QCD at nonzero isospin densities

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in collaboration with Bastian Brandt

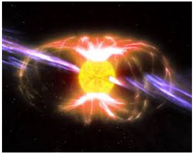
Phase structure of lattice field theories
26. September 2016

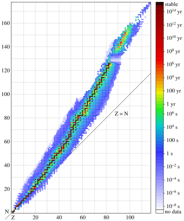
Outline

- introduction: QCD with isospin
- relevant phenomena
  - pion condensation
  - chiral symmetry breaking
- λ-extrapolation
  - naive method
  - new method
- results
  - phase boundary for pion condensation
  - χSB transition line at low μ_I
  - direct check of Taylor-expansion
- outlook

Introduction

- isospin density $n_I = n_u - n_d$
- $n_I < 0 \rightarrow$ excess of neutrons over protons
  $\rightarrow$ excess of π^- over π^+
- applications
  - neutron stars
  - heavy-ion collisions





- chemical potentials (3-flavor)

$$\mu_B = 3(\mu_u + \mu_d)/2 \qquad \mu_I = (\mu_u - \mu_d)/2 \qquad \mu_S = 0$$

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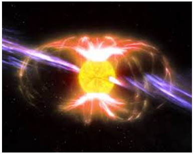
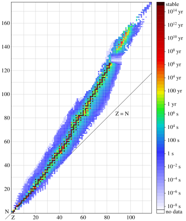




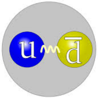
- chemical potentials (3-flavor)

$$\mu_B = 3(\mu_u + \mu_d)/2 \qquad \mu_I = (\mu_u - \mu_d)/2 \qquad \mu_S = 0$$

- here: zero baryon number but nonzero isospin

$$\mu_u = \mu_I \qquad \mu_d = -\mu_I$$

Introduction



- QCD at low energies $\approx$ pions
- on the level of charged pions: $\mu_\pi = 2\mu_I$
  at zero temperature

$\mu_\pi < m_\pi$	vacuum state
$\mu_\pi = m_\pi$	Bose-Einstein condensation
$\mu_\pi > m_\pi$	undefined

- on the level of quarks: lattice simulations
  - no sign problem
  - conceptual analogies to baryon density
    (Silver Blaze, hadron creation, saturation)
  - technical similarities
    (proliferation of low eigenvalues)

Setup

Symmetry breaking

- QCD with light quark matrix

$$M = \not{D} + m_{ud} \mathbb{1}$$

- chiral symmetry (flavor-nontrivial)

$$\mathrm{SU}(2)_V$$

Symmetry breaking

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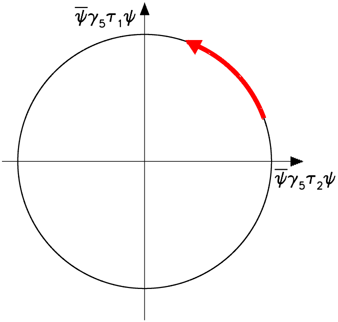

- spontaneously broken by a pion condensate

$$\left\langle \bar{\psi}\gamma_5\tau_{1,2}\psi \right\rangle$$

- a Goldstone mode appears

Symmetry breaking

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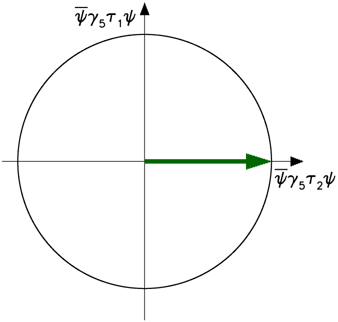

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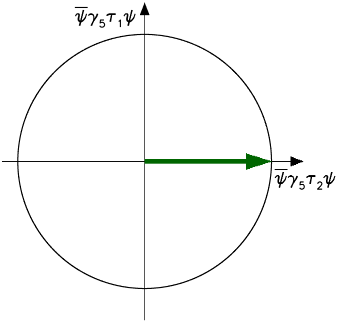

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- extrapolate results $\lambda \to 0$

Simulation details

- staggered light quark matrix with $\eta_5 = (-1)^{n_x+n_y+n_z+n_t}$

$$M = \begin{pmatrix} \not{D}_\mu + m & \lambda\eta_5 \\ -\lambda\eta_5 & \not{D}_{-\mu} + m \end{pmatrix}$$

- we have $\gamma_5\tau_1$-hermiticity

$$\eta_5\tau_1 M \tau_1 \eta_5 = M^\dagger$$

- determinant is real and positive

$$\det M = \det(|\not{D}_\mu + m|^2 + \lambda^2)$$

- pioneering studies [Kogut, Sinclair '02] [de Forcrand, Stephanov, Wenger '07] with unimproved action
- here: $N_f = 2+1$ rooted stout-smeared staggered quarks + tree-level Symanzik improved gluons

Condensates: definition and renormalization

$$\left\langle \bar{\psi}\psi \right\rangle = \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial m} \qquad \langle \pi \rangle = \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial \lambda}$$

- multiplicative renormalization

$$Z_\pi = Z_\lambda^{-1} = Z_m^{-1} = Z_{\bar{\psi}\psi}$$

- convenient normalization

$$\Sigma_{\bar{\psi}\psi} \equiv m \cdot \left\langle \bar{\psi}\psi \right\rangle \cdot \frac{1}{m_\pi^2 f_\pi^2} \qquad \Sigma_\pi \equiv m \cdot \langle \pi \rangle \cdot \frac{1}{m_\pi^2 f_\pi^2}$$

- so that in leading-order chiral PT [Son, Stephanov '00]

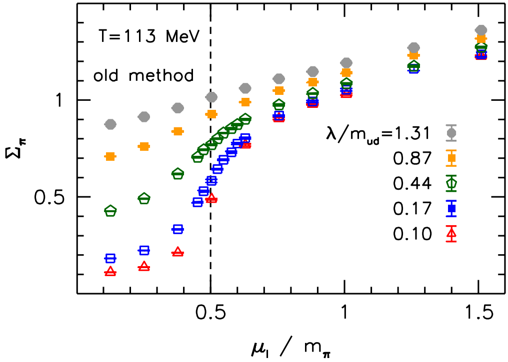
$$\Sigma_{\bar{\psi}\psi}^2(\mu_I) + \Sigma_\pi^2(\mu_I) = 1$$

Condensates: old method

Pion condensate: old method

- traditional method [Kogut, Sinclair '02]
  measure full operator at nonzero λ (via noisy estimators)

$$\Sigma_\pi \propto \left\langle \mathrm{Tr} M^{-1} \eta_5 \tau_2 \right\rangle$$



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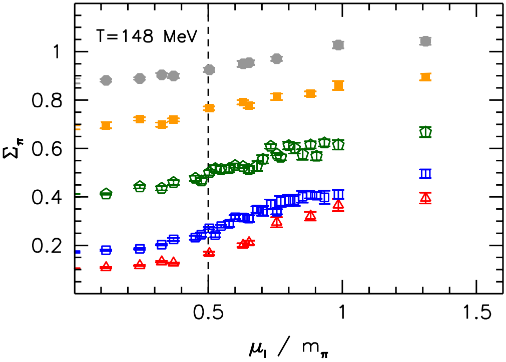

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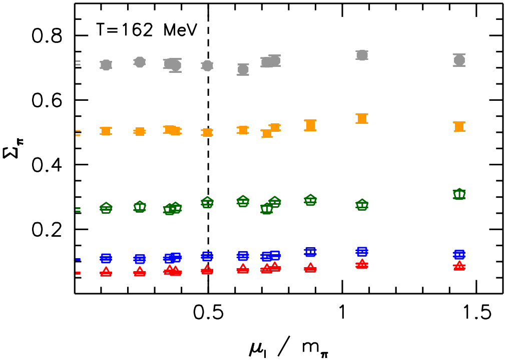

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Chiral condensate: old method

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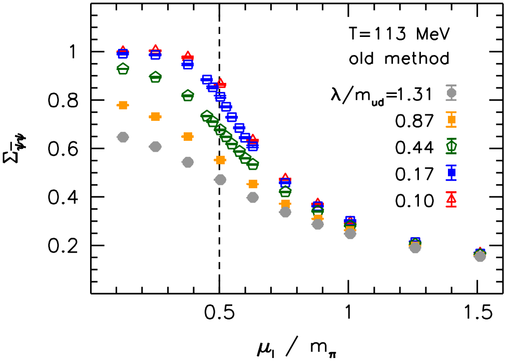

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Pion condensate: new method

Singular value representation

- pion condensate

$$\pi = \frac{\partial}{\partial\lambda} \log\det(|\not{D}_\mu + m|^2 + \lambda^2) = \mathrm{Tr}\frac{2\lambda}{|\not{D}_\mu + m|^2 + \lambda^2}$$

- singular values

$$|\not{D}_\mu + m|^2\,\psi_i = \xi_i^2\,\psi_i$$

- spectral representation

$$\pi = \frac{T}{V}\sum_i \frac{2\lambda}{\xi_i^2 + \lambda^2} = \int \mathrm{d}\xi\,\rho(\xi)\,\frac{2\lambda}{\xi^2+\lambda^2} \xrightarrow{\lambda\to 0} \pi\rho(0)$$

first derived in [Kanazawa, Wettig, Yamamoto '11]

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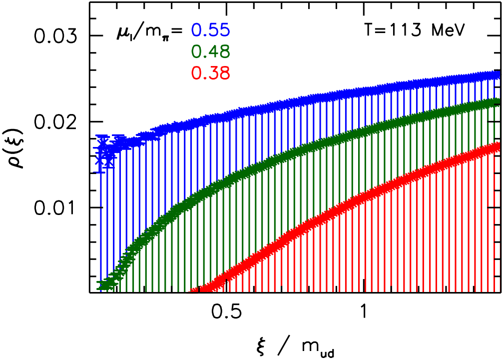
- compare to Banks-Casher-relation at $\mu_I = 0$

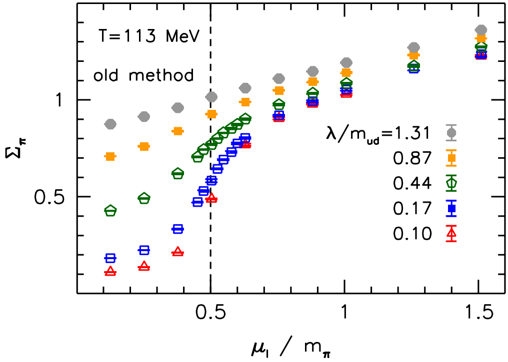
Singular value density

- spectral densities at $\lambda/m = 0.17$



Density at zero

- scaling with λ is improved drastically



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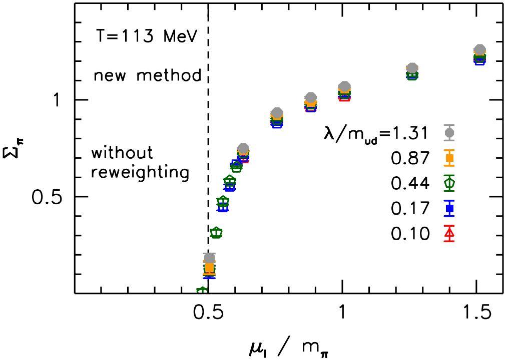

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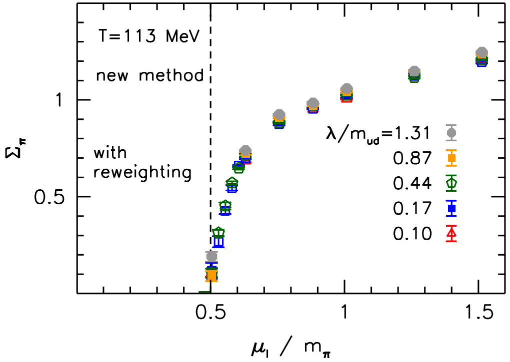
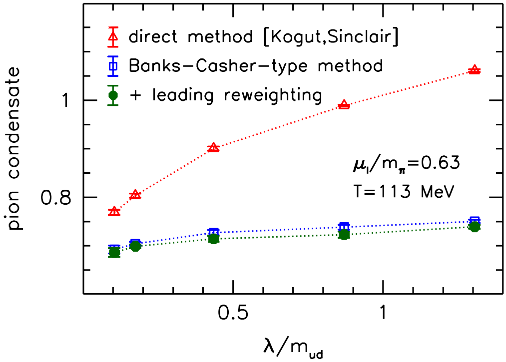


- leading-order reweighting

$$\langle \pi \rangle_{\rm rew} = \langle \pi W_\lambda \rangle \, / \, \langle W_\lambda \rangle \qquad W_\lambda = \exp[-\lambda V_4 \pi + \mathcal{O}(\lambda^2)]$$

Comparison between old and new methods

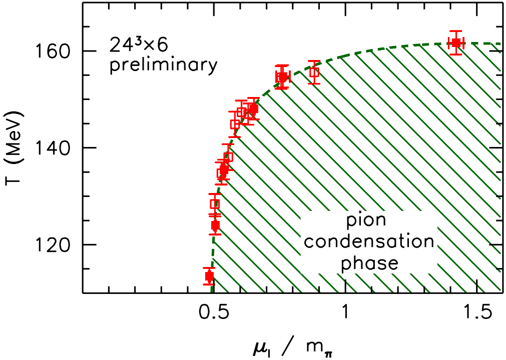
- extrapolation in λ gets almost completely flat



Results

Phase boundary

- interpolate $\rho(0)$ as function of μ_I to find phase boundary



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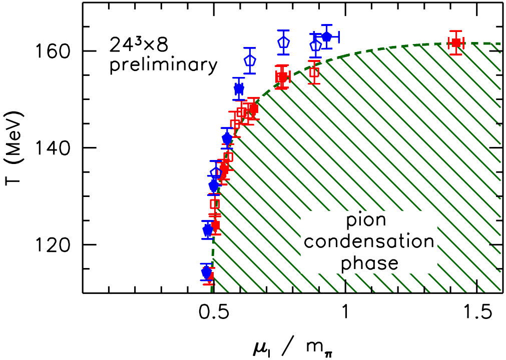

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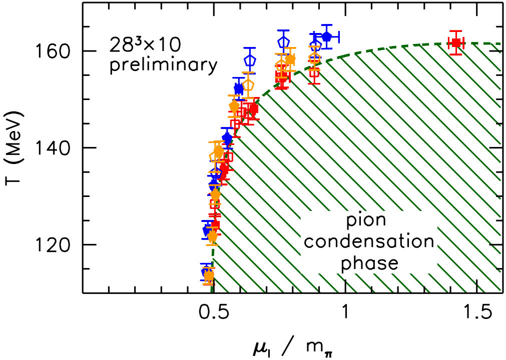

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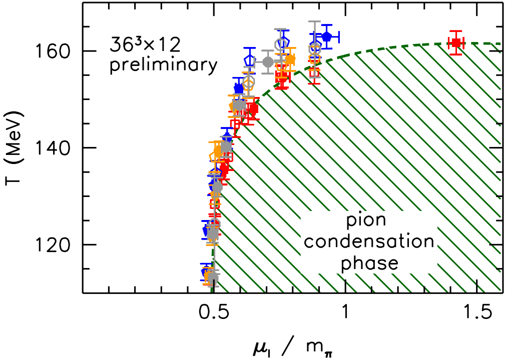

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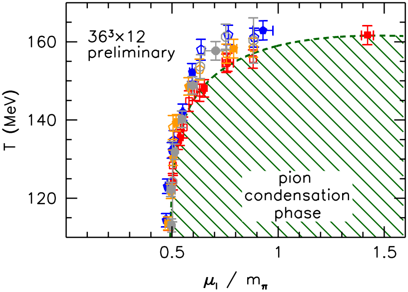

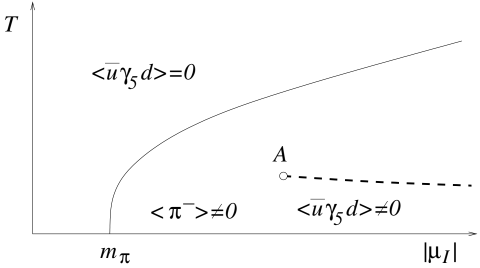

- compare to expectations from χPT [Son, Stephanov '00]

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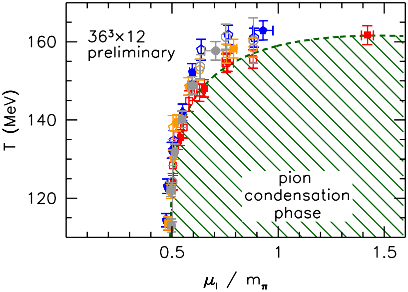

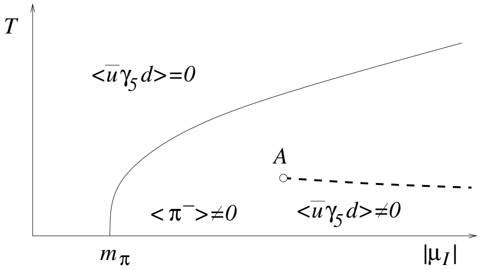

- compare to expectations from χPT [Son, Stephanov '00]
- no pion condensate above $T \approx 160$ MeV

New method for other observables

Singular value representation

- chiral condensate

$$\bar{\psi}\psi = \frac{\partial}{\partial m} \log\det(|\not{D}_\mu + m|^2 + \lambda^2) = \mathrm{Tr}\,\frac{(\not{D}_\mu + m) + (\not{D}_\mu + m)^\dagger}{|\not{D}_\mu + m|^2 + \lambda^2}$$

- singular values

$$|\not{D}_\mu + m|^2\,\psi_i = \xi_i^2\,\psi_i$$

- spectral representation

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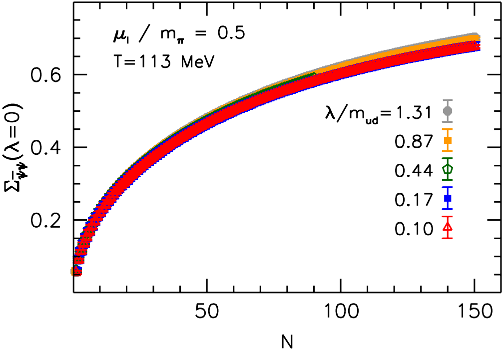
Singular value representation

- spectral representation at $\lambda = 0$

$$\bar{\psi}\psi = \frac{T}{V}\sum_{i=1}^{N} 2\,\mathrm{Re}\,\frac{\langle \psi_i | \not{D}_\mu + m | \psi_i \rangle}{\xi_i^2}$$



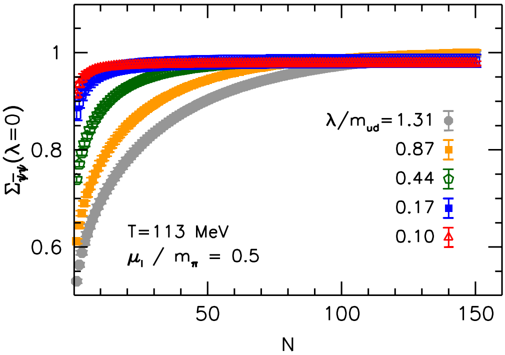
- convergence not visible for $N \leq 150$

Improvement

- work instead with the *difference*

$$\delta_{\bar\psi\psi} \equiv \bar\psi\psi(\lambda=0) - \bar\psi\psi(\lambda) = \frac{2T}{V}\sum_{i=1}^{N} \mathrm{Re}\langle\psi_i|\not{D}_\mu + m|\psi_i\rangle \left[\frac{1}{\xi_i^2} - \frac{1}{\xi_i^2+\lambda^2}\right]$$

$$\bar\psi\psi(0) = \bar\psi\psi(\lambda) + \delta_{\bar\psi\psi}$$



- convergence already for small N

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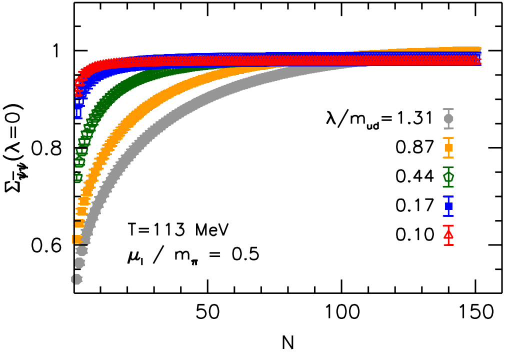
noisy estimators



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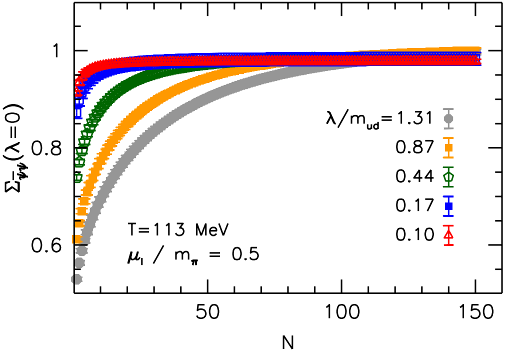
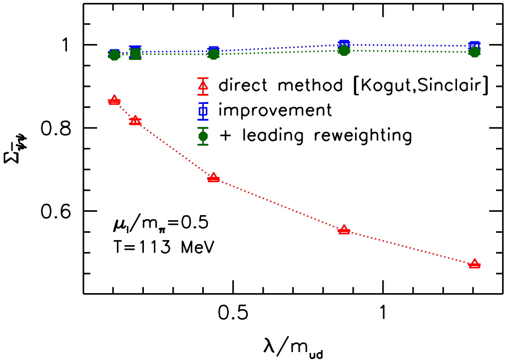


- convergence already for small N

Comparison between old and new methods

- extrapolation in λ gets almost completely flat



Improvement

- the same strategy for the isospin density $n_I = \partial(\log \mathcal{Z})/\partial\mu_I$

$$\delta_{n_I} \equiv n_I(0) - n_I(\lambda) = \frac{2T}{V}\sum_{i=1}^{N} \mathrm{Re}\langle\psi_i|(\not{D}_\mu + m)^\dagger \not{D}'_\mu|\psi_i\rangle \left[\frac{1}{\xi_i^2} - \frac{1}{\xi_i^2+\lambda^2}\right]$$

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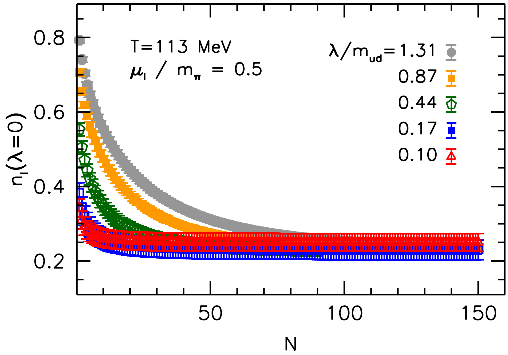

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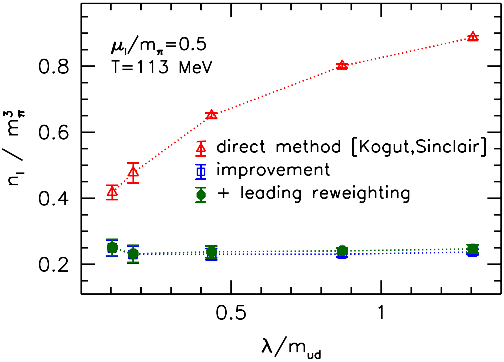
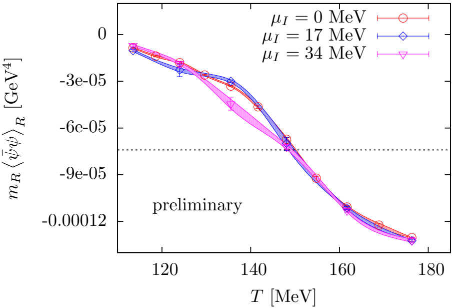


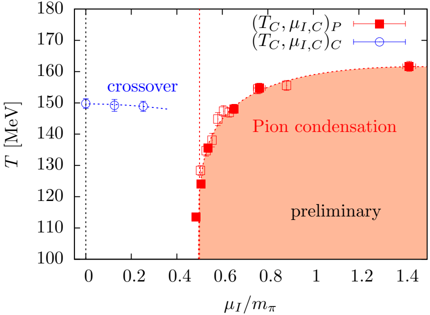
Results

- additively renormalized chiral condensate

$$\Sigma_{\bar{\psi}\psi}(T) - \Sigma_{\bar{\psi}\psi}(T=0)$$

- define T_c using the temperature at a constant value (valid at low μ_I)





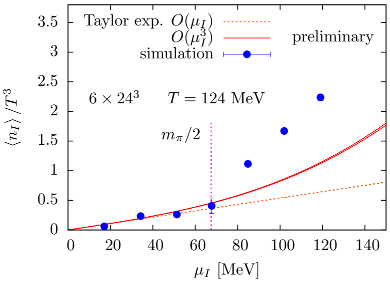
Check Taylor-expansion

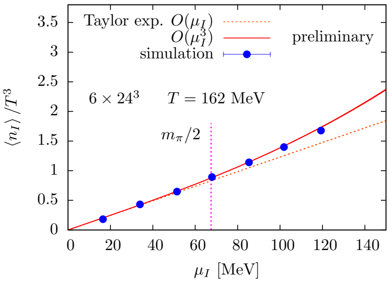
- isospin density via Taylor-expansion at $\mu_I = 0$

$$n_I(\mu_I) = \chi_2^I \cdot \mu_I + \chi_4^I \cdot \mu_I^3 + \ldots$$

using $\chi_{2,4}^I$ from [BMWc, 1112.4416]

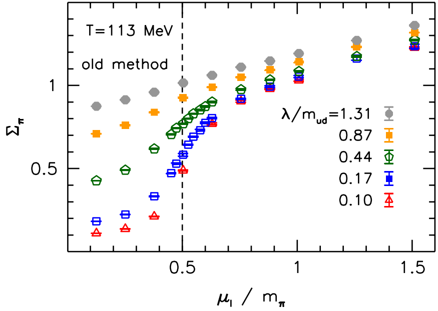
- how far is it reliable?

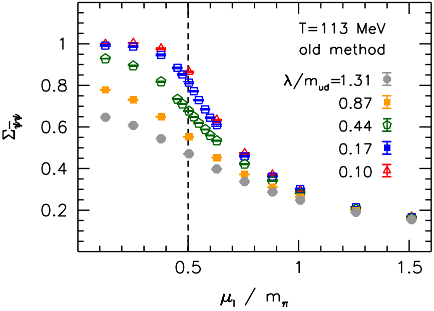




- similar convergence expected for μ_B

- fit transition region using chiral perturbation theory [Splittorff et al '02, Endrődi '14]





Order of the transition – fits

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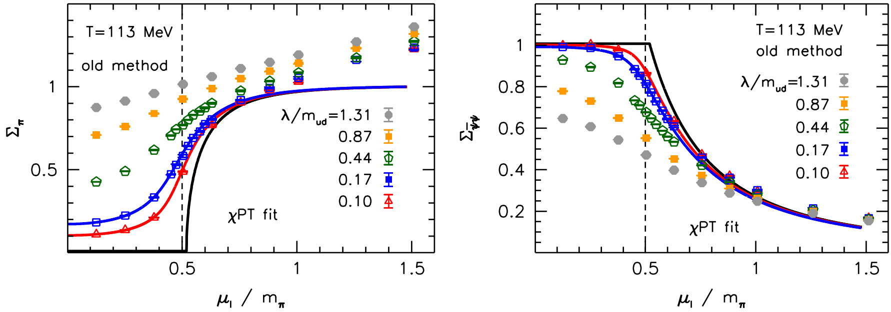
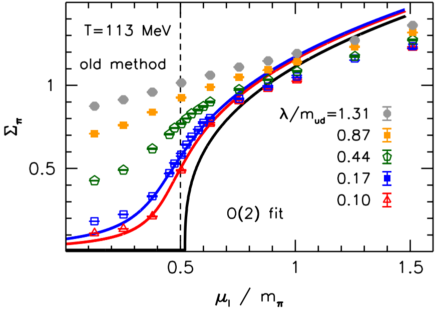
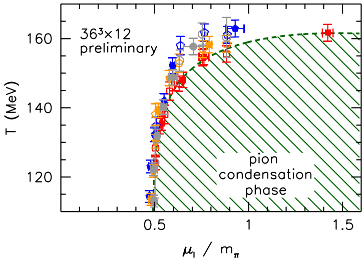


Order of the transition – fits

- fit using $\mathrm{O}(2)$ scaling [Ejiri et al '09]



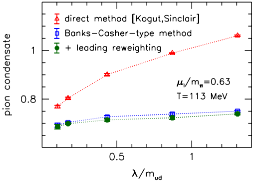
Outlook



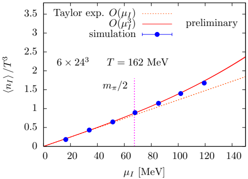
- order of transition?
- deconfinement/chiral symmetry breaking transition
- asymptotic-μ_I limit?
- BCS phase at large μ_I?

Summary

- determine/improve observables using singular values of $\not{D}_\mu + m$
  $\rightsquigarrow$ flat extrapolation in λ



- direct check of Taylor-expansion convergence



- phase boundary surprisingly flat for intermediate μ_I
- chance to test effective theories and low-energy models