Causality and Stability of First Order Relativistic Hydrodynamics

based on [P. Kovtun, JHEP 10 (2019) 034]

Mattis Harhoff

High-energy physics Journal club
Bielefeld University

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Why (relativistic) hydro?

- Hydro = Effective long-distance description of many-body systems at non-zero temperature
- Relativistic Hydro: Microscopic constituents constrained by Lorentz symmetry
- Applications: Description of QGP, quantum magnets, black holes, etc.

Outline

- Introduction to relativistic hydrodynamics
  Conservation laws, constitutive equations, frames, etc.
- Criteria for stability and causality, Landau & Eckart frames
- Stability and causality in the general frame

Introduction to Relativistic Hydro

Conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu = 0$$

Energy-momentum tensor $T^{\mu\nu} \leftrightarrow$ spacetime symmetries,
Conserved current $J^\mu \leftrightarrow$ global $U(1)$-symmetry (e.g. baryon number)

Problem: $(d+2)$ equations
$(d+1)(d+2)/2 + (d+1)$ variables ($T^{\mu\nu}$ is symmetric)

Hydrodynamic assumption

$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, $J^\mu = \langle \hat{J}^\mu \rangle$ can be parametrized by:
Temperature $T(x)$, four-velocity $u^\mu(x)$, chemical potential $\mu(x)$

$\rightarrow$ Derivative expansion

Energy-momentum tensor

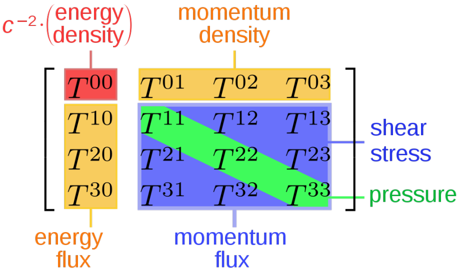

Image taken from: [Wikimedia Commons]

Decomposition using u^μ:

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu$$

Projector $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$, Metric $\eta^{\mu\nu} = \text{diag}(-,+,+,+)$
Scalars; transverse vectors; transverse, symmetric, traceless tensors

Ingredients for a derivative expansion:

- 0thorder: T, μ (equilibrium)
- 1storder: $\dot{T} \equiv u^\lambda \partial_\lambda T$, $\partial_\lambda u^\lambda$, $\dot{\mu}$, $\Delta^{\rho\sigma}\partial_\sigma T$, $\dot{u}^\rho$, $\Delta^{\rho\sigma}\partial_\sigma \mu$, $\sigma^{\mu\nu} = \Delta^{\mu\rho}\Delta^{\nu\sigma}(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{d}\eta_{\rho\sigma}\partial_\lambda u^\lambda)$
- 2ndorder: ...

Constitutive equations (first order)

$$\mathcal{E} = \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2)$$
$$\mathcal{P} = p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2)$$
$$q^\mu = \theta_1 \dot{u}^\mu + \theta_2/T \Delta^{\mu\lambda}\partial_\lambda T + \theta_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2)$$
$$t^{\mu\nu} = -\eta\sigma^{\mu\nu} + \mathcal{O}(\partial^2)$$
$$\mathcal{N} = n + \nu_1 \dot{T}/T + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda(\mu/T) + \mathcal{O}(\partial^2)$$
$$j^\mu = \gamma_1 \dot{u}^\mu + \gamma_2/T \Delta^{\mu\lambda}\partial_\lambda T + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda(\mu/T) + \mathcal{O}(\partial^2)$$

We have:

- At 0^{th}order: ϵ, p, n functions of μ, T
- At 1^{st}order: 16 (a priori independent) **transport coefficients** $\varepsilon_{1,2,3}, \pi_{1,2,3}, \theta_{1,2,3}, \nu_{1,2,3}, \gamma_{1,2,3}, \eta$ functions of μ, T

$\rightarrow$ Input parameters coming from some microscopic theory

Frames & frame transformations

- No first-principles microscopic definition of $T(x), u^\mu(x), \mu(x)$
  - Redefinitions $T' = T + \delta T$, $u'^\mu = u^\mu + \delta u^\mu$, $\mu' = \mu + \delta\mu$ possible
  - Choices must agree in equilibrium $\rightarrow$ δT, δu^μ, $\delta\mu$ have to be $\mathcal{O}(\partial)$
  - Most transport coefficents transform non-trivially under frame transformations!
- There are 7 invariant transport parameters:

$$\eta, \quad f_i \equiv \pi_i - \left(\frac{\partial p}{\partial \epsilon}\right)_n \varepsilon_i - \left(\frac{\partial p}{\partial n}\right)_\epsilon \nu_i, \quad \ell_i \equiv \gamma_i - \frac{n}{\epsilon + p}\theta_i$$

- Freedom in choosing frame but:
  Different differential equations $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$
  $\rightarrow$ Effects on well-posedness of initial value problem or instabilities

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu$$

Landau(-Lifshitz) frame[1]

- Choose $\mathcal{E} = \epsilon$, $\mathcal{N} = n$, $q^\mu = 0$
- Velocity defined by flow of energy: $u_\mu T^{\mu\nu} = \epsilon u^\nu$
- Use "on-shell" relations $\rightarrow$ left with shear viscosity η, bulk viscosity π_2, charge conductivity γ_3

Eckart frame[2]

- $\mathcal{E} = \epsilon$, $\mathcal{N} = n$, $j^\mu = 0$
- Velocity defined by flow of particles: $J^\mu = n u^\mu$
- Use "on-shell" relations $\rightarrow$ left with shear viscosity η, bulk viscosity π_2 and θ_1, $\theta_2 \leftrightarrow$ heat conductivity

[1] L. D. Landau and E. M. Lifshitz, Fluid Mechanics. Pergamon, 1987.

[2] C. Eckart, Phys. Rev. 58, 919 (1940).

Linearized Stability and Causality in the Landau & Eckart Frames

Look at state T, $u^\alpha = (1, \mathbf{v})/\sqrt{1-\mathbf{v}^2}$, μ

- Assume equilibrium background with small perturbations
  $T = T_0 + \delta T$, $\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v}$, $\mu = \mu_0 + \delta\mu$
  Linearize hydrodynamic equations in δT, $\delta \mathbf{v}$, $\delta\mu$
- Ansatz: δT, $\delta \mathbf{v}$, $\delta\mu \propto e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$ (Single Fourier modes)
  $\rightarrow$ Linear system $M^A_B \delta Y^B = 0$
- $\det(M) = 0 \rightarrow$ Set of eigenfrequencies $\omega_a(\mathbf{k})$
  Functions of $T_0, \mathbf{v}_0, \mu_0$ and transport coefficients

Stability: $\operatorname{Im}\omega_a(\mathbf{k}) \leq 0$,

Causality: $0 < \lim_{k\to\infty} \left| \frac{\operatorname{Re}\omega_a(\mathbf{k})}{|\mathbf{k}|} \right| < 1$

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Generic instabilities in first-order dissipative relativistic fluid theories

William A. Hiscock*
Department of Physics, University of California, Santa Barbara, California 93106

Lee Lindblom
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
(Received 27 September 1984)

We consider the stability of a general class of first-order dissipative relativistic fluid theories which includes the theories of Eckart and of Landau and Lifshitz as special cases. We show that all of these theories are unstable in the sense that small spatially bounded departures from equilibrium at one instant of time will diverge exponentially with time. The time scales for these instabilities are very short; for example, water at room temperature and pressure has an instability with a growth time scale of about 10^{-34} seconds in these theories. These results provide overwhelming motivation (we believe) for abandoning these theories in favor of the second-order (Israel) theories which are free of these difficulties.

- Theory that contains the Landau and Eckart as special cases:

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu$$

$\mathcal{E} = \epsilon$, $\mathcal{P} = p - \zeta\partial_\lambda u^\lambda$, $t^{\mu\nu} = -\eta\sigma^{\mu\nu}$, $\mathcal{N} = n$

$q^\mu = -\kappa T \Delta^{\mu\nu}\left[\frac{1}{T}\partial_\nu T + \dot{u}_\nu\right] \to$ Landau: $\kappa = 0$

$j^\mu = -\sigma T^2 \Delta^{\mu\nu}\partial_\nu(\frac{\epsilon+p}{nT} - s(\epsilon, n)) \to$ Eckart: $\sigma = 0$

- Linearization, plane wave Ansatz δT, $\delta\mathbf{v}$, $\delta\mu \propto e^{ikx-i\omega t}$
$\to M^A_B \delta Y^B = 0$, $A, B = 1, ...17$

- Block-diagonal $M = \text{diag}(Q, R, R, \mathbb{I}_{4x4})$)

$$\det(M) = 0 \quad \Rightarrow \quad (\det Q)(\det R)^2 = 0$$

- Identity matrix: $\delta j^y = \delta j^z = \delta t^{yz} = \delta t^{yy} - \delta t^{zz} = 0$

- R: 3x3 matrix $\leftrightarrow$ contains transverse modes

$$\omega_{\pm}(k) = \frac{i}{2\kappa T}\left[(\epsilon + p) \pm \sqrt{(\epsilon + p)^2 + 4\eta\kappa T k^2}\right]$$

  No propagation! Only growth/decay. Stability: $\operatorname{Im}\omega(k) < 0$.
  Calculation in Landau frame $\kappa = 0 \rightarrow \operatorname{Im}\omega(k) < 0$, but:
  $\omega \rightarrow \infty$ in limit $\kappa \rightarrow 0$

- Q: 7x7 matrix, longitudinal modes
  Similar problems with unstable, growing modes

Stability & Causality in the General Frame for Uncharged Fluids

- General frame, 6 transport coefficients: $\varepsilon_{1,2}$, $\pi_{1,2}$, $\theta \equiv \theta_1 = \theta_2$, η
  Goal: Stable & causal subset of parameter space
  ε_1, π_1, θ_1 come with time derivatives $\rightarrow$ relaxation times

- Reminder: Modes $\leftrightarrow F_{\text{shear}}(\omega, \mathbf{k})^{d-1} F_{\text{sound}}(\omega, \mathbf{k}) = 0$

- Stability depends on Lorentz frame $\rightarrow$ analysis at non-zeros $\mathbf{v}_0$

- If linear dispersion relation $\omega(\mathbf{k}) = \pm c_0 |\mathbf{k}|$ at $\mathbf{v}_0 = 0$:
  (generally true at large $\mathbf{k}$)

$$c_v(\phi) = \frac{v_0(1-c_0^2)}{1-c_0^2 v_0^2} \pm \frac{c_0}{1-c_0^2 v_0^2} \sqrt{(1-v_0^2)\left[1 - v_0^2 c_0^2 - v_0^2(1-c_0^2)\cos^2\phi\right]}$$

  ϕ: angle between $\mathbf{k}$, $\mathbf{v}_0$
  $\rightarrow$ If the modes are stable and causal at rest, they remain so at all $\mathbf{v}_0$

Shear channel

Gamma factor $\gamma_0 = 1/\sqrt{1-\mathbf{v}_0^2}$, eq. enthalpy density $w_0 = \epsilon_0 + p_0$

$$F_{\text{shear}} = (\theta - \mathbf{v}_0^2\eta)\omega^2 + \left(\frac{iw_0}{\gamma_0} - 2(\theta-\eta)\mathbf{k}\cdot\mathbf{v}_0\right)\omega - \frac{iw_0}{\gamma_0}(\mathbf{k}\cdot\mathbf{v}_0) - \frac{\mathbf{k}^2\eta}{\gamma_0^2} + (\mathbf{k}\cdot\mathbf{v}_0)^2(\theta-\eta)$$

- Small $\mathbf{k}$:
  - Gapless modes $\omega(\mathbf{k}) = \mathbf{k}\cdot\mathbf{v}_0 - \frac{i\eta}{w_0}\sqrt{1-\mathbf{v}_0^2}(\mathbf{k}^2 - (\mathbf{k}\cdot\mathbf{v})^2) + \mathcal{O}(\mathbf{k}^3)$
  - Gapped modes $\omega(\mathbf{k}) = \frac{iw_0\sqrt{1-\mathbf{v}_0^2}}{\eta\mathbf{v}_0^2-\theta} + \mathcal{O}(\mathbf{k}\cdot\mathbf{v}_0)$

  Stability $(\operatorname{Im}\omega < 0)$ requires $\theta > \eta > 0$
- Large $\mathbf{k}$: $\omega(\mathbf{k}) = c_{\text{shear}}(\phi)|\mathbf{k}|$

  $|c_{\text{shear}}| < \frac{1+|v_0|\sqrt{\theta/\eta}}{|v_0|+\sqrt{\theta/\eta}} < 1$ as long as $\theta > \eta$.

Dispersion relation of shear modes

$v_0 = 0.9$, $\theta/\eta = 2$, different angles ϕ between $\mathbf{k}$ and $\mathbf{v}_0$

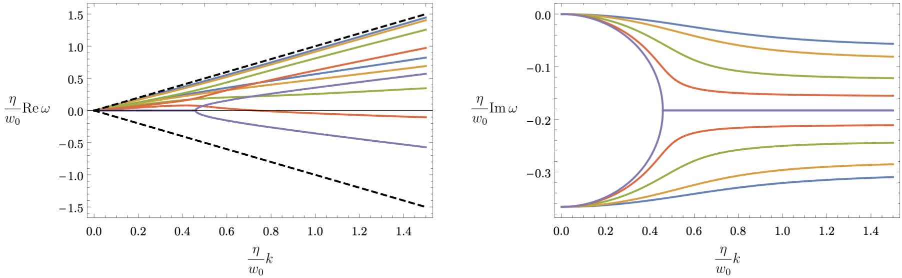


Image taken from: [P. Kovtun, JHEP 10 (2019) 034]

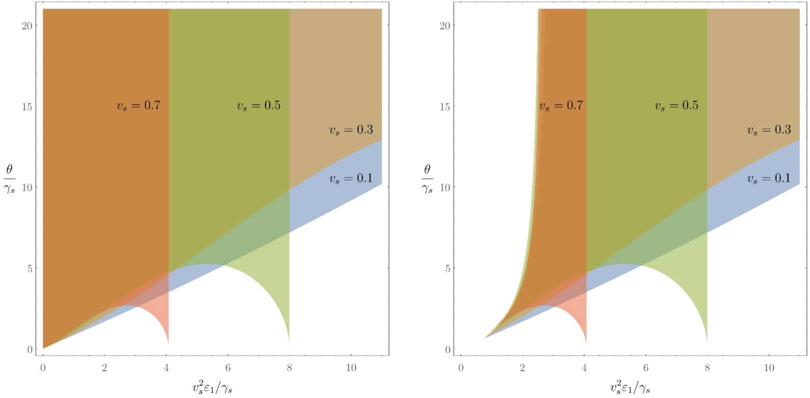
Sound channel

$F_{\text{sound}}(\mathbf{v}_0, \omega, \mathbf{k})$ is a fourth-order polynomial in ω

- At rest ($\mathbf{v}_0 = 0$):
  - Small $\mathbf{k}$: Two linear sound modes & two gapped modes
    Stability: Damping factor $\gamma_s \equiv \frac{4}{3}\eta + \zeta > 0, \varepsilon_1 > 0, \theta > 0$
  - Arbitrary $\mathbf{k}$: Stability $\rightarrow$ Two non-linear relations between $\varepsilon_{1,2}, \pi_1, \theta$
  - Large $\mathbf{k}$: Weaker stability constraints but additional causality constraint

Constraints for transport coefficients θ, ϵ_1 for fluid at rest

with $\varepsilon_2 = 0$, $\pi_1/\gamma_s = 3/v_s^2$. Speed of sound $v_s^2 \equiv \frac{\partial p_0}{\partial \epsilon_0}$



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Left: Region where all modes are stable

Right: Region where all modes are stable and large-$\mathbf{k}$ modes are causal

- Moving fluid $\mathbf{v}_0 \neq 0$
  - Small $\mathbf{k}$: $\omega(k,\phi) = c_s(\phi)k - \frac{i}{2}\Gamma_s(\phi)k^2 + \mathcal{O}(k^3)$

    For $v_s = 1/2$ (c_s solid, Γ_s dashed):

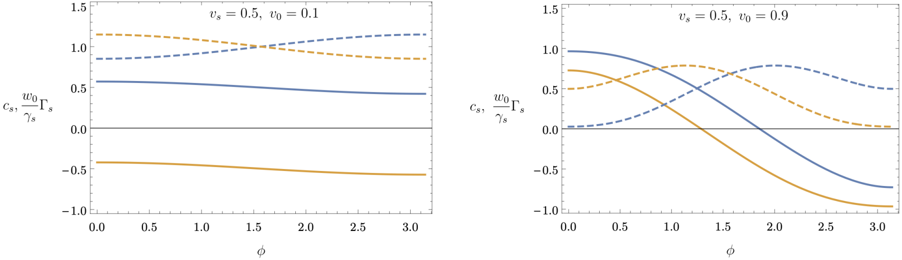
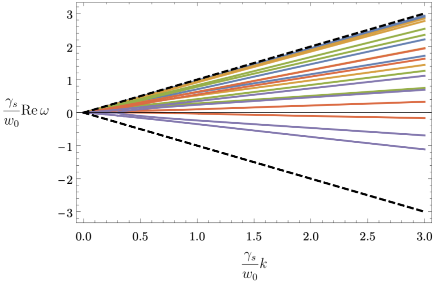


Image taken from: [P. Kovtun, JHEP 10 (2019) 034]

  - Stability & causality at $\mathbf{v}_0 = 0$
    $\rightarrow$ Stability and causality for large $\mathbf{k}$ for $\mathbf{v}_0 \neq 0$

Dispersion relation of sound modes



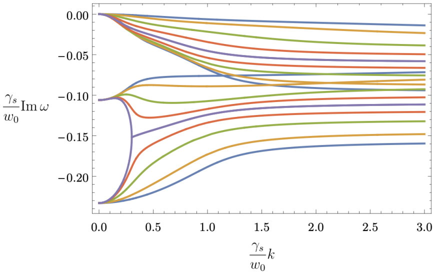


Image taken from: [P. Kovtun, JHEP 10 (2019) 034]

Summary

- First-order relativistic hydrodynamics:
  Derivative expansion in T, u^μ, μ to first order
  In general: More transport coefficients than for Landau/Eckart frames
- Analysis of linear perturbations of thermal equilibrium
- Linear stability by using proper relaxation times
  Finite region in parameter space where lin. perturbations are stable
- Outlook:
  Non-linear perturbations $\rightarrow$ [P. Kovtun, JHEP 06 (2020) 067]
  Charged fluids seem possible