The model is the Lieb-Liniger model of bosons with contact interaction on the unit interval but with an additional external random potential $V_\omega$. The Hamiltonian on the Hilbert space $L^2([0, 1], dz) \otimes s^N$ is

$$H = \sum_{i=1}^N \left(-\partial^2_{z_i} + V_\omega(z_i)\right) + \frac{\gamma}{N} \sum_{i<j} \delta(z_i - z_j)$$

with $\gamma \geq 0$ and Dirichlet boundary conditions.

The random potential will be taken to be

$$V_\omega(z) = \sigma \sum_j \delta(z - z_j^\omega)$$

with $\sigma \geq 0$ independent of the random sample $\omega$ while the obstacles $\{z_j^\omega\}$ are Poisson distributed with density $\nu \gg 1$.

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Why this model?

- BEC in the ground state can be proved in a suitable limit
- $V_\omega$ is simple enough to allow a rigorous analysis of the condensate

Once BEC has been established the main question is:

How do the properties of the condensate depend on the parameters?

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For *fixed* $\gamma$, $\sigma$ and $\omega$ there is complete BEC in the ground state in the sense that the 1-particle density matrix $\rho_N$ converges to a one dimensional projector as $N \to \infty$. The corresponding wave function of the condensate is the normalized minimizer of the GP energy functional

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E_{GP}[\psi] = \int_0^1 \left\{ |\psi'(z)|^2 + V_\omega(z)|\psi(z)|^2 + (\gamma/2)|\psi(z)|^4 \right\} dz
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We want, however, to consider $\gamma$, $\sigma$ and $\nu$ large. Hence it is relevant to estimate the rate of the convergence of the 1-particle density matrix in dependence of the parameters and of $\omega$. 

Jakob Yngvason (Uni Vienna)

Localization and Entanglement
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BEC in the LL model is not expected if $\gamma \gtrsim N^2$. Our proof of BEC covers the range $\gamma \ll N^{2/3}$. The case $\gamma \gg N^2$ corresponds to the Girardeu-Tonks regime.

The proof of BEC for the LL model (for $\gamma$ fixed or slowly increasing with $N$) is simpler than the proof of Lieb and Seiringer (2002) of the corresponding result for 3D bosons with a general (positive, short range) interaction. But it is still nontrivial, and the random potential adds some twist.
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The Proof of BEC (sketch)

Key elements of the proof are energy bounds:

- An upper bound to the many-body ground state energy $E_{QM}^0$ by taking $\psi_0^\otimes N$ as a trial function for $H$ where $\psi_0$ is the minimizer of the GP energy functional, normalized so that $\|\psi_0\|_2 = 1$. This gives
  \[ E_{QM}^0 \leq Ne_0(1 + o(1)) \]
  where $e_0 = E^{GP}[\psi_0]$ is the g.s.e. of the GP functional.

- A lower bound on $H$, up to controlled errors, in terms of the 1-particle mean-field Hamiltonian
  \[ h = -\partial_z^2 + V_\omega(z) + \gamma|\psi_0(z)|^2 - (\gamma/2) \int |\psi_0|^4 \]
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The proof of BEC follows from the upper and lower bounds and the fact that there is an energy gap between \( e_0 \) and the next lowest eigenvalue, \( e_1 \) of the mean-field Hamiltonian \( h \): Let

\[
N_0 = \langle \Psi_0 | a^\dagger (\psi_0) a (\psi_0) | \Psi_0 \rangle
\]

be the occupation number of the GP ground state \( \psi_0 \) in the many-body ground state \( \Psi_0 \). Then the energy bounds give

\[
N_0 e_0 + (N - N_0) e_1 - o(1) Ne_0 \leq E_0^{QM} \leq Ne_0.
\]

This implies an upper bound for the depletion:

\[
\left(1 - \frac{N_0}{N}\right) \leq o(1) \frac{e_0}{e_1 - e_0}
\]

The right-hand side, averaged over \( \omega \), can be shown to tend to 0 if \( N \to \infty \) and the parameters \( \gamma, \nu, \sigma \) do not grow too fast with \( N \).
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Elaboration on the lower bound

Have to estimate the two-body interaction term

$$\frac{\gamma}{N} \sum_{i<j} \delta(z_i - z_j)$$

from below by a sum of one-body terms.

- Replace the delta function with a regular potential of positive type (i.e. with a positive Fourier transform), namely

$$\delta_b(z) := \frac{1}{2b} \exp\left(-|z|/b\right)$$

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Borrow a bit of kinetic energy and write, with $p_i^2 = -\partial_{z_i}^2$,

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$$+ \frac{1}{N} \sum_{i<j} (\varepsilon(p_i^2 + p_j^2) + \gamma \delta(z_i - z_j))$$

Define for $\varepsilon, b > 0$

$$w(z) := \frac{\gamma}{1 + (b\gamma/2\varepsilon)\delta_b(z)).}$$

It can be proved that the operator $\varepsilon p^2 + \gamma \delta(z) - w(z)$ has no bound states. Hence

$$\varepsilon p^2 + \gamma \delta(z) \geq w(z)$$
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Since \( w \) is of positive type we have

\[
\sum_{i<j} w(z_i - z_j) \geq \sum_i N(w \ast \rho)(z_i) - (N/2)w(0) - (N^2/2) \int (\rho \ast w) \rho
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for any density \( \rho(z) \). Take \( \rho(z) = |\psi_0(z)|^2 \) and obtain

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H \geq \sum_{i=1}^{N} \left\{ h^{(i)} - \frac{\varepsilon}{2} p_i^2 - \frac{b\gamma^2}{\varepsilon} e_0^{1/2} - \frac{3\pi^{1/2}}{2^{1/2}} \gamma e_0^{3/4} b^{1/2} - \frac{\gamma}{4Nb} \right\}.
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Optimizing the parameters and tracing with the one-particle reduced density matrix of the ground state gives

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\frac{E_{0}^{\text{QM}}}{N} \geq \frac{N_0}{N} e_0 + \left( 1 - \frac{N_0}{N} \right) e_1 - C e_0 N^{-1/3} \min\{\gamma^{1/2}, \gamma\}.
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Together with the upper bound \( E_0^{QM} / N \leq e_0 \) this gives an estimate on the depletion of the condensate:

**Theorem (BEC)**

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It remains to consider the dependence of the energy gap, \( e_1 - e_0 \) on the random potential.
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The energy gap

Consider a one-dimensional Schrödinger operator $-\partial_z^2 + W(z)$ with Dirichlet boundary conditions and a nonnegative potential $W$.

**Lemma (Gap):** Define $\eta > 0$ by

$$\eta^2 = \pi^2 + 3\int_0^1 W(z)dz.$$ 

Then

$$e_1 - e_0 \geq \eta \ln \left(1 + \pi e^{-2\eta}\right)$$

The proof is based on a modification of a result of Kirsch and Simon (1985), that, however, involves the sup norm of $W$.

In our case $\eta = \sqrt{\pi^2 + 3m_\omega \sigma + 3\gamma}$ where $m_\omega$ is the number of obstacles in $[0, 1]$. 
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The Poisson distribution of the obstacles

Let \( z_1^\omega \leq z_2^\omega \leq \cdots \leq z_m^\omega \) denote those random points which lie in the open interval \([0, 1]\).

The lengths \( \ell_i = z_{i+1}^\omega - z_i^\omega \) are independent random variables with distribution

\[
dP_\nu(\ell) = \nu e^{-\ell_\nu} d\ell
\]

and we are considering the case \( \nu \gg 1 \).

The average length of an interval free of obstacles is \( \nu^{-1} \) and with probability one, \( m_\omega / \nu \to 1 \) and \( \sum_i \ell_i = 1 \).

One consequence is that the energy gap is uniform in the \( L^p \) norm on sample space for any \( p < \infty \).
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While the average length of an interval is $\nu^{-1}$ there is with probability one a unique largest interval of length $\nu^{-1} \ln \nu$.

If there is no interaction, i.e., $\gamma = 0$, and $\sigma \to \infty$, then the ground state wave function will be localized in the longest interval for kinetic energy reasons.

The question is how the situation changes when the interaction, i.e., the term $(\gamma/N) \int |\psi|^4$ comes into play.
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An auxiliary problem

To study the energy in an interval between two obstacles consider first for $\kappa \geq 0$ and $\alpha \geq 0$ the energy

$$e(\kappa, \alpha) = \inf_{\|\psi\|_{L^2[0,1]}=1} \left[ \int_0^1 (|\phi'(x)|^2 + \kappa |\phi(x)|^4) \, dx + \frac{\alpha}{2} (|\phi(0)|^2 + |\phi(1)|^2) \right]$$

The energy in an interval of length $\ell$ and normalization to $n$ is then obtained from $e(\kappa, \alpha)$ by scaling:

$$\inf_{\|\psi\|_{L^2[0,\ell]}=n} \left[ \int_0^\ell (|\psi'(z)|^2 + \gamma |\psi(z)|^4) \, dz + \frac{\sigma}{2} (|\psi(0)|^2 + |\psi(\ell)|^2) \right]$$

$$= \frac{n}{\ell^2} e(\ell n \gamma, \ell \sigma)$$
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Properties of $e(\kappa, \alpha)$

- $e(0, \alpha)$ is a strictly concave function of $\alpha$ with slope not bigger than 1, with $e(0, 0) = 0$ and $e(0, \infty) = \pi^2$.
- $\frac{1}{2} \leq \frac{e(\kappa, \alpha) - e(0, \alpha)}{\kappa} \leq \frac{3}{4}$
- $e(0, \alpha) \geq \frac{c_\alpha}{1 + \alpha}$
- $e(\kappa, \alpha)$ is jointly concave in $\kappa$ and $\alpha$.
- $\kappa \mapsto \kappa e(\kappa, \alpha)$ is convex
- $e(\kappa, \alpha)$ is differentiable in $\kappa$

Lemma (comparison with $e(\kappa, \infty)$)

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq e(\kappa, \infty) \left(1 - C\alpha^{-1/2}\right)$$
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**Lemma (comparison with $e(\kappa, \infty)$)**

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq e(\kappa, \infty) \left( 1 - C\alpha^{-1/2} \right)$$
Introduce a chemical potential $\mu$ and define

$$g(\mu, \alpha) := \inf_{\kappa \geq 0} (\kappa e(\kappa, \alpha) - \mu \kappa) .$$

The infimum is attained for $\kappa$ fulfilling

$$\kappa = [\mu - e(0, \alpha)]_+ + \frac{1}{e'(\kappa, \alpha) + \frac{1}{\kappa}(e(\kappa, \alpha) - e(0, \alpha))}.$$ 

Using the properties of $e(\kappa, \alpha)$ one shows that the solution $\bar{\kappa}(\mu, \alpha)$ satisfies

$$\frac{2}{3} [\mu - e(0, \alpha)]_+ \leq \bar{\kappa}(\mu, \alpha) \leq [\mu - e(0, \alpha)]_+ .$$
A Legendre transform

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$$g(\mu, \alpha) := \inf_{\kappa \geq 0} (\kappa e(\kappa, \alpha) - \mu \kappa) .$$

The infimum is attained for $\kappa$ fulfilling

$$\kappa = [\mu - e(0, \alpha)]_+ + \frac{1}{e'(\kappa, \alpha) + \frac{1}{\kappa}(e(\kappa, \alpha) - e(0, \alpha))} .$$

Using the properties of $e(\kappa, \alpha)$ one shows that the solution $\bar{\kappa}(\mu, \alpha)$ satisfies

$$\frac{2}{3} [\mu - e(0, \alpha)]_+ \leq \bar{\kappa}(\mu, \alpha) \leq [\mu - e(0, \alpha)]_+ .$$
Introduce a chemical potential $\mu$ and define

$$g(\mu, \alpha) := \inf_{\kappa \geq 0} (\kappa e(\kappa, \alpha) - \mu \kappa).$$

The infimum is attained for $\kappa$ fulfilling

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The average energy for $\sigma = \infty$

Define

$$e_0(\gamma, \nu) = \inf \left\{ \nu \int_0^\infty dP_\nu(\ell) \frac{n(\ell)}{\ell^2} e(n(\ell)\ell\gamma, \infty) : \nu \int_0^\infty dP_\nu(\ell) n(\ell) = 1 \right\},$$

where the infimum as over all $n(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying the normalization constraint.

Alternatively we can write

$$e_0(\gamma, \nu) = \sup_{\mu > 0} \left\{ \mu + \nu \int_0^\infty dP_\nu(\ell) \frac{1}{\ell^3} g(\mu \ell^2, \infty) \right\}.$$
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Alternatively we can write

$$e_0(\gamma, \nu) = \sup_{\mu > 0} \left\{ \mu + \nu \int_0^\infty dP_\nu(\ell) \frac{1}{\ell^3\gamma} g(\mu\ell^2, \infty) \right\}.$$
The optimal particle distribution in the definition of $e_0(\gamma, \nu)$ is

$$n(\ell) = (\ell \gamma)^{-1} \kappa(\mu \ell^2, \infty) \sim (\ell \gamma)^{-1} [\mu \ell^2 - \pi^2]_+ .$$

The corresponding optimal $\mu$ satisfies the relation

$$\mu \sim \gamma f(\nu^2 / \gamma)$$

with

$$f(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ \frac{x}{(1+\ln x)^2} & \text{for } x \geq 1. \end{cases}$$

Also

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$$e_0(\gamma, \nu) \sim \gamma f(\nu^2 / \gamma)$$
Let \( e_\omega(\gamma, \sigma, \nu) \) denote the GP energy with the random potential \( V_\omega \).

**Theorem (Convergence of the energy).**

Assume that \( \nu \to \infty \), \( \sigma \to \infty \) and \( \gamma \to \infty \) in such a way that

\[
\gamma \gg \frac{\nu}{(\ln \nu)^2} \quad \text{and} \quad \sigma \gg \frac{\nu}{1 + \ln (1 + \nu^2/\gamma)}.
\]

Then, for almost every sample \( \omega \),

\[
\lim \frac{e_\omega(\gamma, \sigma, \nu)}{e_0(\gamma, \nu)} = 1.
\]
Proof of the limit theorem (sketch)

Consider $\nu$ intervals of length $\ell_i$. The GP energy is bounded from below by

$$\inf_{\{n_i\}} \sum_{i=1}^{\nu} \frac{n_i}{\ell_i^2} e(n_i \ell_i \gamma, \ell_i \sigma)$$

where the infimum is under the constraint $\sum_i n_i = 1$.

It is bounded from above by

$$\inf_{\{n_i\}} \sum_{i=1}^{\nu} \frac{n_i}{\ell_i^2} e(n_i \ell_i \gamma, \infty)$$

By the estimates for $e(\kappa, \alpha)$ the two bounds agree, to leading order, provided $\ell_i \sigma$ is large for all the intervals of length $\ell_i$ for which $n_i > 0$ in the optimal configuration.
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The proof now essentially consist in establishing the last condition, under the stated assumptions on the parameters, and justifying the replacement of the sums by integrals as in the definition of the mean energy.

The calculations are done in a grand canonical ensemble. The normalization of the wave function requires

\[ 1 = \sum_{i=1}^{\nu} n_i \sim \nu \int_0^{\infty} dP(\ell) \frac{1}{\ell \gamma} \left[ \mu \ell^2 - e(0, \ell \sigma) \right]_+ \]

which serves to determine \( \mu \) as function of the parameters.
Discussion of the optimal particle distribution

Since \( n(\ell) \sim (\ell \gamma)^{-1}[\mu \ell^2 - \pi^2]_+ \) the average number of intervals with non-zero occupation numbers is given by

\[

\nu \int_{\pi/\sqrt{\mu}}^{\infty} dP_\nu(\ell) = e^{-\pi \nu/\sqrt{\mu}} \nu.
\]

Since \( \nu \) is the total number of available intervals,

\[

\lambda := e^{-\pi \nu/\sqrt{\mu}} \leq 1
\]

defines the fraction of them which are occupied.

The normalization requires

\[

1 \sim \frac{\mu}{\gamma} e^{-\pi \nu/\sqrt{\mu}} \quad (\ast)
\]

so \( \lambda \) is determined by \( \gamma \) and \( \nu \) via relation

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\frac{\gamma}{\nu^2} \sim \frac{\lambda}{(\ln \lambda^{-1})^2} \quad (\ast\ast)
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\]
Limiting cases

- If $\gamma \gg \nu^2$ then by (**) we get $\lambda \to 1$, i.e., all the intervals are occupied (delocalization). The chemical potential satisfies $\mu \sim \gamma$ in this regime.

- If $\gamma \sim \nu^2$ then $\lambda \sim 1$, but $\lambda$ is strictly less than 1 (transition to localization). Again we have $\mu \sim \gamma$.

- If $\gamma \ll \nu^2$ then $\lambda \ll 1$, i.e., only a small fraction of the intervals are occupied (localization). The relation (*) implies $\mu \sim (\nu/\ln(\nu^2/\gamma))^2$ for the chemical potential.

- If $\gamma \sim \nu/(\ln \nu)^2$ then by (**) the fraction $\lambda$ becomes $O(1/\nu)$, i.e., only finitely many intervals are occupied. In this latter case, $\mu \sim \gamma \nu \sim \nu^2/(\ln \nu)^2$, which corresponds exactly to the inverse of the square of the size of the largest interval.
In particular, $\lambda \nu \gg 1$ only if $\gamma \gg \nu/(\ln \nu)^2$, and hence this condition guarantees that many intervals are occupied. In this case the law of large numbers applies and hence the energy becomes deterministic in the limit.

If $\lambda \nu = O(1)$, on the other hand, the value of $e_\omega(\gamma, \sigma, \nu)$ is random. This shows, in particular, that our condition on $\gamma$ is optimal.

Also the second condition in on $\sigma$ can be expected to be optimal. It can be rephrased as $\bar{\ell} \sigma \gg 1$, where $\bar{\ell}$ is the (weighted) average interval length

$$\bar{\ell} = \nu \int_0^\infty dp\nu(l) l n(l)$$

with $n(l) = (l \gamma)^{-1} \bar{n}(\mu l^2, \infty)$ the optimal distribution. A simple calculation shows that $\bar{\ell} \sim \nu^{-1}(1 + \ln(1 + \nu^2/\gamma))$. 
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\[
\bar{\ell} = \nu \int_0^\infty dp_\nu(\ell) \ell n(\ell)
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with \( n(\ell) = (\ell \gamma)^{-1} \tilde{n}(\mu \ell^2, \infty) \) the optimal distribution. A simple calculation shows that \( \bar{\ell} \sim \nu^{-1}(1 + \ln(1 + \nu^2/\gamma)) \).
In particular, $\lambda\nu \gg 1$ only if $\gamma \gg \nu/(\ln \nu)^2$, and hence this condition guarantees that many intervals are occupied. In this case the law of large numbers applies and hence the energy becomes deterministic in the limit.

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with $n(\ell) = (\ell\gamma)^{-1}\overline{n}(\mu\ell^2, \infty)$ the optimal distribution. A simple calculation shows that $\overline{\ell} \sim \nu^{-1}(1 + \ln(1 + \nu^2/\gamma))$. 
Conclusions

- BEC in the ground state of the interacting gas in the GP regime can survive even in a strong random potential. As far as BEC is concerned the interacting gas in this regime thus behaves in a similar way as an ideal gas at zero temperature. The character of the wave function of the condensate, however, is strongly affected by the interaction.

- A random potential may lead to localization of the wave function of the condensate, even though the density of obstacles is much less than the particle density. The interparticle interaction counteracts this effect, however, and can lead to complete delocalization if the interaction is strong enough.

- In terms of the interaction strength, $\gamma$, and density of scatterers, $\nu$, the transition between localization and dislocalization of occurs in the model considered when $\gamma \sim \nu^2$. For $\gamma \lesssim \nu / (\ln \nu)^2$ the condensate is localized in a small number of intervals with fluctuating particle numbers.
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