Inhomogeneous phases in the Gross-Neveu model (and beyond)

Laurin Pannullo,

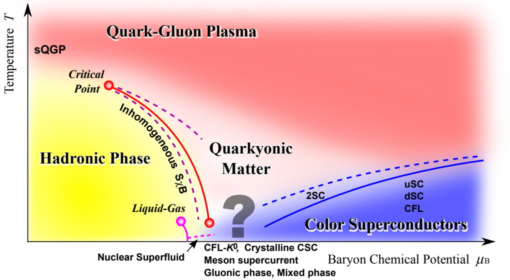
Journal Club, 10.11.2023

Based on

[M. Thies, K. Urlichs, *Phys. Rev. D.* **67** (2003)]
[M. Thies, *Phys. Rev. D.* **69** (2004)]
[O. Schnetz *et al.*, *Ann. Phys.* **314** (2004)]

DFG HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research UNIVERSITÄT BIELEFELD CRC–TR 211 Strong-interaction matter under extreme conditions

Motivation (I) – QCD phase diagram



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- A plot full of conjectures

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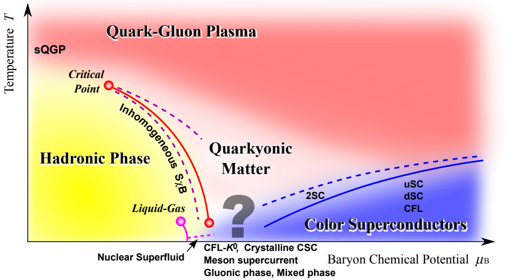

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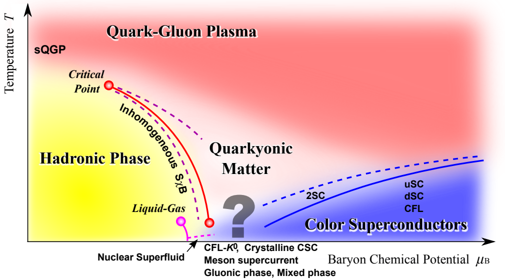

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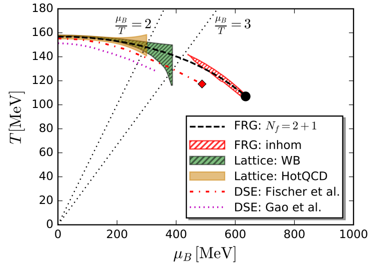

adapted from [W.-j. Fu *et al.*, *Phys. Rev. D.* **101** (2020)]

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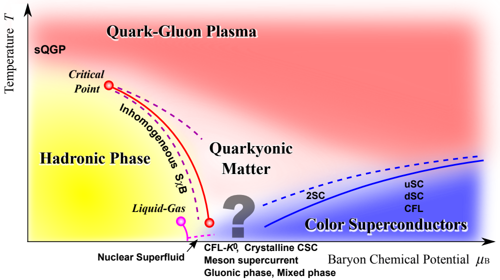

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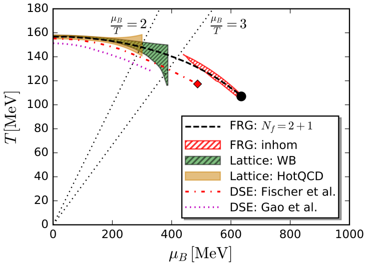

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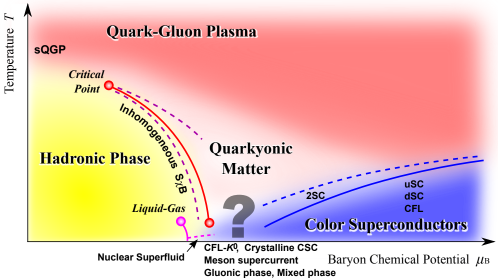

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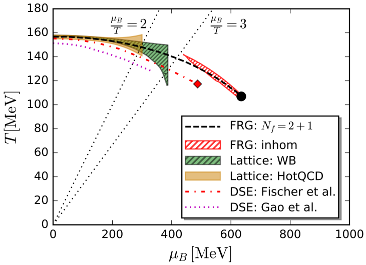

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  - Do first principal calculations $\Rightarrow$ very hard / impossible
  - Use models of QCD
    $\Rightarrow$ a lot easier; questionable physical relevance
  - Maybe chiral inhomogeneous phases?

- What we know about the QCD phase diagram from lattice and functional methods

Motivation (II) – What is a chiral inhomogeneous phase?

- Possible chiral phases
  - $\langle\bar{\psi}\psi\rangle(x) = \text{const.} = 0$: Symmetric phase (SP)
  - $\langle\bar{\psi}\psi\rangle(x) = \text{const.} \neq 0$: Homogeneously broken phase (HBP)

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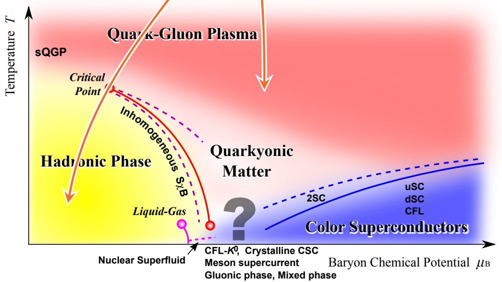

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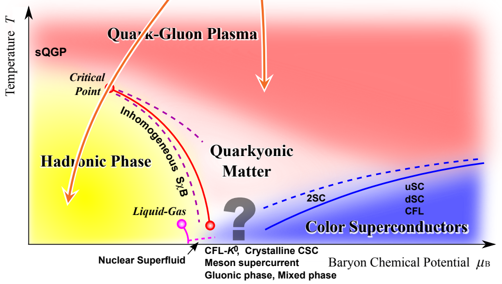

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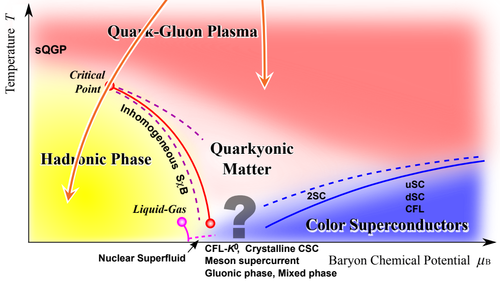
- IP breaks chiral symmetry and translational invariance **(!)**
- Well known in condensed matter, exotic in high energy physics



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Outline

- The $(1+1)$-dimensional Gross-Neveu model: Where?
- Inhomogeneous condensation in the $(1+1)$-dimensional Gross-Neveu model: How?
  [M. Thies, K. Urlichs, *Phys. Rev. D.* **67** (2003)] [O. Schnetz *et al.*, *Ann. Phys.* **314** (2004)]
- Tales from condensed matter physics: Peierls instability: "Why?"
- Where to go from here?

Gross-Neveu model

- Gross-Neveu (GN) model in $1+1$ dimensions in the large-N limit / mean-field approximation
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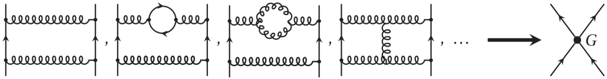
Four-Fermion vertex effectively describes gluonic interactions



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$$Z = \int \mathscr{D}\sigma\, \mathrm{e}^{-NS_{\text{eff}}[\sigma]} \xrightarrow[\text{field}]{\text{mean-}} Z = \sum_i \mathrm{e}^{-NS_{\text{eff}}[\Sigma_i]}$$

$$\Sigma = \underset{\sigma}{\operatorname{argmin}}\, S_{\text{eff}}[\sigma]$$

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Gross-Neveu model: Assuming translational invariance

- assume homogeneous fields $\sigma = \bar{\sigma}$
- Minimize the effective action in $\bar{\sigma}$

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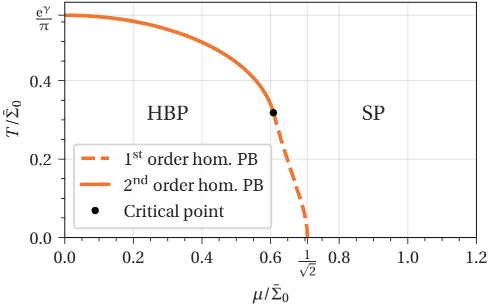

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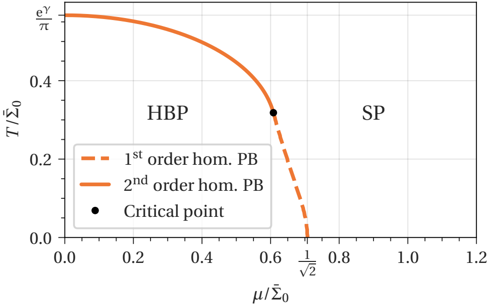
- However, μ_c should be $2/\pi \approx 0.64$ instead of $1\sqrt{2} \approx 0.71$! [R. F. Dashen *et al.*, *Phys. Rev. D.* **12** (1975)]



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- Express spinors as $\psi = (\phi_+, \phi_-)$ to obtain the coupled equations

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which can be decoupled by squaring

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- Lots of analogies with SUSY
  - Schrödinger potentials $U_\pm = \mp\frac{\partial\sigma}{\partial x} + \sigma^2$ with same eigenvalues
  - σ is the so-called superpotential to U

Gross-Neveu model: Parametrization of the scalar field (I)

Most naive parametrization:

- Put system in box $L = Na$ and express σ as Fourier components

$$\sigma(x) = \sum_l \tilde{\sigma}_l \mathrm{e}^{\mathrm{i}2\pi l x/a}.$$

- Find energies via numerical diagonalization and use these in the calculation of the grand-canonical potential
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- There is a better way!

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Choose a better ansatz relying on intuition, SUSY and luck?:

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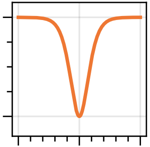

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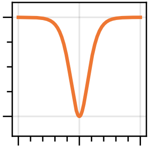

- "stitch" these potential wells together in distance d

$$\sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2(x-nd)} = \text{ complicated stuff}$$

- the resulting Schrödinger potential is of the known Lamé-type with superpotential

$$\sigma(x) = A\kappa^2 \frac{\operatorname{sn}\left(Ax|\kappa^2\right)\operatorname{cn}\left(Ax|\kappa^2\right)}{\operatorname{dn}\left(Ax|\kappa^2\right)}$$

$\operatorname{dn}, \operatorname{cn}, \operatorname{sn}$ are Jacobi elliptic functions. A and κ are parameters of the ansatz.

Gross-Neveu model: Solution of the Hartree-Fock equation

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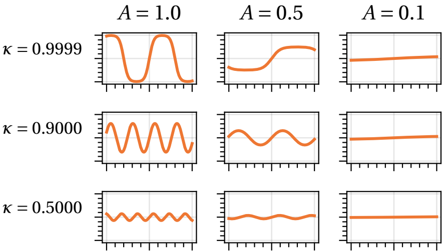

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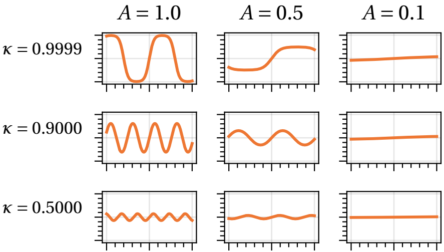

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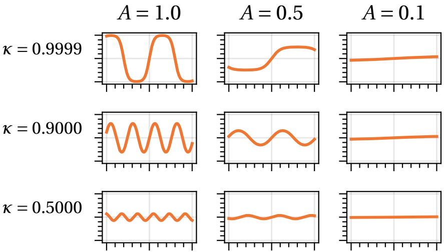

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- Known solution for ϕ_+ and ω!

$$|\omega| = \mathrm{dn}\left(\alpha|\kappa\right), \quad q = -\mathrm{i}AZ(\alpha) + \frac{A\pi}{2\mathbf{K}}, \quad \frac{\mathrm{d}q}{\mathrm{d}\omega} = \pm A\frac{\omega^2 - \mathbf{E}/\mathbf{K}}{\sqrt{(\omega^2-1+\kappa^2)(\omega^2-1)}}$$

Gross-Neveu model: Grand-canonical potential

- Grand-canonical potential:

$$\Omega = \frac{S_{\text{eff}}}{\beta V} =$$

$$= -\frac{1}{\beta \pi} \int_0^{\Lambda/2} \mathrm{d}q \ln\left[\left(1 + \mathrm{e}^{-\beta(E-\mu)}\right)\left(1 + \mathrm{e}^{-\beta(E+\mu)}\right)\right] + \frac{1}{4GV} \int_{-\infty}^{\infty} \mathrm{d}x \sigma^2(x) =$$

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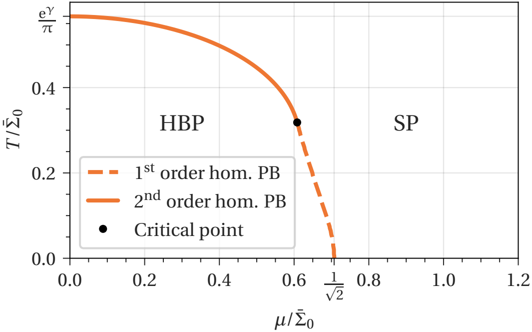
- renormalize Ω and minimize in κ and A (very technical, not discussed here)

Gross-Neveu model: Inhomogeneous phase diagram



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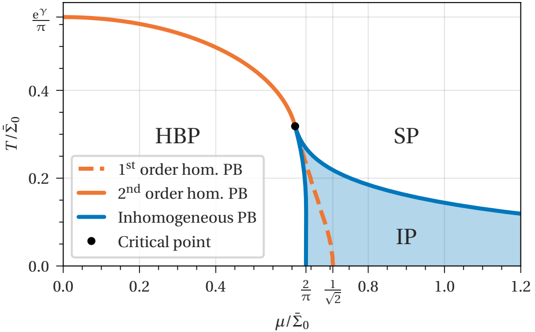

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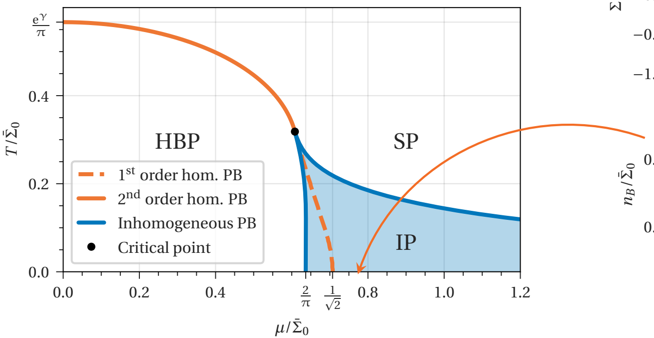

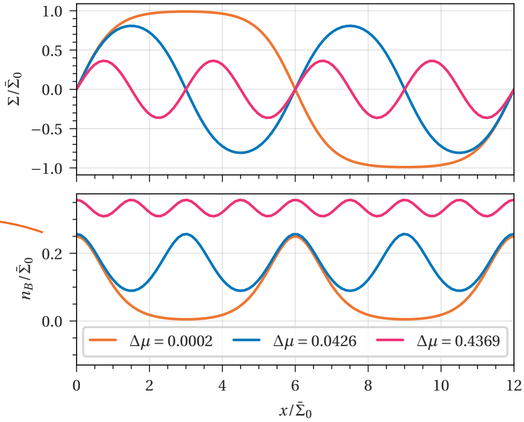

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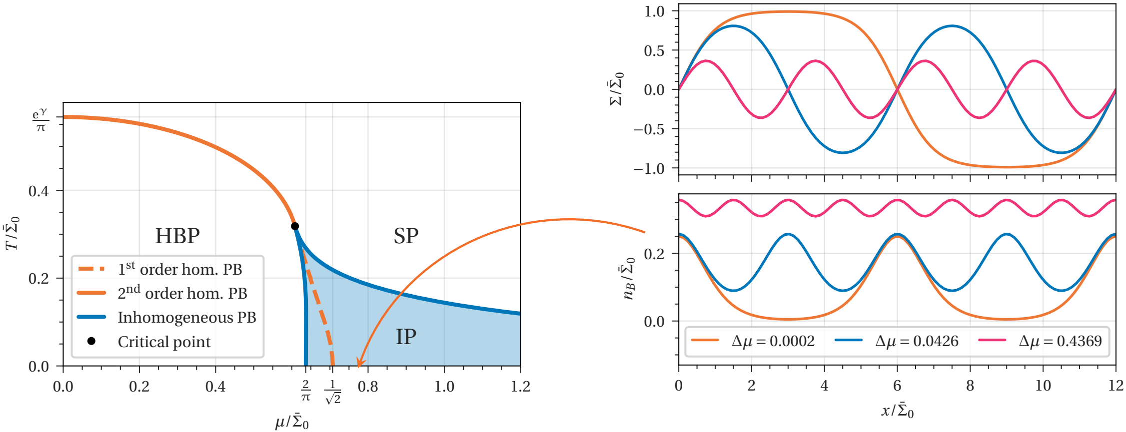

- What is going on? Why does the system break translational invariance?

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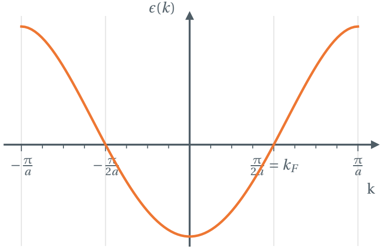

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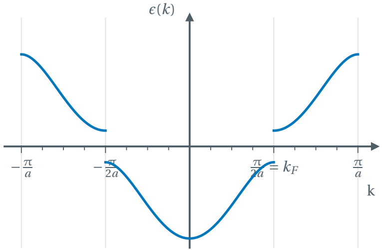

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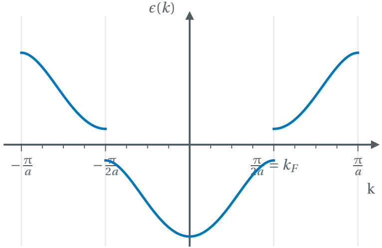

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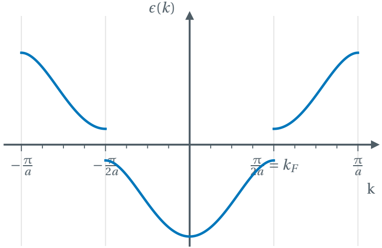

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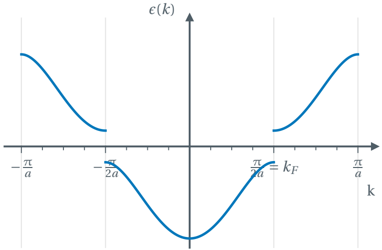

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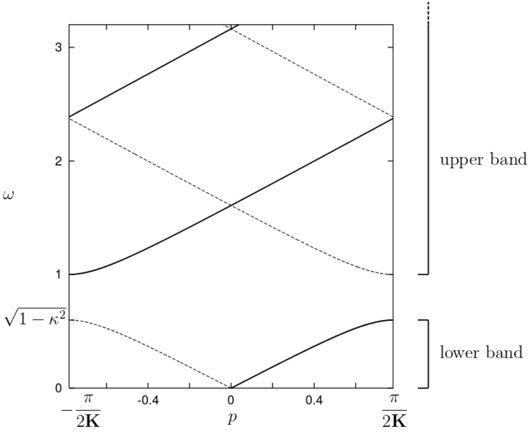
Back to the Gross-Neveu model

- Electrons ≡ Fermions in the Gross-Neveu model
- potential of the atoms ≡ chiral condensate
- dispersion relation has pronounced gap at $p = \frac{\pi}{2\mathbf{K}} = \pi p_f$



[O. Schnetz *et al.*, *Ann. Phys.* **314** (2004)]

Summary of what we learned

- The Gross-Neveu model is a very simple four-Fermion model
- It experiences spontaneous breaking of translation symmetry at finite density
- This is caused by the Peierls instability
- Very specific setting: Mean-field, $1+1$ dimensions, simple model

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  - $d+1$: IP only for $d<2$, IP in $d=3$ regulator dependent ("artifact")
    [L. Pannullo, *Phys. Rev. D.* **108** (2023)]