of type (1) in the physics literature. One example is the growth of rough amorphous surfaces (cf. Section 7.1), which after rescaling to dimension-less form is described by (45). In that model $h = h(t,x)$ is the height profile of a growing surface over $x \in [0,2\pi]^d$, $d = 1,2$. See for example [RLH01, RM+00] for the physical derivation, and [BGR02] or [BlGu] for the existence of global solutions and the uniqueness of local solutions. The noise ξ is Gaussian space-time noise, which in most applications should be space-time white. The model is known to exhibit an instability for large ν leading to the growth of hills (see e.g. [RLH00]).

The model was already proposed in [SP94] for the growth of crystalline surfaces. Another related model is the well known Kuramoto-Shivashinski equation, where the term $\Delta|\nabla h|^2$ is replaced by $|\nabla h|^2$. This model originally describes the propagation of flames, but was recently also proposed to model surfaces obtained by ion-sputtering (see e.g. [CB95, FB+02]).

Our other example is the *Rayleigh-Bénard problem* (see Section 7.2), which is the paradigm of pattern formation in convection problems. It is described by the Navier-Stokes equation coupled to a heat equation (see e.g. [CH93, Ge98, W97]). For simplicity we consider the equation only in a strip. The full three dimensional problem is more technical, but the result is similar. For the set of equations see (47) - (49). One major problem arises due to the fact that the linearised operator L is not self-adjoint.

7.1 Surface Growth Model

Consider the surface growth model with fractional noise of trace-class with Hurst parameter $H \in [\frac{1}{2}, 1)$ given by:

$$\partial_t h = -\Delta_{\text{per}}^2 h - \nu \Delta_{\text{per}} h - \Delta_{\text{per}} |\nabla h|^2 + \varepsilon^{2H+1} \xi \ . \tag{45}$$

Here Δ_{per} is the Laplacian w.r.t. periodic boundary conditions. Suppose initial condition $h(0) = 0$ corresponding to an initially flat surface, and assume for simplicity of presentation that h is a 2π-periodic height profile of the surface over $[0, 2\pi]^2$, which corresponds to the full three-dimensional growth problem.

Let $H^s_{\text{per}} := H^s_{\text{per}}([0, 2\pi]^2) = D((1 - \Delta_{\text{per}})^{s/2})$ for $s \geq 0$ be the standard fractional Sobolev space, and define H^{-s}_{per} as the dual of H^s_{per}. Define $\mathcal{L}_\nu = -\Delta^2_{\text{per}} - \nu \Delta_{\text{per}}$. We consider $\mathcal{L}_\nu$ as an operator on $X = H^{3/2}_{\text{per}}$ with domain $D(\mathcal{L}_\nu) = H^{11/2}_{\text{per}}$. We could also choose the Sobolev space $W^{1,4}_{\text{per}}([0, 2\pi]^2)$ for X. Nevertheless, we work in a Hilbert space setting, where the assumptions are easier to verify.

Obviously, there is an orthogonal basis $\{e_k\}_{k \in \mathbb{Z}^2}$ of eigenfunctions of both $\mathcal{L}_\nu$ and Δ_{per} given by $e_k(x) := \mathrm{e}^{ikx}$ for $k \in \mathbb{Z}^2$. We use complex functions for simplicity of notation. To obtain a real basis, consider products of $\sin(k_j x_j)$ and $\cos(k_j x_j)$ instead. The corresponding eigenvalues are given by

$$\lambda_k(\nu) = |k|^2 (\nu - |k|^2) \quad \text{for} \quad k \in \mathbb{Z}^2 \ .$$

Obviously, $\lambda_0(\nu) = 0$. Furthermore $\lambda_k(\nu) = 0$ for $|k| = 1$ if and only if $\nu = \nu_c := 1$, and these λ_k simultaneously change stability at ν_c. Hence, we have a bifurcation.

We apply the general theory to two different regimes. For $\nu < \nu_c$ one has $\mathcal{N} = \text{span}\{1\}$, and we obtain an SDE for the mean value of the solution. But we do not focus on this example, as in this case the mean value behaves like a rescaled fractional Brownian motion, which follows from $B_s(1, \cdot) \equiv 0$ and $A_c(1) = 0$ leading to $\Gamma = 0$ and $\nu = 0$ in the amplitude equation. Also one could see directly from the equation that the constant mode decouples from the rest of the equation.

More interesting is the case when $\nu \approx \nu_c$ and the distance from the bifurcation point and the noise strength are of a comparable size, for example $\nu = \nu_c + \varepsilon^2 \nu_0$ for some $\nu_0 \in [-1, 1]$. Now

$$L := \mathcal{L}_{\nu_c} = -\Delta^2_{\text{per}} - \Delta_{\text{per}} \quad \text{and} \quad A := -\nu_0 \Delta_{\text{per}} \ .$$

It is standard to verify that Assumptions 3.2 and 3.6 are true, if we choose for instance $Y = H^{-2}_{\text{per}}$. Moreover, we obtain $\mathcal{N} = \text{span}\{e_k \;:\; |k| \in \{0,1\}\}$ and Assumption 3.1 is fulfilled with $n = 5$. With a slight abuse of notation, we fix a real basis of $\mathcal{N}$

$$e = (\cos(x_1),\ \sin(x_1),\ 1,\ \cos(x_2),\ \sin(x_2)).$$

The bilinear operator B is given by

$$B(u, v) = -\Delta_{\text{per}}(\nabla u \cdot \nabla v).$$

It is easy to check that B fulfils Assumption 3.5. First $B_c = 0$ on $P_c X$ is straightforward, and the rest of the assumption follows from the continuity of the mappings

$$X \longrightarrow W^{1,4}_{\text{per}}([0, 2\pi]^2) \stackrel{|\nabla\cdot|^2}{\longrightarrow} L^2([0, 2\pi]^2) \stackrel{-\Delta_{\text{per}}}{\longrightarrow} H^{-2}_{\text{per}}([0, 2\pi]^2).$$

Assumption 3.7 is for instance verified in [BlGu] for noise that is white in time, which nevertheless carries immediately over to our case, as we need only enough regularity of the stochastic convolution. Furthermore Assumption 3.4 is true for trace-class noise with $H \geq \frac{1}{2}$, see for example [DPM02]. We already have $W_L(t) \in H^2_{\text{per}}$.

Using the definitions from (13) and (14), it is an elementary but long computation to derive the amplitude equation (15) explicitly. First

$$\Gamma[a] = -\frac{1}{3}\Big(a_1(a_1^2 + a_2^2),\ a_2(a_1^2 + a_2^2),\ 0,\ a_4(a_4^2 + a_5^2),\ a_5(a_4^2 + a_5^2)\Big)$$

and $\nu(a) = \nu_0 \cdot (a_1, a_2, 0, a_4, a_5)$. It is an interesting observation that the deterministic part decouples into three independent equations. One describing functions constant in x-direction, another constant in y-direction, an the last part describing the constant itself.

The amplitude equation is now given by

$$da(T) = \nu_0 a(T)dT + \Gamma[a(T)]dT + d\beta^H(T), \tag{46}$$

where β^H is a fractional Brownian motion in $\mathbb{R}^5$ with Hurst parameter H and covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$ determined by the covariance operator Q of the original noise process. Here $\Sigma = \Pi Q \Pi^*$. We remark without proof that in the case of spatially homogeneous noise this covariance matrix is diagonal (cf. e.g. [Bl05a] for $H = \frac{1}{2}$). This implies that here also the SDE decouples into three independent parts.

Note that we obtain a stable equation for a, in the sense of Assumptions 3.13, where large deviation estimates are no problem. The *main results* are now the following. The attractivity is trivial with $t_\varepsilon = 0$, as $h(0) = 0$, and the approximation states that we have with high probability $h(t) \approx \varepsilon a(\varepsilon^2 t) \cdot e$ on a time-interval of order $\mathcal{O}(\varepsilon^{-2})$, where a is a solution of (46) with $a(0) = 0$. We refrain from reformulating the abstract result for this case.

7.2 Rayleigh-Bénard Problem

We consider the two dimensional Rayleigh-Bénard problem in a strip, where a fluid is heated from below. The three dimensional problem in a box can be treated similarly, but the notation is much more involved.

In the following denote by (v, w) the velocity field of the fluid in $(y, z) \in D := [0, 2\pi] \times [0, \pi]$, where z is the vertical direction. Hence, the fluid is heated at $z \equiv 0$. Let p be the pressure and θ the normalised temperature, which means that $\theta \equiv 0$ and $(v, w) \equiv 0$ is heat transport without motion.

In dimension-less form the governing Navier-Stokes and heat equations are given by (see e.g. [Ge98] or [W97])

$$\partial_t(v, w) + ((v, w) \cdot \nabla)(v, w)) = -\nabla p + (0, 1)\frac{R}{P}\theta + \Delta(v, w) \qquad (47)$$

$$\partial_t \theta - w + ((v, w) \cdot \nabla)\theta = \frac{1}{P}\Delta\theta + \varepsilon^{2H+1}\xi \qquad (48)$$

$$\operatorname{div}(v, w) = 0 \qquad (49)$$

We suppose periodic boundary conditions in y both for θ and (v, w). Moreover $\partial_z v = w = \theta = 0$ for $z = 0, \pi$. The noise ξ is trace-class fractional noise, corresponding to fluctuations in the temperature. We could also incorporate fluctuations in the velocity field, but we neglect this for simplicity.

In order to rule out motion of the whole fluid in the y-direction, we suppose vanishing mean flux $\int_0^\pi v dz$. We use the following constants: R is the Rayleigh number, P the Prantl number, and the quotient $\rho = R/P$ is the Reynolds number. The Rayleigh number is a dimension-less measure of the heat difference between top and bottom of the strip, while the Prantl number depends only on the properties of the fluid.

Consider the following operator

$$\mathcal{L}u := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} u + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} u = \begin{pmatrix} \Delta v \\ \Delta w + \rho\theta \\ \frac{1}{P}\Delta\theta + w \end{pmatrix}$$

with domain

$$\mathcal{D} = \Big\{ u = (v, w, \theta) \in H^2(D, \mathbb{R}^3) : \ 2\pi - \text{periodic in } y,$$

$$\partial_z v = w = \theta = 0 \ \text{ for } z = 0, \pi, \quad \operatorname{div}(v, w) = 0, \ \int_0^\pi v dz = 0 \Big\}.$$

Let $\mathcal{Q}$ be the L^2-projection onto the functions that are divergence-free in the first two components, and fulfil the boundary conditions of the problem. It is easy to check that $\mathcal{Q}$ projects to the space $\{(v, w) \in H^2(D, \mathbb{R}^2) : (v, w, 0) \in \mathcal{D}\}$ which is spanned by the following orthogonal basis of eigenfunctions of the Laplacian:

$$\begin{pmatrix} \cos(mz) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} m\cos(mz)\cos(ky) \\ k\sin(mz)\sin(ky) \end{pmatrix}, \quad \begin{pmatrix} m\cos(mz)\sin(ky) \\ -k\sin(mz)\cos(ky) \end{pmatrix}$$

for $k, m \in \mathbb{N}$. Hence, $\mathcal{Q}$ and Δ commute, and

$$\mathcal{QL} := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} + \mathcal{Q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} .$$

Define furthermore for $u = (v, w, \theta)$

$$B_0(u, \tilde{u}) = -\mathcal{Q}[(v\partial_y + w\partial_z)\tilde{u}] \quad \text{and} \quad B(u, \tilde{u}) = \tfrac{1}{2}B_0(u, \tilde{u}) + \tfrac{1}{2}B_0(\tilde{u}, u). \quad (50)$$

Now the equations (47) - (49) are given as an equation in $\mathcal{D}$ by

$$\partial_t u = \mathcal{QL}u + B(u, u) + \varepsilon^2(0, 0, \xi) .$$

In the following we explicitly give the eigenfunctions of $\mathcal{QL}$ and the projection P_c. Major difficulties arise due to the fact that $\mathcal{QL}$ is in general not self-adjoint. Hence, the eigenfunctions are not orthogonal. Consider for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\varphi^{\pm}(k, m)e^{iky} = \Big(-ik\cos(mz), \quad -\frac{k^2}{m}\sin(mz), \quad -\frac{s(s + \lambda^{\pm})}{\rho m}\sin(mz) \Big) e^{iky} ,$$

where $s = k^2 + m^2$ and

$$\lambda^{\pm} = -\frac{1+P}{2P}s \pm \sqrt{\frac{k^2\rho}{s} + \frac{1}{4}\Big(\frac{1-P}{P}\Big)^2 s^2}.$$

We use complex-valued functions for simplicity of notation. It is easy to check that $\varphi^{\pm}(k,m)\mathrm{e}^{iky}$ is divergence-free and fulfils the boundary conditions. Moreover, it is an eigenfunction of $\mathcal{QL}$ to the eigenvalue $\lambda^{\pm}$. We obtain a basis of eigenfunctions, if we include the functions $(\cos(mz), 0, 0)$ corresponding to eigenvalues $-m^2$ for $m > 1$.

For the eigenvalues we see that as long as $\rho \le s^3/Pk^2$ always $\lambda^- < 0$ and $\lambda^+ < 0$. It is well know that the critical Reynolds number for the unbounded domain is $\rho_c = 27/4P$ with unstable wavenumber $s_c = \pm 1/\sqrt{2} \notin \mathbb{Z}$ (see e.g. [Ge98]). As we have a bounded domain with $s \in \mathbb{Z}$ our result is slightly different. We obtain:

$$\rho_c = \frac{8}{P}, \quad R_c = 8, \quad \text{and} \quad (m_c, k_c) = (1, \pm 1).$$

Define $\rho = \rho_c + \varepsilon^2\rho_0$ with $\rho_0 \in [-1, 1]$ and

$$L := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} + \mathcal{Q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_c \\ 0 & 1 & 0 \end{pmatrix}, \quad A := \mathcal{Q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{N} = \mathrm{span}\{\varphi^+(1,1)\mathrm{e}^{iy},\ \varphi^+(1,-1)\mathrm{e}^{-iy}\} = \mathrm{span}\{e_1, e_2\}$ fulfilling Assumption 3.1 with

$$e_1 := \begin{pmatrix} \cos(z)\sin(y) \\ -\sin(z)\cos(y) \\ -P\sin(z)\cos(y)/2 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} -\cos(z)\cos(y) \\ -\sin(z)\sin(y) \\ -P\sin(z)\sin(y)/2 \end{pmatrix}.$$

Now define the projection P_c to $\mathcal{N}$ as a projection along the other eigenspaces. It involves the dual basis of eigenfunctions of the adjoint L^*, and ensures that P_c and L commute. The main advantage is that Assumption 3.2 is easy to verify, as the operators P_c, L, and e^{tL} are given in explicit series expansions with respect to the same eigenfunctions.

For simplicity, we use $X = \mathcal{D}$ with high spatial regularity. However, it is possible to relax this constraint significantly. Here $D(L) = H^4(D, \mathbb{R}^3) \cap X$, and $Y = \mathbb{L}^2(D)$ is the closure of X in $L^2(D, \mathbb{R}^3)$. Now Assumption 3.2 is straightforward to check.

Next we verify that Assumption 3.6 is also true. Moreover, as e_j is divergence-free and $\mathcal{Q}$ and the Laplacian commute (i.e., they exhibit joint invariant eigenspaces), it is a straightforward calculation to verify

$$Ae_j = \frac{\rho_0}{4}\frac{P}{1+P}e_j \quad \text{for} \quad j = 1,2. \quad \text{Hence,} \quad \nu(a) = \frac{\rho_0}{4}\frac{P}{1+P}a.$$

Obviously, B from (50) fulfils Assumption 3.5, where a rather lengthy computation verifies $P_cB(e_k, e_j) = 0$. Note that the nonlinearity without $\mathcal{Q}$ does not map into the right space.

It is an interesting observation that $P_sB(e_k, e_k) = -\frac{P}{4}(0, 0, \sin(2z))$ and $P_sB(e_k, e_l) = (0, 0, 0)$ for $k \neq l$. Hence, all the coupling of the dominant modes is caused by the heat equation.

Using $L_s(0, 0, \sin(2z)) = -\frac{4}{P}(0, 0, \sin(2z))$, it is now straightforward to compute $-2P_cB(e_j, L_s^{-1}P_sB(e_k, e_l))$. We end up with (cf. (13))

$$\Gamma(a_1, a_2) = -\frac{\pi}{32}\frac{\sqrt{2}P^2\sqrt{8+P^2}}{1+P}(a_1^2 + a_2^2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

Hence, the amplitude equation is given by

$$\partial_T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{\rho_0}{8}\frac{P}{P+1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \frac{\pi}{32}\frac{\sqrt{2}P^2\sqrt{8+P^2}}{1+P}(a_1^2 + a_2^2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \hat{\xi} ,$$

where $\hat{\xi}$ is some fractional noise in $\mathbb{R}^2$, which is derived from ξ by projection and rescaling of time.

The amplitude equation is for $\rho_0 < 0$ a stable equation. Thus the approximation result shows that although the modes e_1 and e_2 dominate the behaviour, but they are frequently near 0. Note that $\rho_0 < 0$ corresponds to a Reynolds number slightly below threshold of instability. On the other hand for $\rho_0 > 0$ there is a circle of deterministically stable solutions, i.e. they are stable for the amplitude equation without noise. Thus we expect the dominant modes to be roughly of that order for most of the times.

Let us finally comment on the remaining assumptions. Assumption 3.7 is standard, provided we have enough regularity of the stochastic convolution, which is assured by Assumption 3.4. The latter is also easy to verify, as the stochastic convolution with respect to L is just $(0, 0, W_{\Delta_\theta})$. Where W is the Wiener process in $H^2(D, \mathbb{R})$ corresponding to the noise ξ, and Δ_θ is the Laplacian in $L^2(D, \mathbb{R})$ subject to the boundary conditions of the temperature equation. Hence, we do not have a vector-valued stochastic convolution, and the probability estimates are well known for that case.

8 Technical Lemmas

This section provides technical lemmas needed in the proof of the main theorems. The first lemma is a generalised Gronwall-type estimate.

Lemma 8.1 *Suppose we have two functions $b, d : \mathbb{R} \to \mathbb{R}$ and a positive function $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ with $\varepsilon^2 \int_0^{T_0/\varepsilon^2} \gamma(t)dt \leq K_\gamma$. Suppose moreover that for some positive constants ω, C_i, C_b and C_d and $\alpha \in [0,1)$ we have for some $\varepsilon > 0$ and all $t \in [0, T_0\varepsilon^{-2}]$*

$$0 \leq d(t) \leq C_1\varepsilon^2 \int_0^t [d(\tau)(1+\gamma(\tau)) + b(\tau)]d\tau + C_d \tag{51}$$

$$0 \leq b(t) \leq C_2 \int_0^t (1+(t-\tau)^{-\alpha})\mathrm{e}^{-(t-\tau)\omega}[d(\tau) + \varepsilon b(\tau)]d\tau + C_b. \tag{52}$$

Define $K_{\alpha,\omega} = \int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{\tau\omega}d\tau$. If $\varepsilon \leq 1/(2C_2K_{\alpha,\omega})$ then

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} |d(t)| \leq [C_d + 2C_bC_1T_0]\mathrm{e}^{C_1T_0(K_\gamma + 2C_2K_{\alpha,\omega})}$$

and

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} |b(t)| \leq 2C_b + 2C_2K_{\alpha,\omega}[C_d + 2C_bC_1T_0]\mathrm{e}^{C_1T_0(K_\gamma + 2C_2K_{\alpha,\omega})}.$$

Proof: Define $S_b(t) := \sup_{s\in[0,t]} b(s)$ and similarly S_d. From (52) we obtain

$$b(t) \leq C_b + C_2K_{\alpha,\omega}[S_d(t') + \varepsilon S_b(t')]$$

for all $0 \leq t \leq t' \leq T_e\varepsilon^{-2}$. Hence

$$(1 - C_2K_{\alpha,\omega}\varepsilon)S_b(t) \leq C_b + C_2K_{\alpha,\omega}S_d(t),$$

and for $\varepsilon \leq 1/(2C_2K_{\alpha,\omega})$

$$S_b(t) \leq 2C_b + 2C_2K_{\alpha,\omega}S_d(t). \tag{53}$$

Now for $t \leq T_0\varepsilon^{-2}$

$$\begin{aligned} d(t) &\overset{(51)}{\leq} C_d + C_1\varepsilon^2 \cdot \int_0^t (S_d(\tau)\gamma(\tau) + S_b(\tau))d\tau \\ &\overset{(53)}{\leq} C_d + 2C_bC_1T_0 + C_1\varepsilon^2 \cdot \int_0^t (\gamma(\tau) + 2C_2K_{\alpha,\omega})S_d(\tau)d\tau. \end{aligned}$$

As the right hand side is monotone in t, we obtain

$$S_d(t) \le C_d + 2C_bC_1T_0 + C_1\varepsilon^2 \cdot \int_0^t (\gamma(\tau) + 2C_2K_{\alpha,\omega})S_d(\tau)d\tau.$$

Now Gronwall's inequality implies for $t \le T_0\varepsilon^{-2}$

$$S_d(t) \le [C_d + 2C_bC_1T_0]\mathrm{e}^{C_1(K_\gamma + 2C_2K_{\alpha,\omega}T_0)}.$$

Using (53) the claim follows. □

The following lemma gives Hölder estimates for convolutions with the semigroup of L. Similar results are well known (see e.g. [L95, Prop. 4.2.1]). Nevertheless, we give a simple proof, as we need explicitly the dependence of the constants on ε.

Lemma 8.2 *Consider L as in Assumption 3.2. Then for all $h \in L^\infty([0,T], P_sX)$ and $p \in (0,1)$ we obtain*

$$\Big\| t \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^p([0,T],X)} \le C\varepsilon^{2-2p}\|h\|_{L^\infty([0,T],X)}, \qquad (54)$$

and for all $h \in C^p([0,T\varepsilon^{-2}], P_sX)$ with $h(0) = 0$

$$\Big\| t \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^p([0,T],X)} \le C\varepsilon^{2}\|h\|_{C^p([0,T],X)}, \qquad (55)$$

where the constants depend only on p, ω, and M.

We can also prove an intermediate result, which is useful for Hurst parameter $H = \frac{1}{2}$.

Corollary 8.3 *Under the assumptions of Lemma 8.2. For all $\alpha \in (0, \frac{1}{4})$ and $p \in (0, (1-4\alpha)/(2\alpha))$ we obtain for all $h \in C^p([0,T\varepsilon^{-2}], P_sX)$ with $h(0) = 0$*

$$\Big\| t \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^{\frac{1}{2}+\frac{2\alpha}{p+1}}([0,T],X)} \le C\varepsilon^{2-\frac{1}{p+1}-2\alpha\frac{p+2}{p+1}} \cdot \|h\|_{C^{\frac{1}{2}-\alpha}([0,T],X)}, \qquad (56)$$

where the constant $C > 0$ depends only on p, α, ω, and M.

Proof: Define $\delta = \frac{1}{2} - \alpha$ and $\gamma = \frac{1}{2} + \alpha(p+2)$, which are both in $(0,1)$. Then we can use the following obvious interpolation formula for Hölder-norms:

$$\|u\|_{C^{\frac{p\delta+\gamma}{p+1}}([0,T],X)} \le 2\|u\|^{\frac{p}{p+1}}_{C^\delta([0,T],X)} \cdot \|u\|^{\frac{1}{p+1}}_{C^\gamma([0,T],X)}.$$

Now the claim follows immediately by using (54) and (55). □

Proof of Lemma 8.2: First by (7) it is straightforward to verify

$$\Big\| \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \leq M\varepsilon^2 \int_0^\infty \mathrm{e}^{-\omega t} dt \|h\|_{L^\infty([0,T],X)}.$$

This is the L^∞-bound. Let us now turn to the Hölder semi-norm. Consider for $\eta > 0$ and $t \in [0,T]$ such that $t+\eta \leq T$

$$\begin{aligned}
&\Big\| \int_0^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds - \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \\
&\leq \Big\| \int_t^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds \Big\| + \Big\| (\mathrm{e}^{\eta L\varepsilon^{-2}} - 1) \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \\
&\leq M\|h\|_{L^\infty([0,T],X)} \Big[\int_t^{t+\eta} \mathrm{e}^{-(t+\eta-s)\omega\varepsilon^{-2}} ds + M \int_0^\eta s^{-1+p} \mathrm{e}^{-s\omega\varepsilon^{-2}} ds \int_0^t s^{-p} \mathrm{e}^{-s\omega\varepsilon^{-2}} ds \Big] \\
&\leq \varepsilon^2 M\|h\|_{L^\infty([0,T],X)} \Big[\int_0^{\eta\varepsilon^{-2}} \mathrm{e}^{-s\omega} ds + M \int_0^{\eta\varepsilon^{-2}} s^{p-1} ds \int_0^\infty s^{-p} \mathrm{e}^{-s\omega} ds \Big] \\
&\leq C\varepsilon^{2(1-p)} \eta^p \|h\|_{L^\infty([0,T],X)}.
\end{aligned}$$

This implies the first assertion.

The second result is proved similarly. First the L^∞-bound in $C^0([0,T],X)$ is completely analogous. Moreover, as $h(0) = 0$

$$\int_0^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds - \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds$$

$$= \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} [h(s+\eta) - h(s)]ds + \int_0^\eta \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} [h(s) - h(0)]ds.$$

Now it is straightforward to verify the second assertion. □

The following lemma relies on fractional integration by parts and it is useful for bounding stochastic integrals path-wise:

Lemma 8.4 *For $p_1 \in (0,1)$ let $g \in C^{p_1}([0,T],\mathbb{R})$ and $\Phi \in C^1([0,T],X)$. Then*

$$\Big\| \int_0^T g(s)\partial_s \Phi(s)ds \Big\| \leq C\|g\|_{C^{p_1}([0,T])} \|\Phi\|_{C^{p_2}([0,T],X)}$$

with $p_1 + p_2 > 1$, where the constant $C = C(T, p_1, p_2)$ grows monotone in T.

We do not give a detailed proof of this lemma, as it is straightforward. We can for example first use the fractional integration by parts formula (see e.g. [Zä98]). This is to our knowledge only developed for real-valued functions, but it is easy to generalise this to our case. Estimating the integral like for instance in [MN03] we obtain first fractional Sobolev norms, which then, by crude estimates on the fractional derivatives, give the Hölder norms.

Lemma 8.5 *Let Assumption 3.2 be true. Given $f \in C^q([0,T],\mathbb{R})$ and $g \in C^p([0,T],P_sY)$ with $q+p>1$, we obtain*

$$\begin{aligned} \|J_2\| &:= \|\int_0^T f(t)\Big(\varepsilon^{-2}L\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)ds - L^{-1}g(t)\Big)dt\| \\ &\leq C\varepsilon^2\|f\|_{C^q([0,T])}\|g\|_{C^p([0,T],Y)}. \end{aligned}$$

For $r>1$ define $p=\frac{1}{2}-1/(4r(r+2))$ and $q=\frac{1}{2}-(r+1)/(4r(r+2))$. Now

$$\|J_2\| \leq C\varepsilon^{2-\frac{3r+1}{2r(r+1)}} \cdot \|f\|_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T],\mathbb{R})} \cdot \|g\|_{C^{\frac{1}{2}-\frac{r+1}{4r(r+2)}}([0,T],Y)}.$$

Proof: As $g \in C^q([0,T],Y)$ and hence $L^{-1}g \in C^q([0,T],X)$ we know (e.g. [L95, Thm. 4.3.1]) that $\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)ds$ is in $C^q([0,T],D(L))$ and in $C^{1+q}([0,T],X)$. We obtain

$$\begin{aligned} \|J_2\| &= \|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)dsdt\| \\ &\leq \|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}[g(s)-g(0)]dsdt\| \qquad (57) \\ &\quad +\|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}dsdtL^{-1}g(0)\|, \qquad (58) \end{aligned}$$

where we introduce $g(0)$ to apply Lemma 8.2. As

$$\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}ds = \partial_t\int_0^t \mathrm{e}^{sL\varepsilon^{-2}}ds = \mathrm{e}^{tL\varepsilon^{-2}},$$

the second term is easily bounded by

$$(58) \leq C\|f\|_{L^\infty(0,T)} \cdot \|g\|_{L^\infty([0,T],Y)} \cdot \varepsilon^2 \cdot \int_0^\infty \mathrm{e}^{-t\omega}dt.$$

For (57) we use fractional integration by parts (see Lemma 8.4) to obtain

$$\begin{aligned} (57) \quad &\le \quad C\|f\|_{C^q([0,T])} \cdot \Big\| t \mapsto \int_0^t \mathrm{e}^{-(t-s)L\varepsilon^{-2}} L^{-1}[g(s)-g(0)]ds \Big\|_{C^p([0,T],X)} \\ &\le \quad C\varepsilon^2 \|f\|_{C^q([0,T])} \cdot \|g\|_{C^p([0,T],Y)} \end{aligned}$$

by Lemma 8.2, and the first claim follows.

The second assertion is analogous. In the application of Lemma 8.4 we use $q = \frac{1}{2} - \frac{\alpha}{r+1}$ and $p = \frac{1}{2} + \frac{2\alpha}{r+1}$ and then we apply Corollary 8.3 to obtain

$$\|J_2\| \le C\varepsilon^{2-\frac{1}{r+1}-2\alpha\frac{r+2}{r+1}} \cdot \|f\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \|g\|_{C^{\frac{1}{2}-\alpha}([0,T],Y)}.$$

The claim follows by fixing $\alpha = (r+1)/(4r(r+2))$. The condition $r > 1$ ensures that all conditions of Corollary 8.3 are fulfilled. □

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