In Memoriam - Friedhelm Waldhausen

Friedhelm Waldhausen, professor of mathematics at the University of Bielefeld from 1971 until his retirement in 2004, passed away following a short illness on November 21st 2024.

Brave new rings and higher algebra

In his memory, the department, together with his former students Oliver Röndigs and Stefan Schwede, is organising an international conference from August 30th to September 3rd, 2027, celebrating his contributions and charting the current developments he originally inspired. Its website can be found here.

It will be accompanied by a special volume of the Documenta Mathematica serial in his honour.

Obituary

Friedhelm Waldhausen was born in December 9th 1938 in Millich. He studied mathematics in Göttingen, Munich, and Bonn until 1962. In 1966 he received his doctorate in Bonn under Friedrich Hirzebruch with the now-famous thesis "Eine Klasse von 3-dimensionalen Mannigfaltigkeiten" (A Class of 3-Dimensional Manifolds). The class in the title consists of what he called graph manifolds: those that can be cut along tori, in a particular way, into circle bundles or more precisely, Seifert fibered spaces; a graph then records where exactly the cuts are made. Herbert Seifert had begun classifying these pieces in the 1930s and Waldhausen's work not only put the finishing touches on that classification, it also showed that the decomposition of a graph manifold is unique, up to a few explicit exceptions. Most graph manifolds are therefore topologically rigid.

Rigidity, a notion which will form a leitmotif of this text, is the property that every homotopy equivalence of a manifold can be realised by an actual symmetry. At the time it was a well-studied property, putting Waldhausen's work into illustrious company: For surfaces it had been known since the 1920's by the famous theorem of Max Dehn, Jakob Nielsen and Reinhold Baer. Generalising this into higher dimensions, Armand Borel had asked about the rigidity of aspherical manifolds in general as early as 1953, and the graph manifolds to which

Waldhausens result applied are indeed all aspherical. Simultaneously, George Mostow's (even geometric) rigidity of hyperbolic (and thus aspherical) manifolds in dimension 3 and above appeared at the same time as Waldhausen's work. And Stephen Smale had just received a Fields Medal for his h-cobordism theorem and the resulting rigidity of spheres (which are indeed not aspherical) in dimensions greater than 4.

After Bonn, Waldhausen spent two years in the United States, with stops in Princeton, Urbana-Champaign, and Ann Arbor. During this time he extended his rigidity theorem to the class of Haken manifolds, in what remains his most-cited paper. Haken manifolds are those three-dimensional manifolds that can be cut along surfaces of arbitrary genus into balls, which makes them considerably more general than graph manifolds. They are named after Wolfgang Haken, who showed in 1962 that the existence of a single incompressible surface suffices for the existence of such a decomposition. Waldhausen's rigidity theorem and the techniques he developed to prove it are still regarded as a milestone of three-dimensional topology. They were arguably among the field's deepest insights until Thurston revolutionised it in the 1980s, quite some time after Waldhausen's interest had shifted away from low-dimensional topology (more on this below).

Waldhausen returned to Germany in 1968 with the fame of this extraordinarily productive period unter his belt - in addition to those mentioned above he made several further fundamental contributions to three-dimensional topology. He completed his habilitation in Kiel in 1969 and was appointed to a professorship in Bochum in 1970. The following year he moved to the newly founded University of Bielefeld, where he held the chair for topology until his retirement. In Bielefeld, he played a key role in building up the Institute of Mathematics and made it one of Europe's centres for topology. He was famous both as a character and for the number of guests he invited to Bielefeld, from 1989 onward as part of the newly established Collaborative Research Center SFB 343 "Discrete Structures in Mathematics", which one so frequently encounters in the acknowledgements of topology papers of the time. There are many anecdotes among topologists worldwide of their encounters in Bielefeld, many retold in the outpouring on the global Algebraic-Topology-mailing-list, following the news of his death. The hospitality he provided together with his wife Waltraut, affectionately known as Traute even by visitors still on honorific terms (per Sie) with Waldhausen himself, is nothing short of legendary: He would often house guests not in a hotel but in the summer house (the "Gartenhaus") of his wonderfully green property on the edge of town - though frequently also outside of summer. Among topologists, having stayed there was a badge of honour, and stories are told of conference nights when the number of guests and their desire to stay at Waldhausen's house sufficed for established professors to sleep beneath his grand piano.

His first doctoral student in Bielefeld was Klaus Johannson, who in his 1976 dissertation generalised the uniqueness of Waldhausen's graph manifold decomposition to arbitrary

three-dimensional manifolds. The American mathematicians William Jaco and Peter Shalen established the existence of this decomposition at almost the same time and so it is now usually referred to as the JSJ decomposition. It remains one of the backbones of three-dimensional topology. In general, besides Seifert fibered pieces, one must also allow atoroidal building blocks: pieces that contain no further tori along which they could be cut. These remained mysterious for a long time. Around 1980, Thurston proposed that all of them could be equipped with a hyperbolic geometry. Building on the techniques Waldhausen had developed for his rigidity theorem, Thurston proved this for Haken manifolds, giving in particular a direct bridge between the rigidity theorems of Waldhausen and Mostow. Thurston received the Fields Medal for this insight in 1982 and Grigori Perelman was awarded one in 2006, almost 30 years after the discovery of the JSJ decomposition, for proving Thurston's geometrisation conjecture in full (though Perelman famously declined the honour).

Johannson was to remain Waldhausen's sole student in low-dimensional topology as over the course of the 1970s, Waldhausen's mathematical interests changed. Namely, an obstruction to rigidity had long been known to homotopy theorists: A homotopy equivalence must first satisfy Whitehead's condition, that it must be possible to assemble it from elementary collapses and expansions. Whitehead's obstruction to this lies in a quotient of the first algebraic K-group of the manifold's fundamental group. The only reasonable way for this not to contradict Waldhausen's rigidity theorem is for this purely algebraically defined group to vanish. Waldhausen proved that it indeed does shortly after arriving in Bielefeld. In the following years he generalised this vanishing theorem in two papers on higher algebraic K-theory in the Annals of Mathematics. These papers set off a revolution in the field, which was driven by his ideas. The first version of Waldhausen's vanishing theorem in fact appeared in the same 1973 conference proceedings as Daniel Quillen's groundbreaking construction of the higher algebraic K-groups, for which he was awarded the Fields medal in 1978, simultaneous to the appearance of Waldhausen's general vanishing result. Quillen had been in letter exchanges with Graeme Segal about possible constructions ultimately choosing what is known as the Q-construction; it is somewhat ironic hat Waldhausen's influence on the subject goes so far that Segal's construction, which Waldhausen revived for his work, is now often referred to as Waldhausen's S-dot-construction.

In these foundational papers, Waldhausen did not just prove a vanishing theorem for Whitehead's groups, but formulated and proved a description of all of Quillen's algebraic K-groups for the fundamental group of a Haken manifold in terms of the so-called assembly map. The connection between rigidity and assembly subsequently took on a life of its own, and was developed by many topologists over the years. About ten years after Waldhausen's vanishing theorem appeared, Tom Farrell and Lowell Jones showed that the assembly map is an equivalence for the fundamental group of every negatively curved manifold and deduced Borel's rigidity conjecture for such manifolds, so long as they are of dimension at least 5: In these dimensions the connection between rigidity and assembly is tight enough to deduce

the former from the latter (granted that one also considers assembly in the closely related L-groups). Their approach was then prominently taken up by the German topology community under Waldhausen's watchful eye, and indeed nowadays rigidity is known for many further classes of manifolds. Interestingly, for the four-dimensional case, where the connection between assembly and rigidity breaks down, a counterexample was proposed just this fall, in 2026, as part of the AI-written papers circulated by OpenAI; at the time of writing it has not yet been verified.

Waldhausen himself did not publish further on rigidity: While studying the vanishing of Whitehead groups, he had recognised an even deeper connection between geometric topology and algebraic K-theory, and he devoted the rest of his mathematical life to it. A brief digression is in order here. Aside from Whitehead's work on elementary homotopy equivalences, the basic idea of algebraic K-theory came from Grothendieck in the 1950s. He assembled vector bundles over an algebraic variety into a new object, the projective class group, for which he chose the letter K (Klasse is German for class) from which the entire theory derives its name. Waldhausen's advisor Hirzebruch, together with Sir Michael Atiyah, had carried this idea over to the topology of manifolds in the late 50's. They noticed that in this setting there was a comparatively obvious definition of higher K-groups. Inspired by the success of K-theory in topology, Quillen then managed, with considerable effort, to transfer these topologically defined higher K-groups back into algebraic geometry.

Having arrived at the topic of K-theory, which had already brought such fame to his advisor, through a very circuitous route, Waldhausen further extended Quillen's definition: Until then, vector bundles had been its fundamental building blocks in every case, where he replaced them with arbitrary sufficiently finite maps among topological spaces. Using the resulting object, which he called the algebraic K-theory of topological spaces, and building on earlier work of Allen Hatcher and Kyoshi Igusa, he described the moduli space of h-cobordisms of a manifold in a certain range. At the level of path components, this recovers Smale's h-cobordism theorem from the second paragraph of this text. At the level of fundamental groups, it is closely tied to the rigidity of a manifold. But the topologists of the day immediately recognised the significance of this description in higher degrees as well. Farrell and Wu-Chung Hsiang, for instance, promptly derived descriptions of the rational homotopy groups of the diffeomorphism groups of spheres up to roughly a third of the dimension. Many of the results from the flurry of work that came out of this last of Waldhausen's great theorems were surpassed only in the past three years, some 40 years after they appeared, with completely new methods. The publication history of his theorem is, however, as remarkable as its genesis: After the initial announcements Waldhausen only published sketches of the proof. Where this would ordinarily call into the doubt the validity of a result, he had developed such an outstanding reputation for his precise writing and the reliability of his work, that this question simply never arose. A complete write-up ultimately appeared only in 2013, some 30 years after the announcement and 9 years after his retirement

(!), with John Rognes and BjÃ¸rn Jahren as coauthors.

It is probably fair to say that by discarding the vector space structure, Waldhausen had removed the last semblance of algebra from algebraic K-theory. At first glance, then, the name he gave his construction might seem more than a little ironic. He pointed out right away, however, that his theory could be treated on an equal footing with Quillen's: One only had to generalise the basic notions of algebra, in particular the notion of a ring, to homotopy theory in a sufficiently structured way. He called these new partially conjectural objects "brave new rings". His ideas about them, pursued in parallel by topologists in the American Midwest, notably Peter May with whom Waldhausen famously had a somewhat rivalrous relationship, are likely to be his most far-reaching legacy: their development dominated homotopy theory for the next 30 years. The theory has since been renamed "higher algebra" and is now well-developed. It reached its (for now) definitive form only in the 2010s, when Jacob Lurie formulated it in the language of higher category theory. Waldhausen's vision of founding his version of K-theory in this way is nowadays fully realised and in this form has entered the mathematical canon, often touched upon in the advanced courses of a topology curriculum (a rare accomplishment for such comparatively young mathematics).

Higher algebra has turned out to be a foundation for many further developments. In his very first publication on the algebraic K-theory of topological spaces, for example, Waldhausen observed that it has a map to stable homotopy. This map is analogous to the trace map from Quillen's algebraic K-theory to Hochschild homology, a genuinely algebraic construction that can be applied to any ring. Following ideas of Waldhausen and Tom Goodwillie, Marcel Bökstedt, then a postdoc of Waldhausen's, generalised this to higher algebra in the mid-1980s and introduced a refined trace map. He called the result topological Hochschild homology (it is about as topological as algebraic K-theory is algebraic). Its further refinement topological cyclic homology, introduced by Bökstedt, Hsiang and Ib Madsen in 1993 is to this day an active are of research and the best tool for computing algebraic K-groups. With its help, Lars Hesselholt and Madsen computed the algebraic K-groups of the p-adic numbers in 2003, a year before Waldhausen's retirement. This was arguably the greatest advance since Quillen's computation of the K-groups of finite fields in 1970. Like Hesselholt, Waldhausen predicted at the time that these new invariants had great potential in number theory, though in characteristic fashion he published nothing further on the subject. Around 2015, Thomas Nikolaus and Peter Scholze recognised topological cyclic homology, very much in the spirit of Waldhausen's philosophy, as the fixed points of a higher-algebra version of the Frobenius operator, which is so fundamental in algebra. This viewpoint allowed Scholze in joint work with Barghav Bhatt, and very much in the spirit of Waldhausen's predictions, to indeed extract a new purely algebraic cohomology theory ("prismatic cohomology"), which has since had its own triumphs in arithmetic geometry.

This connection in turn set off an explosion of results on algebraic K-theory in recent years, which relates to one last prediction of Waldhausen's. In the early 1980s, Douglas

Ravenel had made a series of conjectures about the nature of brave new rings (and more generally the stable homotopy category). In particular, every brave new ring should decompose into pieces, that give rise to different periodic patterns in its homotopy groups, much as Fourier analysis decomposes a signal into periodic functions of different wavelengths. Because of this analogy, Ravenel's idea became known as the chromatic picture of homotopy theory. Devinatz, Hopkins, and Smith settled all but one of Ravenel's conjectures in two papers in 1988 and 1998.

As early as 1982, Waldhausen had considered how this decomposition interacts with his algebraic K-theory of brave new rings - recall that at the time, neither the existence of the decomposition nor the foundations of that K-theory had been fully established. Rognes took up his speculations and, on the basis of a surprisingly small set of computations, formulated the redshift conjecture: algebraic K-theory increases the chromatic wavelength in a tightly controlled way. A large collaboration of young mathematicians gave a proof of this conjecture in 2022, spread over several papers. Moreover, the connection to arithmetic geometry has made algebraic K-theory so computable, that applied to brave new rings of chromatic interest, it finally produced counterexamples to the last of the Ravenel conjectures in 2023, in work of Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer Schlank. That conjecture had arguably been one of the biggest open problems in stable homotopy theory.

Waldhausen passed away shortly after these result appeard, just shy of his 86th birthday. With him, modern topology has lost one of its great visionaries, in the algebraic and geometric directions and in low and high dimensions alike.