# QCD in background electric fields

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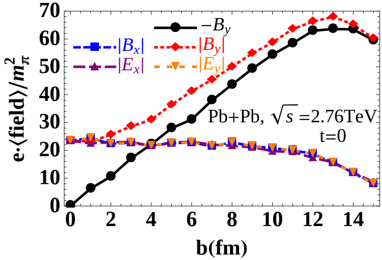
# Outline

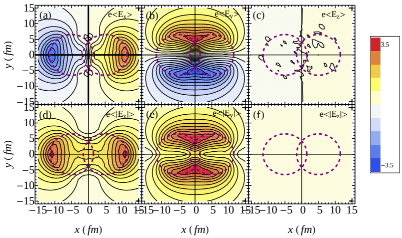
- introduction: electric fields
- Taylor expansion
- susceptibility via vacuum polarization
- results: electric and magnetic susceptibility
- conclusions

# Introduction

# Electric fields

- electromagnetic fields in the early stage of heavy-ion collisions [Deng et al '12]

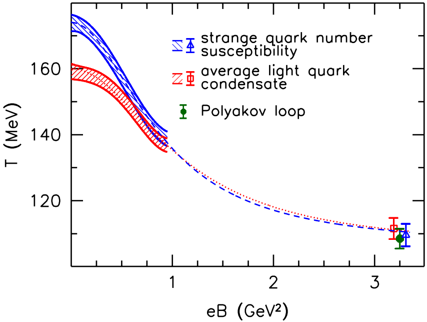



- impact of electric field enhanced for asymmetric systems (for example Cu+Au at RHIC) [Voronyuk et al '14]

# Magnetic/electric fields

- effect of magnetic fields on QCD thermodynamics well understood [Bali et al '11, '12] [D'Elia et al '11] [Endrődi '15]



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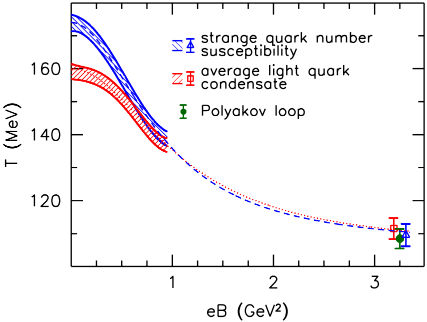



- effect of electric fields:
  NJL model calculations [Suganuma et al '91, Babansky et al '98]
  lattice QCD with opposite charges [Yamamoto '13]
  electric polarizability of hadrons [Engelhardt et al '07] [Alexandru et al '14]

# Setup

# Euclidean electric fields

- covariant derivative (electric charge $q$)

$$D_\mu = \partial_\mu + iqA_\mu + i\mathcal{A}_\mu^{\mathrm{gluon}}$$

- Minkowskian electric field

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i$$

- Wick rotation

$$\partial_0 \to i\partial_4, \qquad A_0 \to iA_4$$

- Euclidean electric field

$$E_i^{\mathrm{Eucl}} = \partial_i A_4 - \partial_4 A_i \equiv -iE_i$$

# Sign problem

- consider constant $\mathbf{E} = E\,\mathbf{e_1}$ in a static gauge

$$A_4 = -iE\,x_1$$

- $\mathrm{U}(1)$ links

$$u_4 = \exp\left[iaqA_4\right] = \exp\left[aqEx_1\right]$$

- hermiticity relation

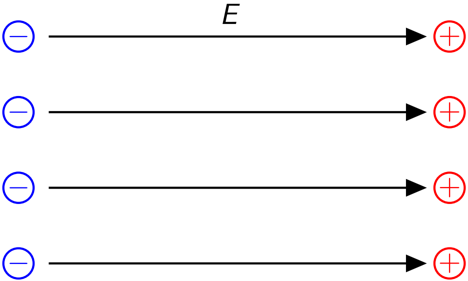
$$\not{D}^\dagger(E) = -\not{D}(-E^*)$$

- determinant

$$\det \not{D}(E) \in \mathbb{C}, \qquad \det \not{D}(E^{\mathrm{Eucl}}) \in \mathbb{R}$$

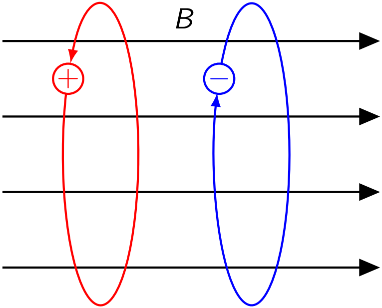
# Sign problem

- electric fields cause a sign problem



# Sign problem

- electric fields cause a sign problem
- magnetic fields cause no sign problem



# Avoiding the sign problem

# Taylor expansion

- circumvent sign problem via Taylor-expansion

$$f(E) \approx f(0) + \mathcal{O}(E^2) + \mathcal{O}(E^4) + \ldots$$

  odd powers vanish due to rotational symmetry

- cannot account for non-perturbative phenomena like Schwinger pair production

$$\operatorname{Im} f(E) \propto \exp(-1/E)$$

- can give access to the change in the permittivity of QCD matter
  $\rightsquigarrow$ linear response

# Linear response

- free energy density $f$ in small background fields

$$f^{\mathrm{tot}} = \frac{B^2}{2\mu} + \epsilon\frac{E^2}{2}$$

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$$\begin{aligned} B &= \mu H & \qquad B &= H + M & \qquad M &= \chi B \\ D &= \epsilon E & \qquad D &= E + P & \qquad P &= \xi E \end{aligned}$$

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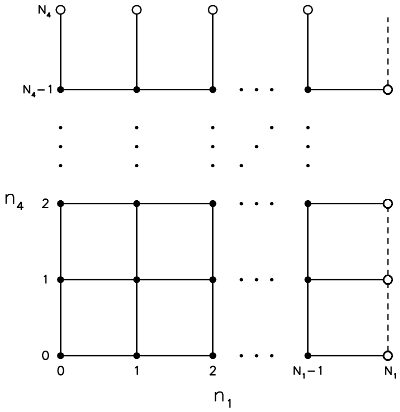
$$f = -\chi\frac{B^2}{2} + \xi\frac{E^2}{2} = -\chi\frac{B^2}{2} - \xi\frac{(E^{\mathrm{Eucl}})^2}{2}$$

# Constant fields on the lattice

- constant Euclidean electric field

$$A_4 = E^{\mathrm{Eucl}}\, x_1, \qquad u_4 = \exp\left[i\eta n_1\right], \qquad \eta = a^2 q E^{\mathrm{Eucl}}$$



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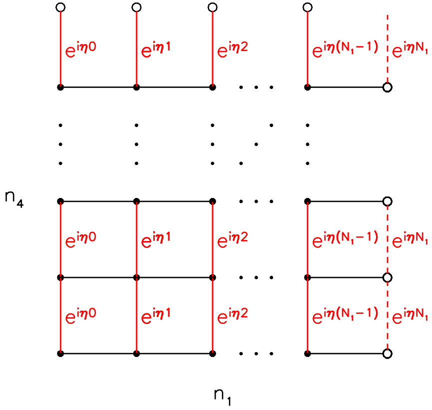


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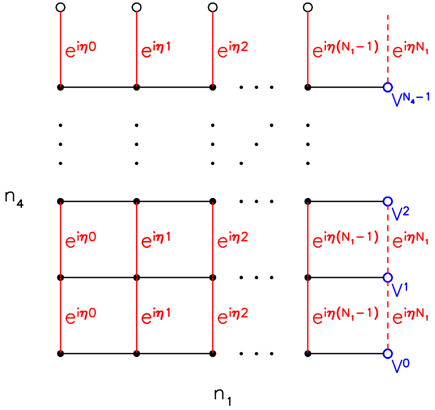


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- local $\mathrm{U}(1)$ gauge trafo
  $V = \exp(i\eta N_1)$

  $\psi(N_1, n_4) \to \psi(N_1, n_4) \cdot V^{n_4}$

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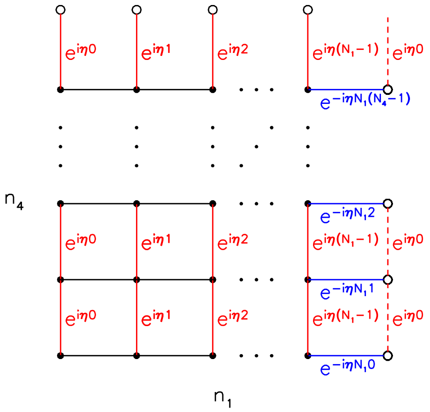


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  $V = \exp(i\eta N_1)$

  $\psi(N_1, n_4) \to \psi(N_1, n_4) \cdot V^{n_4}$
- restores periodicity
- periodic in $x_4$-direction if
  $\exp(-i\eta N_1 N_4) = 1$

# Flux quantization

- constant electric fields represented by periodic $u_\mu$ if

$$\exp\left(-ia^2 qE^{\mathrm{Eucl}} N_1 N_4\right) = 1$$

therefore

$$qE^{\mathrm{Eucl}} \cdot L_1 L_4 = 2\pi N_E, \qquad N_E \in \mathbb{Z}$$

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- on a finite lattice the ‘flux’ of $E^{\mathrm{Eucl}}$ is a discrete variable (the same is true for $B$)
- Taylor expansion <u>not</u> applicable

# Avoiding the quantized flux-problem

# Susceptibility as a limit

- quantization due to $p = 0$ Fourier transform in finite volume

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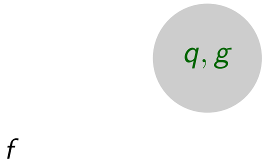

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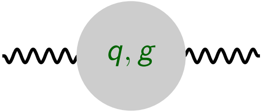


$$-\left.\frac{\partial^2 f}{\partial E^2}\right|_{E=0} = \xi$$

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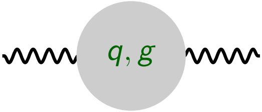


$$-\left.\frac{\partial^2 f}{\partial E^2}\right|_{E=0} = \qquad \xi \qquad = \frac{-1}{2}\left.\frac{\partial^2 \Pi_{44}(p_1)}{\partial p_1^2}\right|_{p_1=0}$$

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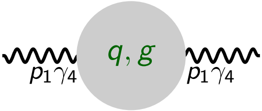


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# Susceptibility and vacuum polarization: proof

- vector potential interacts with current

$$i\int \mathrm{d}^4x\, A_4\, j_4, \qquad j_\mu = \sum_f q_f \bar{\psi}\gamma_\mu\psi$$

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- photon vacuum polarization

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see also [Weldon '82]

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$$\xi = \int \mathrm{d}z_1 \, \frac{z_1^2}{2} G_{44}(z_1), \qquad G_{\mu\nu}(z_1) = \int \mathrm{d}z_2 \, \mathrm{d}z_3 \, \mathrm{d}z_4 \, \langle j_\mu(z) j_\nu(0) \rangle$$

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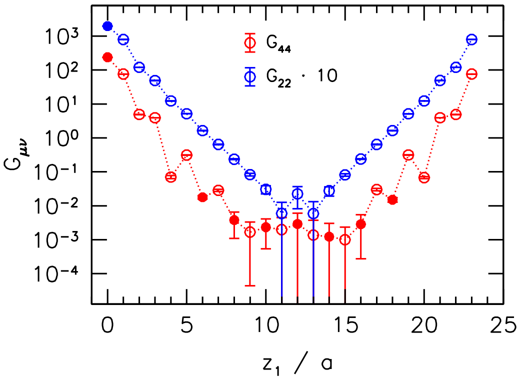
- similarly magnetic susceptibility

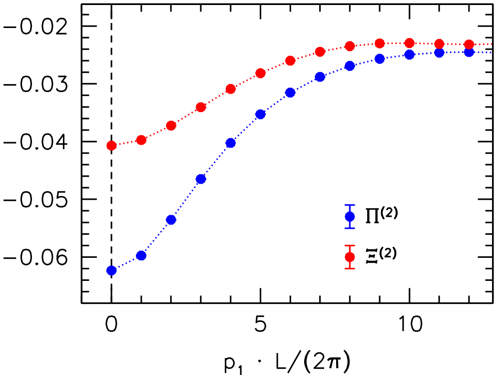
$$\chi = \int \mathrm{d}z_1 \, \frac{z_1^2}{2} G_{22}(z_1)$$

# Results




- on $24^3 \times 6$ lattice, at temperature $T = 176$ MeV
- connected as well as disconnected contributions included
- filled (empty) points: positive (negative)

# Vacuum polarization




- $p_1 > 0$: convolute $G_{\mu\nu}(z_1)$ with $[1 - \cos(p_1 z_1)]/p_1^2$
- $p_1 = 0$: convolute $G_{\mu\nu}(z_1)$ with $z_1^2/2$

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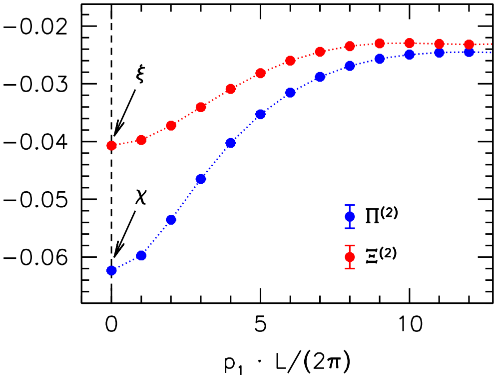


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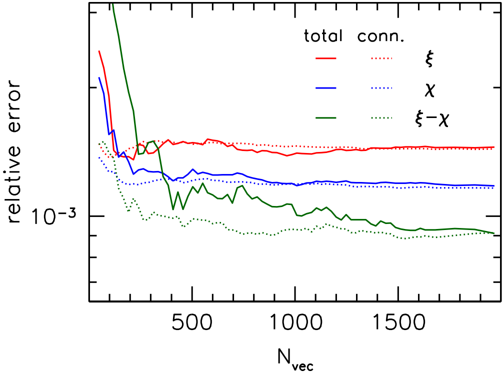
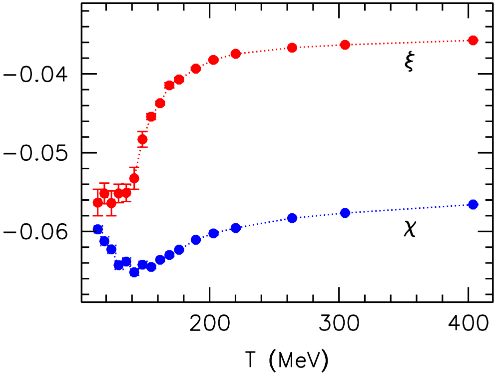
# Precision




- $j_2 j_2$ and $j_4 j_4$ correlate

  $\Rightarrow \xi - \chi$ can be measured more precisely

# Temperature-dependence



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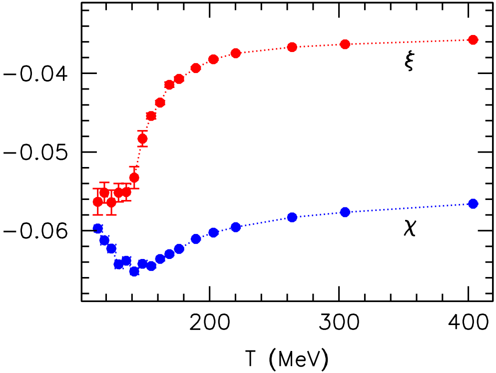


- but in the vacuum $\epsilon = \mu = 1$

  $\Rightarrow \xi(T = 0) = \chi(T = 0) = 0$ should hold

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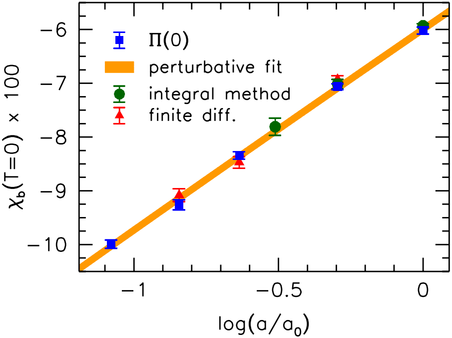


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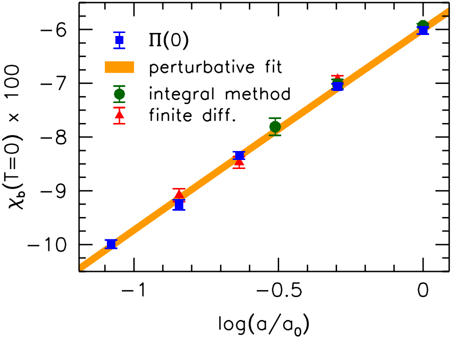


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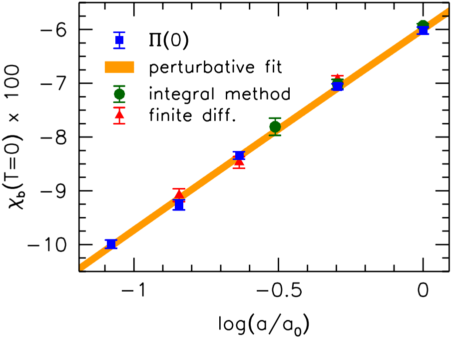


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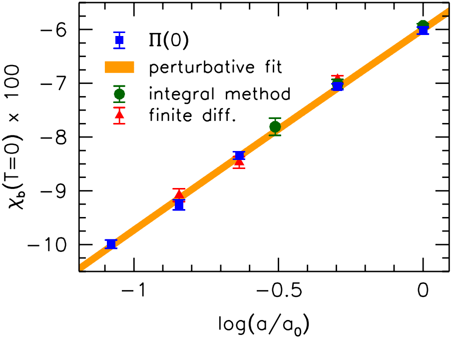


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- comparison to other approaches [Bonati et al '13, Bali et al '14]

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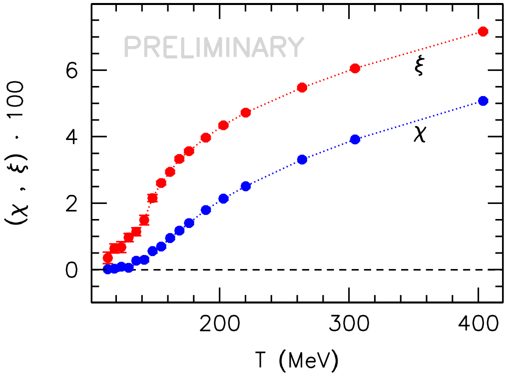
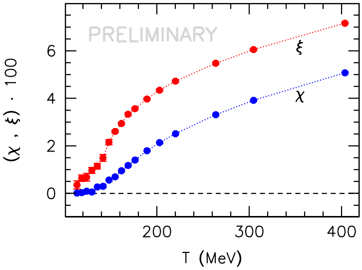


# Interpretation

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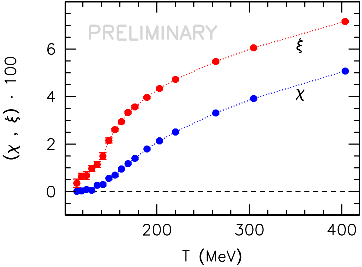


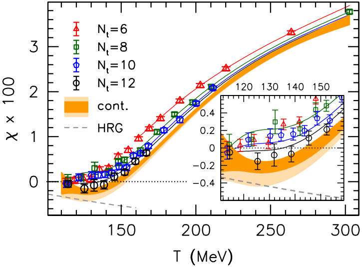


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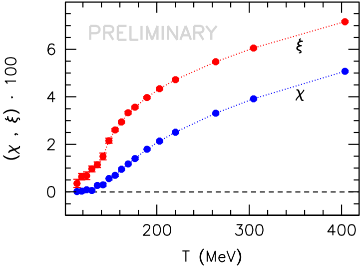


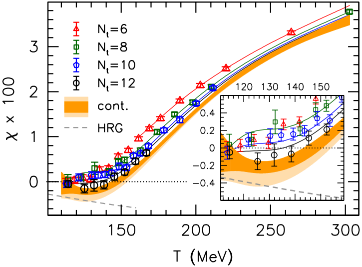


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- $\xi > 0$: permittivity $> 1$ (holds for all dielectrics [Landau Vol 8.])

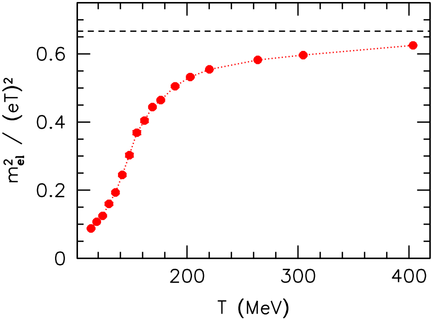
# Electric mass

- photon mass at finite temperature

$$m_{\mathrm{el}}^2 = \int \mathrm{d}z_1 \, G_{44}(z_1)$$

- free quark gas

$$m_{\mathrm{el}}^2 = \frac{e^2 T^2}{3} \cdot N_c \sum_f (q_f/e)^2$$

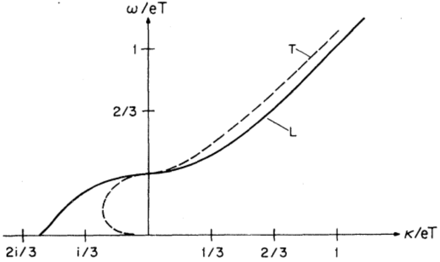


# Dispersion relation

- photon propagator inverse for transverse/longitudinal oscillations

$$\omega^2 - k^2 - \Pi_{T/L}(\omega, k)$$

- in perturbation theory [Weldon '82]



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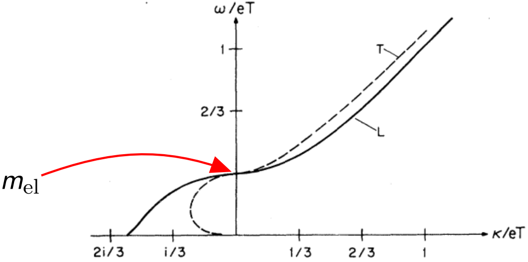

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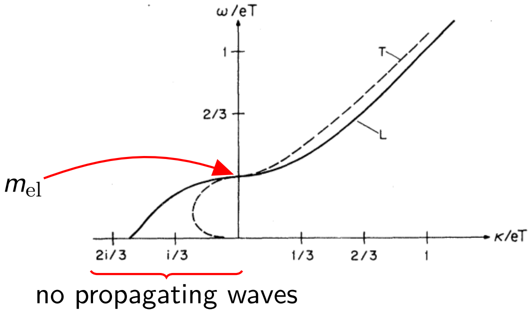

# Summary

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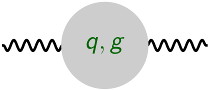
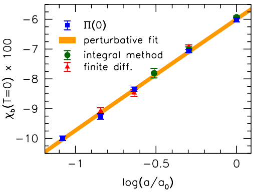
- electric fields induce sign problem and quantization problem

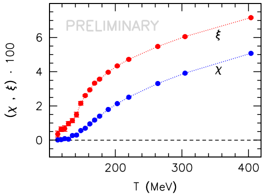
  $\Rightarrow$ solved via vacuum polarization approach




- divergence due to charge renormalization




- para/diamagnetism and paraelectricity



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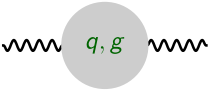


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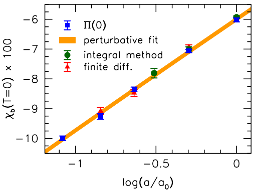


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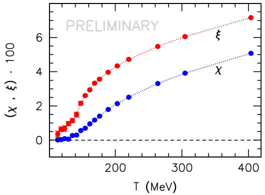
  outlook: generalize to polarizabilities



# Backup

# Electric and magnetic fields

- compare $\mathbf{E} = E\,\mathbf{x_1}$ and $\mathbf{B} = B\,\mathbf{x_3}$

$$A_4 = -iE\,x_1, \qquad A_2 = B\,x_1$$

- equivalence at <u>zero</u> temperature

$$L_4 = L_2 = \infty : \qquad \log \mathcal{Z}(E = i \cdot x) = \log \mathcal{Z}(B = x)$$

- effect of finite temperatures

$$L_4 < L_2 : \qquad \log \mathcal{Z}(E = i \cdot x) \neq \log \mathcal{Z}(B = x)$$

# Trick

- consider cos-type and sin-type fields

$$E^{\cos}(x_1) = \cos(p_1 x_1) \cdot E, \qquad A_4^{\cos}(x_1) = \frac{\sin(p_1 x_1)}{p_1} \cdot E$$

$$E^{\sin}(x_1) = \sin(p_1 x_1) \cdot E, \qquad A_4^{\sin}(x_1) = -\frac{\cos(p_1 x_1)}{p_1} \cdot E$$

- corresponding susceptibilities in infinite volume

$$\xi_{\cos}^{(p_1)} \xrightarrow{p_1 \to 0} \xi, \qquad \xi_{\sin}^{(p_1)} \xrightarrow{p_1 \to 0} -\frac{m_{\rm el}^2}{p_1^2}$$

- combine as

$$\xi_{\cos}^{(p_1)} + \xi_{\sin}^{(p_1)} + \frac{m_{\rm el}^2}{p_1^2}$$

$$= -\frac{T}{V} \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, \frac{\overbrace{\sin(p_1 x_1)\sin(p_1 y_1) + \cos(p_1 x_1)\cos(p_1 y_1)}^{\cos[p_1(x_1 - y_1)]} - 1}{p_1^2} \langle j_4(x) j_4(y) \rangle$$