Macroeconomics with Household Heterogeneity
An Introduction

Dirk Krueger

University of Pennsylvania, CEPR, and NBER

Winter 2025

Why Macro with Heterogeneous Households or Firms?

- Data: lots of observed and unobserved heterogeneity among households (Heckman, Wolpin, etc.). Large literature in empirical microeconomics
- Models: Household heterogeneity affects:
  - Aggregate quantities (Aiyagari '94) and their evolution (Krusell and Smith '98). Asset prices (Huggett '93, Storesletten et al. '06).
  - Answers to counterfactual policy (effect of temporary tax cuts, see Heathcote '05, Kaplan and Violante '12, '13)
  - Answers to normative questions (e.g. Lucas '87 vs Krusell-Smith '99 and Krebs '07 on the cost of business cycles)
- Many questions involve distributional aspects that cannot be studied in representative agent world:
  - What explains trends in inequality (e.g. Krueger and Perri '06)
  - Redistributive consequences of change in tax progressivity, social security (e.g. Auerbach and Kotlikoff '87)
- Note: strong overlap in questions (and sometimes, techniques) with labor economics, public finance, etc.

A Broad Classification and Overview of Models

- Environment: shock $s, s' \in S$ affects household incomes $y(s), y(s')$
- Complete markets models

$$c + \sum_{s' \in S} q(s') a'(s') \leq y(s) + a(s)$$

  - with enforcement, informational frictions: $a'(s') \geq -\bar{A}(s')$
- Standard incomplete markets model

$$\begin{aligned} c + q a'(s) &\leq y(s) + a \qquad \text{or:} \\ c + a'(s) &\leq y(s) + (1+r)a \end{aligned}$$

  - with borrowing constraints: $a' \geq -\bar{A}$
- Autarky (Hand to Mouth Consumers)

$$c \leq y(s)$$

Organization of Facts

- Prototypical Budget constraint

$$c + (a' - a) = \sum_i w_i h_i + ra + T^p + T^g - \tau$$

- Variables of interest:

$$w_i, h_i, w_i h_i, ra, c, a, y^m = \sum_i y_i + ra + T^p, y^d = y^m + T^g - \tau$$

- Dimensions of the data:

	μ	σ	$\Delta\sigma$
age a	means (1)	See next box $\rightarrow$	var. over lc (2)
time t	RA macro (0)	ineq. levels (3)	Ineq. trends (4)

- Aggregate time series of income and consumption (0)
- Follow a cohort over time as it ages: Life cycle means (1) and cross-cohort variances (2)
- Cross-section of population at point in time (3)
- Evolution of the cross-section over time (inequality trends), (4)

Micro Data Sources: (Incomplete) Overview

- Micro data for the U.S.
  - Household surveys
    - Current Population Survey (CPS)
    - American Community Survey (ACS)
    - Consumer Expenditure Survey (CE or CES or CEX)
    - Panel Study on Income Dynamics (PSID)
    - Survey of Consumer Finances (SCF)
  - Administrative data sets
    - Tax data (IRS)
    - Social Security data (SSA)
- Micro data for other individual countries
  - Italy (SHIW)
  - Germany (GSOEP)
  - UK (FES)
  - More recently: poorer economies (India, Thailand, China, Malawi)
- Cross-country data sets
  - Global Repository on Income Dynamics (GRID)
  - World Inequality Database (WID)
  - Luxembourg Income/Wealth Study (LIS/LWS)
  - ECB Household Finance and Consumption Survey (HFCS)

U.S. Micro Data Sources: Consumer Expenditure Survey (CEX) or (CE)

- Conducted by the U.S. Bureau of the Census and sponsored by the Bureau of Labor Statistics.
- Detailed consumption expenditure data.
- Yearly from 1980 onward.
- Initially ca. 5000 representative households; now roughly 21,000 interviewed households.
- Rotating panel (no real panel dimension to speak of).
- Income and wealth data not great. Concerns even about coverage of consumption.
- http://www.bls.gov/cex/

U.S. Micro Data Sources: Survey of Consumer Finances (SCF)

- Conducted by the National Opinion Research center at the University of Chicago and sponsored by the Federal Reserve System.
- Detailed information about households' income and wealth
- Triennial, available surveys are from 1989, 1992, 1995, 1998, 2001, 2004, 2007, 2010, 2013, 2016, 2019, 2022
- Kuhn, Schularick and Steins (2018): historical versions of SCF from 1949 on.
- About 5,000-6,000 households, a representative part and a part that oversamples rich households (key for wealth data)
- No consumption data, no panel dimension
- http://federalreserve.gov/econresdata/scf/scfindex.htm

U.S. Micro Data Sources: Panel Study of Income Dynamics (PSID)

- Conducted by the Survey Research Center, University of Michigan, sponsored by NSF.
- Detailed information about income and wealth (although inferior to SCF).
- Annual from 1968; from 1997 biannual.
- Started with representative sample of 5,000 households, expanded since then.
- In 1990 representative sample of Latinos added.
- The same individuals followed over years.
- New households are added, now size is about 8700 households
- Originally limited information on consumption (food, housing). Last three rounds: more comprehensive coverage.
- http://psidonline.isr.umich.edu/

U.S. Micro Data Sources: Current Population Survey (CPS)

- Conducted by the U.S. Bureau of the Census and sponsored by the Bureau of Labor Statistics
- Detailed information about household income
- Annual survey (the so-called March supplement), started in 1948, but comprehensive information only since 1970's
- Representative sample of 40,000 to 70,000 households
- No consumption or wealth information
- No panel dimension
- http://www.census.gov/cps/

Administrative Data Sets

- Survey data typically designed for scientific use.
- Big plus: often very substantial information about each survey unit (typically the household)
- Big minuses: a) Small sample size; can't slice data too finely; b) Often miss the top of the distribution; c) Measurement error
- Recent alternative: Administrative data sets
    - IRS income data (basis of Piketty and Saez' work).
    - SSA earnings data (Jay Song and various co-authors).
    - For recent use (and comparison), see Guvenen and Kaplan (2017).
    - Other countries (e.g. Norway, Sweden collect wealth data for tax purposes)

Micro Data Sources for Other Countries (Very Selected)

- Family Expenditure Survey (FES) for the U.K. Very similar in scope to the American CEX.
- Socio-Economic Panel (SOEP) for Germany. Very similar in scope to the American PSID.
- Einkommens- und Verbrauchsstichprobe for Germany. Similar to CEX, but less frequent.
- Italian Survey of Household Income and Wealth (SHIW). Biannual panel data for income, consumption and wealth.
- Eurosystem Household Finance and Consumption Survey (by ECB)
- Many other national data sets for other countries. See RED 2011, Vol. 1.

Micro Data Sources for Other Countries (Very Selected)

Country	Researchers	Paper	Data	Last update
Canada	Matthew Brzozowski, Martin Gervais, Pul Klein, Michio Suzuki	Consumption, Income, and Wealth Inequality in Canada		June 2009
Germany	Dirk Krueger, Nicola Fuchs-Schündeln, Mathias Sommer	Inequality Trends for Germany in the Last Two Decades: A Tale of Two Countries		June 2009
Italy	Tullio Jappelli, Luigi Pistaferri	Does Consumption Inequality Track Income in Italy?		May 2009
Mexico	Orazio Attanasio, Chiara Binelli	Mexico in the 1990s: the main cross sectional facts		April 2009
Russia	Yuriy Gorodnichenko, Klara Sabirianova Peter, Dmitriy Stolyarov	Inequality and Volatility Moderation in Russia: Evidence from Micro-Level Panel Data on Consumption and Income		May 2009
Spain	Josep Pijoan-Mas, Virginia Sanchez-Marcos	Spain is Different: falling trends of inequality		June 2009
Sweden	David Domeji and Martin Floden	Inequality Trends in Sweden 1978-2004		June 2009
United Kingdom	Richard Blundell and Ben Etheridge	Consumption, Income and Earnings Inequality in the UK		June 2009
United States	Jonathan Heathcote, Fabrizio Perri and Gianluca Violante	Unequal We Stand: An Empirical Analysis of Economic Inequality in the United States, 1967-2006		August 2009

Summary Statistics from Different Micro Data Sets

Table 1: Sample A, Summary Statistics, 2018

	CPS	ACS	PSID	CES	SCF
Number of households	66,929	1,215,264	8,422	14,793	5,813
Average household size	2.44	2.45	2.14	2.21	2.44
Wages/salaries below $10,000 (%)	30.5	30.4	34.2	38.9	31.8
Wages/salaries above $200,000 (%)	4.6	4.4	3.9	4.3	4.9
Average age	51.7	52.3	53.9	52.6	50.4
White (non-hispanic) (%)	79.6	77.2	79.5	82.9	66.6
College (%)	37.7	35.6	37.9	37.6	37.5
Female (%)	52.8	53.1	53.4	53.0	53.2

Source: Heathcote, Perri, Violante and Zhang [HPVZ] (2023 NBER)

Time Series of Aggregates (Averages): Labor Income

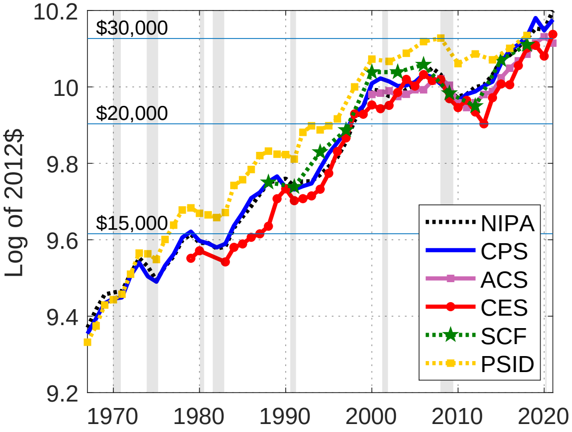


Figure 1: Wage and salary income per capita

- Labor income per capita is growing
- Micro data sources line up reasonably well with NIPA data

Time Series of Aggregates (Averages): Consumption

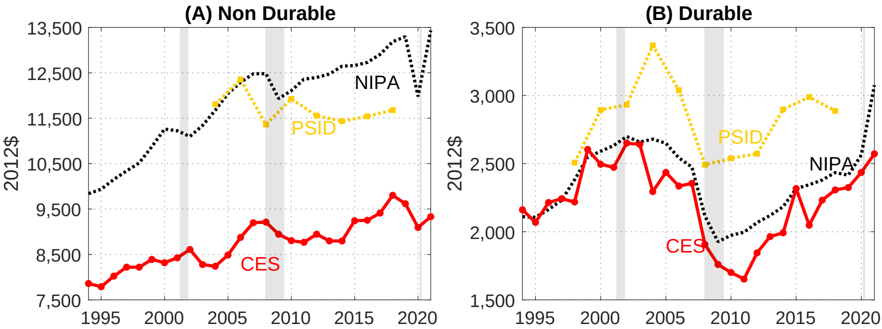
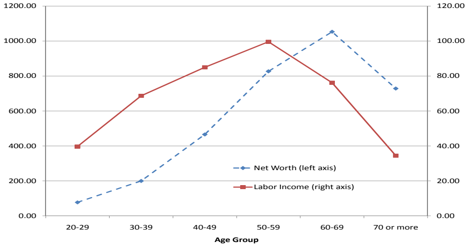


Figure 4: Consumption expenditure per capita

- Only CES and and PSID (from 2000's) record consumption
- Overlap with NIPA data worse than for income
- Expenditures on nondurables and services larger, less cyclical, growing faster than for durables. Quantities or relative prices?

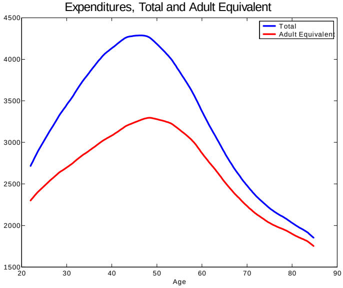
Means over Life Cycle: Labor Earnings, Net Worth

Figure: Labor Income and Net Worth by Age, SCF 2007 ($1,000)



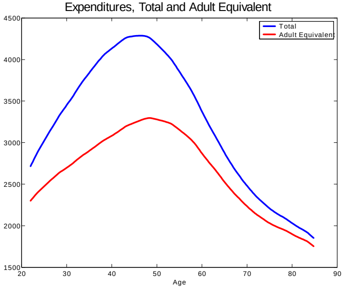
- Labor income follows a hump over life cycle, peaks at age 45 to 50.
- Average earnings at age 45 is almost 2.5 times as at age 25. At age 65 only about 60-70% of earnings at age 45.
- Net worth is accumulated over the life cycle, then de-cumulated in retirement. But not enough (lack of decumulation puzzle)
- Note: Age vs. cohort effects

Means over Life Cycle: Consumption Excess Sensitivity



- Consumption follows hump over the life cycle. Consumption tracks income over the life cycle.
- *Excess sensitivity* puzzle: consumption excessively sensitive to predicted changes in income.
- But: family size is also hump-shaped over the life cycle. Puzzle?

Consumption Hump: Explanations



- Age-varying mortality risk
- Consumption vs. Expenditures (Aguiar and Hurst, JPE 2005). Mainly explanations for expenditure drop at retirement.
- Non-separability in $c, 1-l$. Consumption is high when labor is high if c and leisure are substitutes. Heckman, AER 1974).
- Liquidity constraints when young: Zeldes QJE '89, Deaton EC '91
- Precautionary saving (Gourinchas and Parker, EC 2002).

Consumption Response to Income Shocks

- Theory: response depends on market structure and persistence of shocks
  - Complete markets: all idiosyncratic income shocks fully insured
  - Standard incomplete markets (PIH): transitory shocks (almost) fully insured. Permanent shocks not insured at all.
- Empirics: *Excess Smoothness* puzzle: consumption responds less to permanent shocks than predicted by the PIH
  - Blundell-Preston (QJE, 1998), Blundell, Pistaferri and Preston (AER, 2008), Blundell, Pistaferri and Saporta-Eksten (AER, 2016).
  - What are the additional insurance mechanisms?

Inequality over the Life Cycle

- Deaton and Paxson (JPE, 1994)
- Storesletten, Telmer and Yaron (JME, 2004)
- Heathcote, Storesletten and Violante (JEEA, 2005)

K. Storesletten et al. / Journal of Monetary Economics 51 (2004) 609–633 613

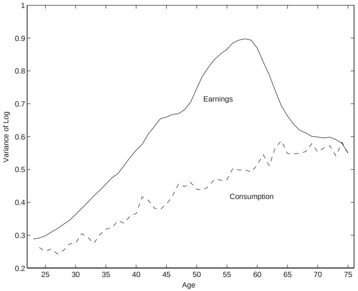
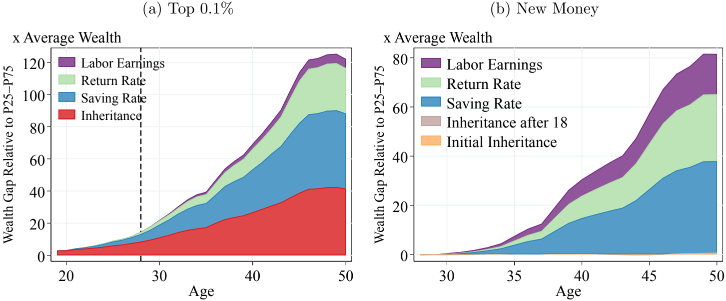


Fig. 1. The graphs represent the cross-sectional variance of the logarithm of earnings and consumption. The basic data unit is the household. Consumption data are from the CEX and are taken directly from Deaton and Paxson (1994). Earnings data are taken from the PSID. The variances are net of 'cohort effects:' dispersion which is unique to a group of households with heads born in the same year. This is accomplished, as in Deaton and Paxson (1994), via a cohort and age dummy-variable regression. The graphs are the coefficients on the age dummies, scaled so as to mimic the overall level of dispersion in the data. Further details are in Appendix A.

Wealth Inequality over the Life Cycle

- Halvorson, Hubmer, Ozkan and Salgado (WP 2024)

Figure 1 – Determinants of the Top 0.1% Wealth Accumulation



Notes: Figures 1a and 1b decompose the excess wealth accumulation of top 0.1% households aged 50—54 and of the New Money within the top 0.1% group relative to median-wealth households, respectively. The vertical axes show the wealth gap in multiples of the economy-wide average wealth (AW). In Figure 1a, for ages to the right of the vertical line, we follow the same group of households across all ages. Because of data limitations, data to the left of the vertical black line is obtained from households with similar age, wealth, and wealth growth. Additional details in Appendix OA.2.

Inequality at Point in Time

- Quadrini and Rios-Rull (with various co-authors), based on SCF
- Inequality in 1998

Measures of U.S. Earnings, Income, and Wealth

Table 1 Concentration

Variable	Gini Index	Coefficient of Variation	Top 1% to Bottom 40% Ratio
Earnings	.611	2.65	158
Income	.553	3.57	73
Wealth	.803	6.53	1,335

- Inequality in 2007

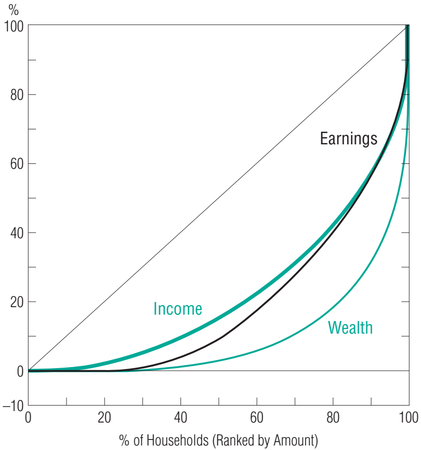
Concentration and Skewness of the Distributions

	Earnings	Income	Wealth
Coefficient of variation	3.60	4.32	6.02
Variance of the logs	1.29	0.99	4.53
Gini index	0.64	0.58	0.82
Top 1% / lowest 40%	183	88	1,526
Location of mean (%)	69	74	80
Mean / median	1.72	1.77	4.61

Inequality at Point in Time (Here 1998)

The Lorenz Curves for the U.S. Distributions
of Earnings, Income, and Wealth

What % of All Households Have
What % of All Earnings, Income, or Wealth



Source: 1998 Survey of Consumer Finances

Inequality Trends over Time

- Wages, Earnings and Income: Heathcote, Storesletten and Violante (2009), Heathcote, Perri and Violante (2011, 2023).

- Consumption: Krueger and Perri (2006), Attanasio, Battistin and Ichimura (2007), Aguiar and Bils (2013).

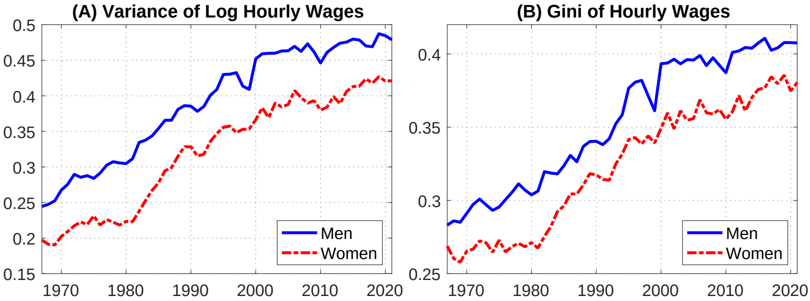
- Wealth: Favilukis (2007), Kuhn et al. (2018)

- Inequality at the very top (the Top 1%): Piketty and Saez (2013), Piketty (2014).

- Inequality and Mobility: Krueger (2011), Chetty et al. (2017).

- Very nice synthesis and update of the literature by Heathcote, Perri, Violante, Zhang (2023 NBER)

Inequality Trends over Time: Individual Wages



- Increase in wage inequality, especially 1980's and 1990's.
- Mostly driven by the top of the wage distribution (see next slide).
- Wage inequity driven by observables or residuals?

Inequality Trends over Time: Individual Wages

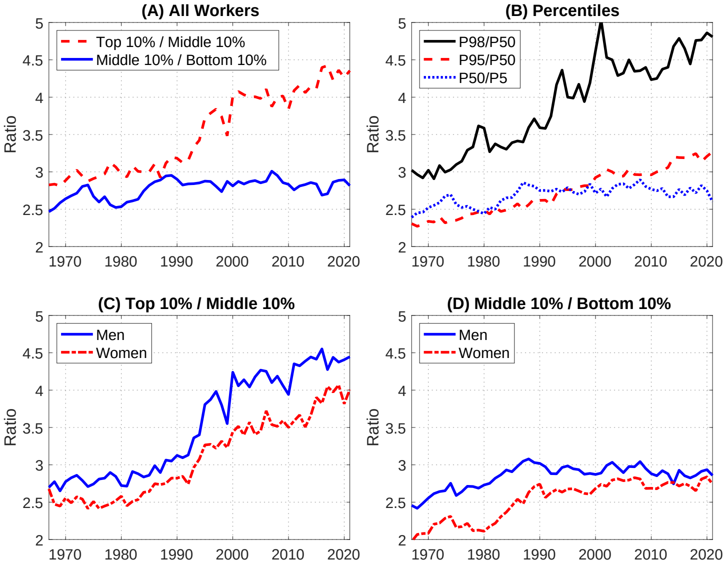
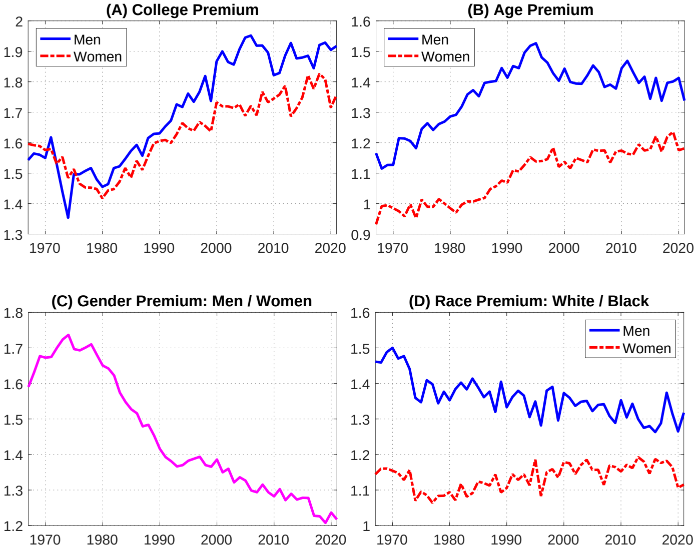
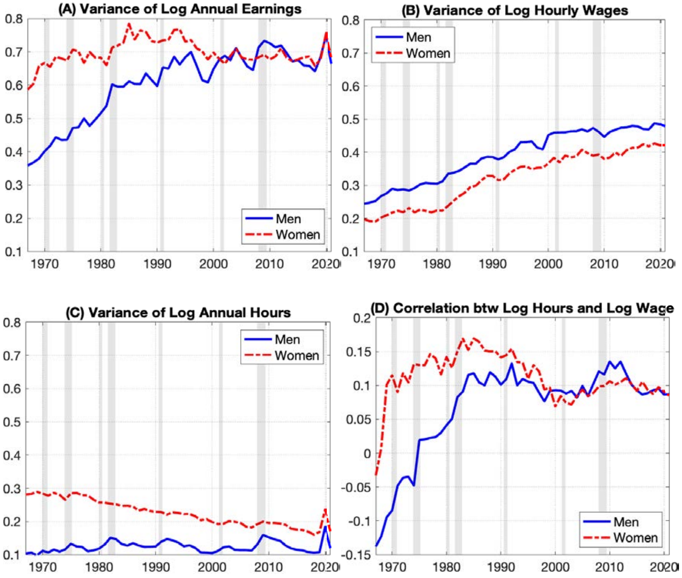


Figure 7: Wage inequality in different parts of the distribution

Inequality Trends: Wage Gaps Across Observable Characteristics



Inequality Trends: From Wages to Earnings



Inequality Trends: Wages, Earnings, Weeks Worked

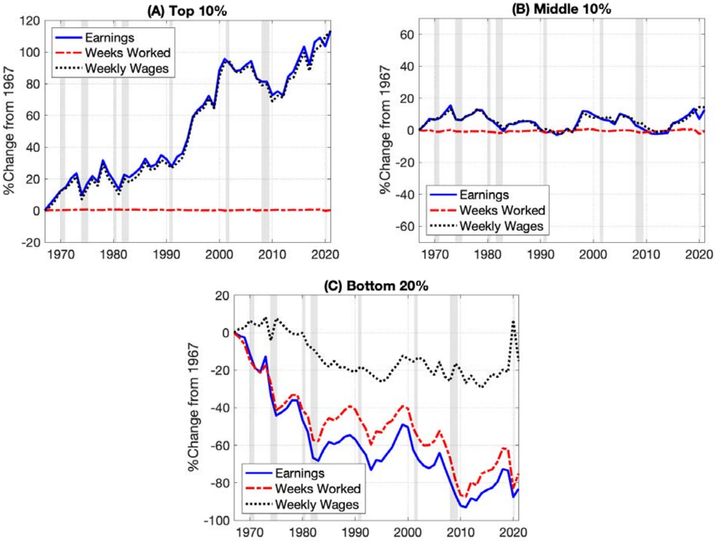


Figure 11: Earnings, wages, and weeks worked in three slices of the earnings distribution

Inequality Trends: Income Insurance in the Household

- Define average earnings in the household as $\bar{y}_{it} = \frac{1}{N_i} \sum_{i \in H_{it}} y_{it}$.
- Measure of household income pooling $HP_t = \frac{Var(y_{it}) - Var(\bar{y}_{it})}{Var(y_{it})}$.

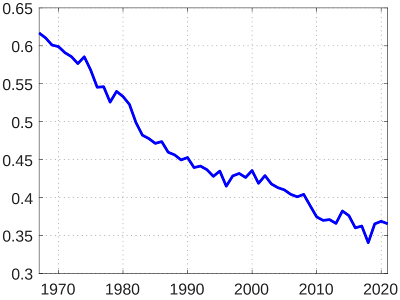


Figure 13: Household income pooling index, HP_t
CPS, Sample B

Decomposition of Income Insurance in the Household

$$\begin{aligned} Var(y_{it}) - Var(\bar{y}_{it}) &= \frac{1}{4}(Var(y_{it}^{c,m}) + Var(y_{it}^{c,f})) \\ &- \frac{1}{2}Cov(y_{it}^{c,m}, y_{it}^{c,f}) + \frac{1}{2}(Y_t^{c,m} - Y_t^{c,f})^2 \\ &- \pi_t^s(Var(y_{it}^c) - Var(\bar{y}_{it}^c)) \end{aligned}$$

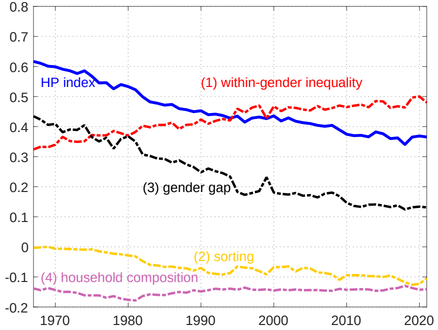


Figure 14: Household income pooling decomposition

Inequality Trends: From Earnings to Disposable Income

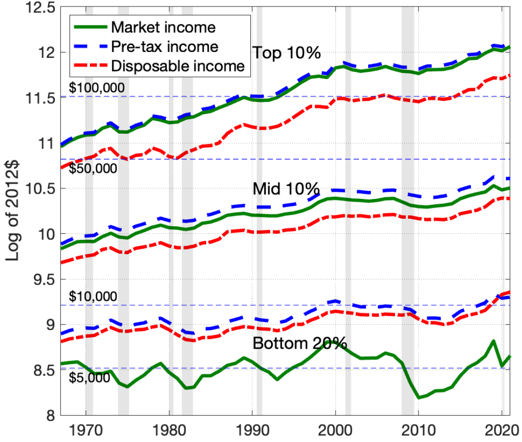


Figure 16: Different income measures across the distribution

Inequality Trends: Private vs Public Insurance

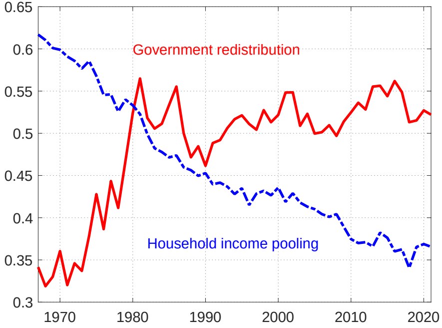


Figure 19: Inequality reduction mechanisms

- Over time, government-provided insurance has replaced household income pooling as key inequality-mitigating force.

Inequality Trends: Consumption Inequality

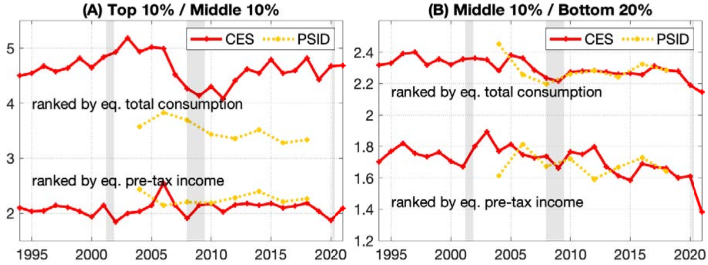


Figure 22: Consumption inequality in different data sets

- Little evidence that consumption inequality has followed wage- and market income inequality trend.

Inequality Trends: Wealth Inequality

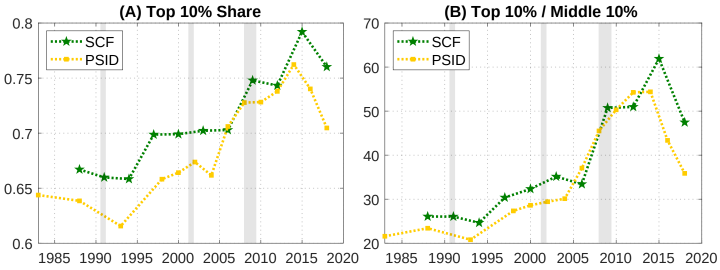


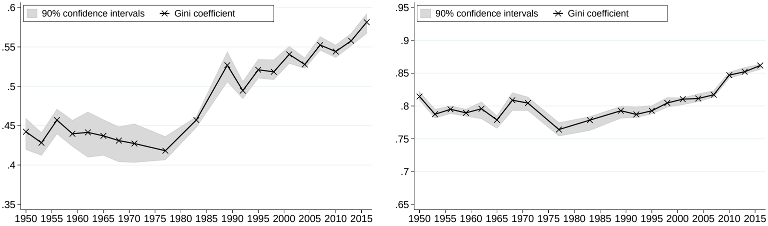
Figure 23: Wealth inequality

- Substantial increase in wealth inequality over the long run, noticeable decline between 2015 and 2018.

Inequality Trends: Summary from Heathcote et al. (2023)

- Increase in income inequality has moderated; however, inequality at the top still increasing
- Growth of college premium and gender/race equalization have stopped
- Bottom 20% of market income distribution in 2021 still at 1967 level (after GR roller coaster)
- Great recession: increase in income inequality that, over the recovery, reversed at the bottom but not at the top
- COVID: historically different, first recession when disposable income inequality declined
- Consumption expenditures inequality still flat throughout
- Wealth inequality increase around great recession, declines after

Income and Wealth Inequality in the Very Long Run (SCF) from Kuhn et al. (2018)



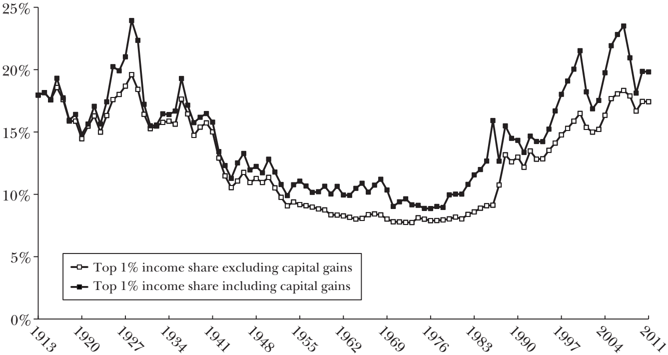
(a) Income

(b) Wealth

- Confirms that wealth inequality has increased since late 1970's, after a period of declining wealth inequality post WW II.

Inequality Trends in Very Long Run: The Top 1%

Top 1 Percent Income Share in the United States

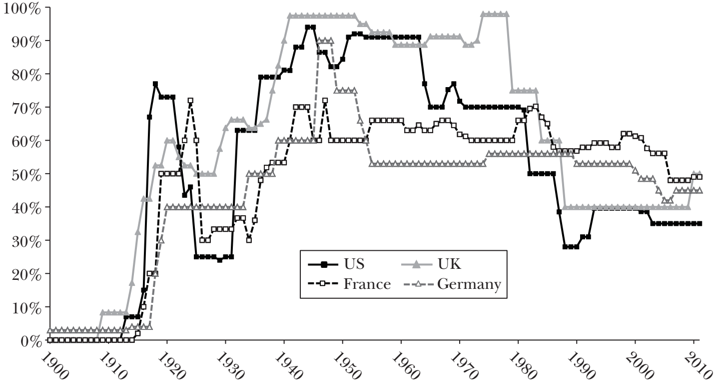


Source: Source is Piketty and Saez (2003) and the World Top Incomes Database.

- Guvenen and Kaplan (2017): Since 2000 rise in Top 1% all due to Top 0.01%. Mainly income from pass-through entities (partnerships, S-corporations).

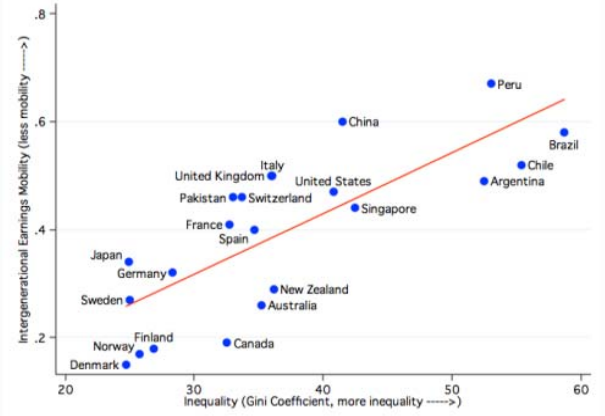
Inequality Trends over Time: Top Marginal Tax Rates

Top Marginal Income Tax Rates, 1900–2011



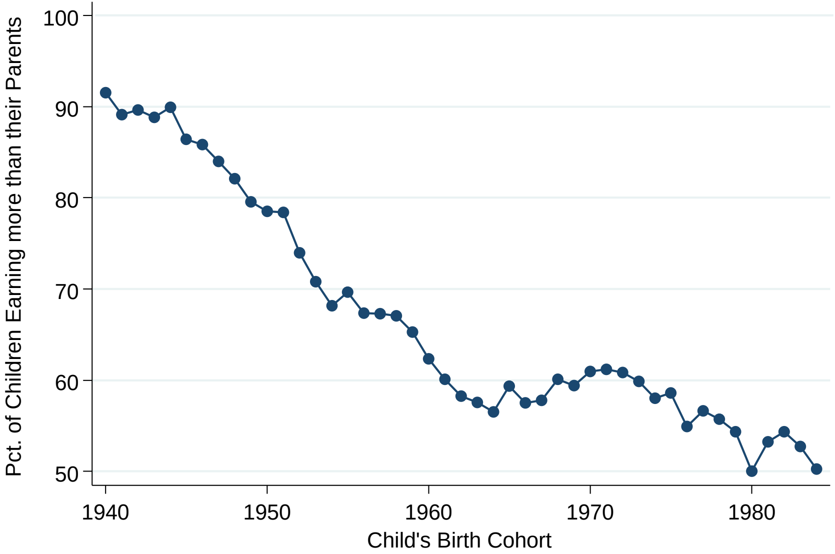
Source: Piketty and Saez (2013, figure 1).

Inequality & Intergenerational Mobility: Great Gatsby Curve

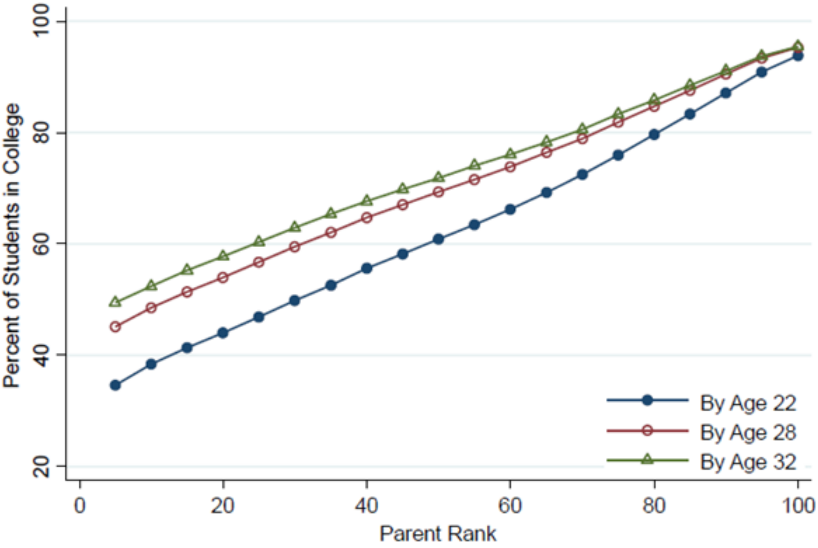


- Cross-country comparison of inequality and persistence: Guvenen, Pistaferri and Violante (NBER 2023) based on the GRID data set.

Intergenerational Mobility over Time: The American Dream is Disappearing (Chetty et al.)

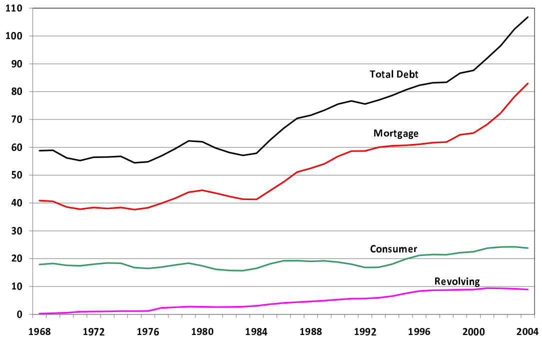


Intergenerational Persistence in Education



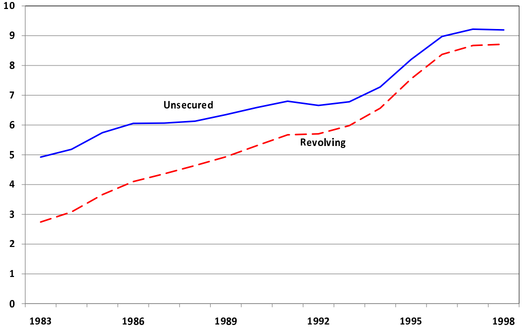
Facts/Puzzles from Household Finance

- Most household debt is secured mortgage debt.
- Household mortgage debt is substantial and increasing over time (as a fraction of household income).



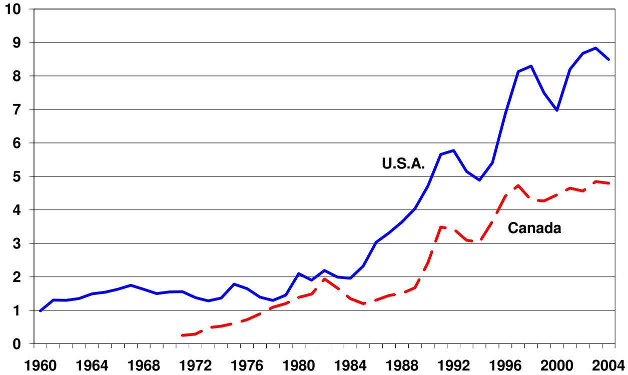
Other Interesting Facts/Puzzles: Household Finance

- Unsecured debt (credit card debt, student loans) is rising, too.



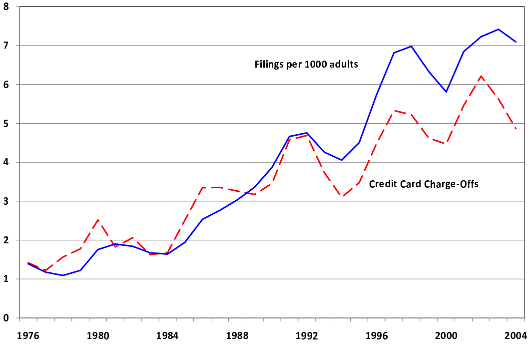
Other Interesting Facts/Puzzles: Household Finance

- Personal bankruptcy filings (chapter 7, 13) increasing over time.
- Why the increase?
- Why do not more people file (White, 1998)?
- Need models of unsecured debt and default (Chatterjee et al., 2007, Livshits et al., 2006).



Other Interesting Facts/Puzzles: Household Finance

- Increase in personal bankruptcy rates over time (extensive margin).
- More unsecured household debt is being defaulted on (intensive margin).



- After financial crisis of 2007-2009: significant deleveraging (Raveendranathan and Stefanides, 2023)

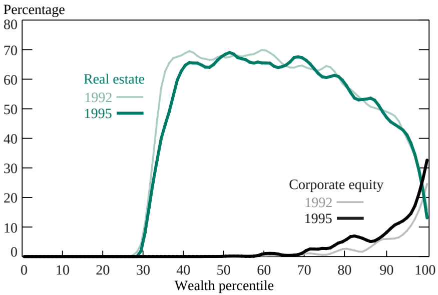
Other Interesting Facts/Puzzles: Household Finance

- Median wealth US household has portfolio concentrated in its own home, rest in low-return checking or savings accounts (not stocks).
- Strong correlation between house prices, foreclosures (defaulting on mortgage debt and have house taken away by bank).
- Gross and Souleles (2000): a significant fraction of the population has simultaneously high-interest credit card debt and liquid, low return assets.
- Fegereng, Guiso, Malacrino and Pistaferri (2018): large cross-household heterogeneity in (realized) investment returns.
- 10 million U.S. households use payday loans (at interest rates of 7000% per year): Tobacman et al. (2009, 2015).

Household Portfolios by Wealth

- Middle class: most wealth in real estate. Financial wealth only dominant at the very top (Tracy, Schneider and Chan, 1999).

Portion of Household Assets in Corporate Equity and Real Estate by Wealth Percentile, 1992 and 1995

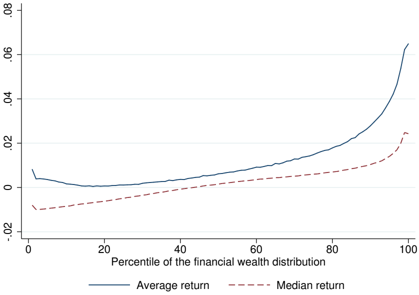


Source: Survey of Consumer Finances.

Wealth and Heterogeneity of Financial Asset Returns

- Strong positive correlation between level of financial wealth and realized returns (Fagereng et al., 2018 for Norway).

Figure 2. The correlation between financial wealth and its return



Other Interesting Facts/Puzzles: Household Finance

- Strong time series correlation between house prices, foreclosures

Figure 2: The housing boom and bust

Source: The real home price index is the US S&P/Case-Shiller Index. Foreclosure starts are from the Mortgage Bankers Association's National Delinquency Survey and are the reported number of mortgages for which foreclosure proceedings are started in a given quarter divided by the initial stock of mortgages.

Benchmark Complete Markets Model: Physical Set-Up

- N individuals, indexed by $i \in I = \{1, 2, \ldots N\}$.
- Each individual lives for T periods, where $T = \infty$ is allowed
- One nonstorable consumption good in each period.
- Individual has a stochastic endowment process $\{y_t^i\}$ of this consumption good.
- Underlying stochastic structure: aggregate state of the world $s_t \in S$, S finite-dimensional.
- Fix s_0 and let
$$s^t = (s_0, \ldots s_t) \in S^t$$
- Agents probability beliefs $\pi_t(s^t)$ are assumed to coincide with objective probabilities.
- A consumption allocation $\{(c_t^i(s^t))_{i \in I}\}_{t=0, s^t \in S^t}^T$ maps aggregate event histories s^t into consumption of agents $i \in I$ at time t.

Benchmark Complete Markets Model: Preferences and Endowments

- Preferences over consumption allocations that are assumed to permit von Neumann Morgenstern expected utility representation

$$u^i(c^i) = \sum_{t=0}^{T} \sum_{s^t \in S^t} \pi_t(s^t) U_t^i(c^i, s^t)$$

- Assume that preferences are identical across agents, additively time-separable and that agents discount the future at common subjective time discount factor $\beta \in (0, 1)$

$$u^i(c^i) = \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U^i(c_t^i(s^t), s^t)$$

- Denote by $\rho = \frac{1}{\beta} - 1$ the subjective time discount rate: $\beta = \frac{1}{1+\rho}$.

Pareto Efficient Allocations

Definition

A consumption allocation $\{c_t^i(s^t)\}$ is feasible if

$$
\begin{aligned}
c_t^i(s^t) &\geq 0 \text{ for all } i, t, s^t \\
\sum_{i=1}^{N} c_t^i(s^t) &= \sum_{i=1}^{N} y_t^i(s^t) \text{ for all } t, s^t
\end{aligned}
$$

Definition

A consumption allocation is Pareto efficient if it is feasible and there is no other feasible consumption allocation $\{\hat{c}_t^i(s^t)\}$ such that

$$
\begin{aligned}
u^i(\hat{c}^i) &\geq u^i(c^i) \text{ for all } i \in I \\
u^i(\hat{c}^i) &> u^i(c^i) \text{ for some } i \in I
\end{aligned}
$$

Market Structure

- Key question: what is the set of assets agents can trade to hedge against income risk?
- Complete markets (Standard Arrow Debreu): complete set of contingent consumption claims
- Complete Markets AD budget constraint

$$\sum_{t=0}^{T} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{T} \sum_{s^t \in S^t} p_t(s^t) y_t^i(s^t)$$

where $p_t(s^t)$ is the period 0 price of one unit of period t consumption, delivered if event history s^t has realized

Arrow Debreu Equilibrium

Definition

An Arrow Debreu competitive equilibrium consists of allocations $\{c_t^i(s^t)\}$ and prices $\{p_t(s^t)\}$ such that

1. Given $\{p_t(s^t)\}$, for each $i \in I$, $\{c_t^i(s^t)\}$ maximizes lifetime utility subject to the AD budget constraint and nonnegativity constraint on consumption
2. $\{c_t^i(s^t)\}$ satisfies

$$\sum_{i=1}^{N} c_t^i(s^t) = \sum_{i=1}^{N} y_t^i(s^t) \text{ for all } t, s^t$$

Assumption 1: Well-Behaved Utility Function

- The period utility functions U^i are twice continuously differentiable, strictly increasing and strictly concave in its first argument.

- The period utility function and satisfy the Inada conditions:

$$\begin{aligned} \lim_{c \to 0} U_c^i(c, s^t) &= \infty \\ \lim_{c \to \infty} U_c^i(c, s^t) &= 0 \end{aligned}$$

where U_c^i is the derivative of U with respect to its first argument.

Social Planner Problem and Equilibrium Allocations

Proposition

Every competitive equilibrium allocation is the solution to the social planners problem of

$$\max_{\{(c_t^i(s^t))_{i\in I}\}_{t=0,s^t\in S^t}^T} \sum_{i=1}^N \alpha^i u^i(c^i)$$

subject to the allocation being feasible, for some Pareto weights $(\alpha^i)_{i=1}^N$ *satisfying* $\alpha^i \geq 0$ *and* $\sum_{i=1}^N \alpha^i = 1$.

Proof

- First welfare theorem applies, thus every competitive equilibrium allocation is Pareto efficient
- Every Pareto efficient allocation is a solution to the social planners problem, for some Pareto weights α.

Characterizing Equilibrium (Efficient) Allocations

- First order conditions

$$\alpha^i \beta^t \pi_t(s^t) U_c^i(c_t^i(s^t), s^t) = \lambda_t(s^t)$$

- Hence for $i, j \in I$

$$\frac{U_c^i(c_t^i(s^t), s^t)}{U_c^j(c_t^j(s^t), s^t)} = \frac{\alpha^j}{\alpha^i}$$

  for all dates t and all states s^t
- Ceteris paribus, agents with higher Pareto weights have higher consumption in each state of the world

Full Consumption Insurance

Definition

A consumption allocation $\{c_t^i(s^t)\}$ is said to satisfy perfect consumption insurance if the ratio of marginal utilities of consumption between any two agents is constant across time and states of the world.

Proposition

With complete markets, every equilibrium allocation exhibits perfect consumption insurance.

Assumption 2: CRRA Utility

- All agents have identical CRRA utility, separable in consumption, i.e.
$$U^i(c, s^t) = \frac{c^{1-\sigma} - 1}{1 - \sigma} + v^i(s^t)$$
with $\sigma \geq 0$.

- For $\sigma = 1$ it is understood that utility is logarithmic.

Full Consumption Insurance with CRRA Utility

- Full consumption insurance: $i, j \in I$

$$\frac{U_c^i(c_t^i(s^t), s^t)}{U_c^j(c_t^j(s^t), s^t)} = \frac{\alpha^j}{\alpha^i}$$

- With CRRA utility:

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}}$$

- Ratio of consumption between any two agents is constant across time, states.
- Thus there exist weights $(\theta^i)_{i \in I} \geq 0$ with $\sum_{i \in I} \theta^i = 1$ s.t

$$c_t^i(s^t) = \theta^i \sum_{i \in I} y_t^i(s^t) \equiv \theta^i y_t(s^t) = \theta^i c_t(s^t)$$

Full Consumption Insurance with CRRA Utility: Implications

- Individual consumption at each date, in each state of the world is a constant fraction of aggregate income (or consumption).
- Individual consumption need *not* be constant across time and states of the world, because it still varies with aggregate income.
- Consumption among agents also need *not* be equalized.
- Since the weights $(\alpha^i)_{i \in I}$ corresponding to a competitive equilibrium are functions of the income processes $\{y_t^i(s^t)\}$, the level of consumption of agent i depends on her income process.
- But

$$\log\left(\frac{c_{t+1}^i(s^{t+1})}{c_t^i(s^t)}\right) = \log\left(\frac{c_t(s^{t+1})}{c_t(s^t)}\right)$$

  Thus individual consumption growth is perfectly correlated with and predicted by aggregate consumption growth.
- Individual income growth should not help to predict individual consumption growth once aggregate consumption growth is accounted for.

Efficient and Equilibrium Allocations: Negishi (1960)

1. Solve social planner problem for efficient allocations indexed by α

$$c_t^i(s^t) = \theta^i c_t(s^t) = \frac{\alpha_i^{\frac{1}{\sigma}}}{\sum_{j=1}^N \alpha_j^{\frac{1}{\sigma}}} c_t(s^t) = c_t^i(s^t, \alpha).$$

2. Transfers indexed by α necessary to make efficient allocation affordable (with LM's from planner problem as prices):

$$\begin{aligned} t^i(\alpha) &= \sum_t \sum_{s^t \in S^t} \lambda_t(s^t, \alpha) \left[c_t^i(s^t, \alpha) - y_t^i(s^t) \right] \text{ with} \\ \lambda_t(s^t, \alpha) &= \left[\sum_{j=1}^N \alpha_j^{\frac{1}{\sigma}} \right]^\sigma \beta^t \pi_t(s^t) c_t(s^t)^{-\sigma} \end{aligned}$$

3. Find $\hat{\alpha}$ s.t. $t^i(\alpha) = 0$. $N-1$ equations in $N-1$ unknowns. Note: Solving for equilibrium directly would involve system of infinitely many equations in infinitely many unknowns.

4. Equilibrium prices, allocations: $c_t^i(s^t, \hat{\alpha})$ and $p_t(s^t) = \lambda_t(s^t, \hat{\alpha})$.

Equilibrium Prices

- First order conditions from household problem

$$\begin{aligned}\beta^t \pi_t(s^t) U_c^i(c_t^i(s^t), s^t) &= \mu p_t(s^t) \\ \beta^{t+1} \pi_{t+1}(s^{t+1}) U_c^i(c_{t+1}^i(s^{t+1}), s^{t+1}) &= \mu p_{t+1}(s^{t+1})\end{aligned}$$

- Combining

$$\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{U_c^i(c_{t+1}^i(s^{t+1}), s^{t+1})}{U_c^i(c_t^i(s^t), s^t)}$$

- With CRRA preferences

$$\begin{aligned}\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} &= \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \left(\frac{c_{t+1}^i(s^{t+1})}{c_t^i(s^t)} \right)^{-\sigma} \\ &= \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma}\end{aligned}$$

- Hence equilibrium Arrow Debreu prices can be written as functions of aggregate consumption only.

Equilibrium Prices and the Representative Agent

Proposition

Suppose allocations $\{c_t^i(s^t)\}$ and prices $\{p_t(s^t)\}$ are an AD equilibrium. With assumption 2 the allocation $\{c_t(s^t)\}$ defined by

$$c_t(s^t) = \sum_{i \in I} c_t^i(s^t)$$

and prices $\{p_t(s^t)\}$ is an AD equilibrium for the single agent economy in which the representative agent CRRA utility and endowments

$$y_t(s^t) = \sum_{i \in I} y_t^i(s^t).$$

- Key Implication: To derive Arrow-Debreu (and all other asset) prices, with complete markets sufficient to study the RA economy
- Assumption 2 can be weakened (CARA utility works too), although the construction of the utility function of the RA becomes more involved. Koulovatianos et al. (2019).

Endogenous Labor Supply

- Household produces $w_t^i(s^t)$ units of the consumption good per unit of time worked. $w_t^i(s^t)$ is stochastic.
- As long as markets are complete, equilibrium allocations can still be determined by solving a social planner problem with appropriate social welfare weights.
- Resource constraints read as

$$\sum_{i=1}^{N} c_t^i(s^t) = \sum_{i=1}^{N} w_t^i(s^t) l_t^i(s^t)$$

- Period utility function $U(c_t^i(s^t), l_t^i(s^t))$, where $l_t^i(s^t) \in [0, 1]$.
- Key efficiency conditions

$$\begin{aligned} \frac{U_c(c_t^i(s^t) l_t^i(s^t))}{U_c(c_t^j(s^t), l_t^j(s^t))} &= \frac{\alpha^j}{\alpha^i} \\ -\frac{U_l(c_t^i(s^t) l_t^i(s^t))}{U_c(c_t^i(s^t), l_t^i(s^t))} &= w_t^i(s^t) \end{aligned}$$

Endogenous Labor Supply: Separable Utility

- Suppose

$$U(c_t^i(s^t), l_t^i(s^t)) = v(c_t^i(s^t)) - g(l_t^i(s^t))$$

where v is strictly concave and g is strictly convex.

- Implications for efficient consumption risk sharing are same as before.
- Efficient allocation of labor supply:

$$\frac{g'(l_t^i(s^t))}{g'(l_t^j(s^t))} = \frac{\alpha^j}{\alpha^i} \frac{w_t^i(s^t)}{w_t^j(s^t)}$$

- Since g is strictly convex more productive households work harder.
- While labor supply does respond to idiosyncratic productivity shocks, consumption does not (as before).

Endogenous Labor Supply: Non-Separable Utility

$$U(c_t^i(s^t), l_t^i(s^t)) = \frac{\left[c_t^i(s^t)^{\alpha}(1 - l_t^i(s^t))^{1-\alpha}\right]^{1-\sigma}}{1-\sigma}$$

- Efficiency conditions yield

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} \left(\frac{w_t^i(s^t)}{w_t^j(s^t)}\right)^{(1-\alpha)\left(1-\frac{1}{\sigma}\right)}$$

$$\frac{1 - l_t^i(s^t)}{1 - l_t^j(s^t)} = \left(\frac{\alpha^i}{\alpha^j}\right)^{\frac{1}{\sigma}} \left(\frac{w_t^i(s^t)}{w_t^j(s^t)}\right)^{(1-\alpha)\left(1-\frac{1}{\sigma}\right)-1}$$

- Since $\sigma > 0$ and $\alpha > 0$ more productive households (for fixed α's) consume less leisure and work more.
- If $\sigma > 1$ more productive households consume more: $U_{c,1-l} < 0$, and marginal utility of consumption rises with l. If $\sigma < 1$ then $U_{c,1-l} > 0$. More productive households work more, consume less.
- Regardless of σ, with $U_{c,1-l} \neq 0$, efficient risk sharing features consumption response to idiosyncratic productivity shocks.

Sequential Markets Equilibrium

- One period contingent IOU's, financial contracts bought in period t, that pay out one unit of the consumption good in $t+1$ only for a particular realization of $s_{t+1} = \eta^j$ tomorrow.
- Let $q_t(s^t, s_{t+1} = \eta^j)$ denote the price of such contract.
- Sequential Budget Constraint

$$c_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq y_t^i(s^t) + a_t^i(s^t)$$

Sequential Markets Equilibrium

Definition

A SM equilibrium is allocations $\hat{c}_t^i(s^t), \hat{a}_{t+1}^i(s^t, s_{t+1})$ and prices $\{\hat{q}_t(s^t, s_{t+1})\}$ such that

1. Given prices, allocations maximize utility subject to sequential budget constraint and

$$a_{t+1}^i(s^t, s_{t+1}) \geq -\bar{A}^i(s^{t+1}, q, c^i)$$

2.

$$\begin{aligned} \sum_{i=1}^{I} \hat{c}_t^i(s^t) &= \sum_{i=1}^{I} y_t^i(s^t) \\ \sum_{i=1}^{I} \hat{a}_{t+1}^i(s^t, s_{t+1}) &= 0 \end{aligned}$$

No Ponzi Scheme Condition

- Date 0, event s_0 present value of one unit of the consumption good at event history s^t

$$\begin{aligned} v_0(s_0) &= 1 \\ v_0(s^{t+1}) &= v_0(s_0)q_0(s^0, s_1) * \ldots * q_t(s^t, s_{t+1}) \end{aligned}$$

- No Ponzi condition

$$-v_0(s^T)a_T^i(s^{T-1}, s_T) \leq \lim_{N\to\infty} \inf S_N(s^T)$$

where $S_N(s^T)$ is the present value of future savings up to date N:

$$S_N(s^T) = \sum_{t=T}^{N} \sum_{s^t|s^T} v_0(s^t)\left(y_t^i(s^t) - c_t^i(s^t)\right)$$

- No Ponzi condition rules out asset sequences for which the present value of future debt is bounded above zero at some s^T:

$$\lim_{N\to\infty} \inf \sum_{s^N|s^T} v_0(s^N) \sum_{s_{N+1}} q_N(s^{N+1})a_{N+1}^i(s^{N+1}) \geq 0$$

Equivalence between Arrow-Debreu and Sequential Markets

Proposition

(a) If (c, a, q) *is a sequential markets equilibrium, then* (c, p) *is an Arrow-Debreu equilibrium where prices are given as*

$$p_t(s^t) = v_0(s^t)$$

(b) If (c, p) *is an Arrow-Debreu equilibrium, then there exists asset holdings a such that* (c, a, q) *is a sequential markets equilibrium, with*

$$q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}$$

Alternative Equivalent No Ponzi Condition

- Denote
$$W(s^T) = \sum_{t=T}^{\infty} \sum_{s^t|s^T} v_0(s^t) y_t^i(s^t)$$
as the Arrow-Debreu future wealth of the agent.
- No Ponzi scheme condition can also be formulated as:
$$-v_0(s^T) a_T(s^{T-1}, s_T) \leq W(s^T)$$
- The no Ponzi condition, stated this way, is sometimes called the natural debt limit.

Recursive Formulation of SM Equilibrium

- Suppose stochastic process $\{s_t\}$ is Markov, and let $\pi(s'|s)$ denote the transition probabilities of the Markov chain.
- Also suppose (which is true with CRRA preferences)

$$q_t(s^t, s_{t+1}) = q(s_{t+1}|s_t)$$

- Bellman equation

$$\begin{aligned} v(a, s) &= \max_{\{a'(s')\}_{s' \in S}} U\left(y(s) + a - \sum_{s'} a'(s') q(s'|s)\right) \\ &+\beta \sum_{s'} \pi(s'|s) v(a'(s'), s') \end{aligned}$$

Recursive Formulation of SM Equilibrium

- The first order and envelope condition are

$$q(s'|s)U'\left(y(s)+a-\sum_{s'}a'(s')q(s'|s)\right)=\beta\pi(s'|s)v_a(a'(s'),s')$$

$$v_a(a,s)=U'\left(y(s)+a-\sum_{s'}a'(s')q(s'|s)\right)$$

- Combining yields a recursive version of the Euler equation, to be solved for the policy functions $a'(a,s;s'),c(a,s)$

$$\begin{aligned}
U'\left(c(a,s)\right) &= \frac{\beta\pi(s'|s)}{q(s'|s)}U'\left(c(a'(a,s;s'),s')\right) \\
c(a,s) &= y(s)+a-\sum_{s'}a'(a,s;s')q(s'|s) \\
c(a'(a,s;s'),s') &= y(s')+a'(a,s;s') \\
&\quad -\sum_{s''}a'(a'(a,s;s'),s';s'')q(s''|s')
\end{aligned}$$

Empirical Implications for Asset Pricing

- Prices as function of $y_t(s^t) = c_t(y^t)$

$$q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma}$$

- Cum dividend time zero price of asset j with dividends $\{d_t^j(s^t)\}$

$$P_0^j = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) d_t^j(s^t)$$

- (Ex dividend) price of asset at node s^t in terms of period t consumption good

$$P_t^j(s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) d_\tau^j(s^\tau)}{p_t(s^t)}$$

- One-period gross realized real return between s^t and s^{t+1}

$$R_{t+1}^j(s^{t+1}) = \frac{P_{t+1}^j(d; s^{t+1}) + d_{t+1}^j(s^{t+1})}{P_t^j(d; s^t)}$$

Examples

- Arrow security that pays in $\hat{s}^{t+1}$
  - Price in terms of consumption at s^t

$$P_t^A(d; s^t) = \frac{p_{t+1}(\hat{s}^{t+1})}{p_t(s^t)} = q_t(s^t, \hat{s}_{t+1})$$

  - Associated gross realized return between s^t and $\hat{s}^{t+1} = (s^t, \hat{s}_{t+1})$

$$R_{t+1}^A(\hat{s}^{t+1}) = \frac{0+1}{p_{t+1}(\hat{s}^{t+1})/p_t(s^t)} = \frac{p_t(s^t)}{p_{t+1}(\hat{s}^{t+1})} = \frac{1}{q_t(s^t, \hat{s}_{t+1})}$$

  and $R_{t+1}^j(s^{t+1}) = 0$ for all $s_{t+1} \neq s_{t+1}$.
- One-period risk free bond:
  - Price

$$P_t^B(d; s^t) = \frac{\sum_{s^{t+1}} p_{t+1}(s^{t+1})}{p_t(s^t)} = \sum_{s^{t+1}} q_t(s^t, s_{t+1})$$

  - Realized return

$$R_{t+1}^B(s^{t+1}) = \frac{1}{P_t^B(d; s^t)} = \frac{1}{\sum_{s^{t+1}} q_t(s^t, s_{t+1})} = R_{t+1}^B(s^t)$$

Examples

- Stock (Lucas tree) that pays as dividend the aggregate endowment in each period

$$P_t^S(d; s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) e_\tau(s^\tau)}{p_t(s^t)}$$

- Call option: buy one share of the Lucas tree at time T (at all nodes) for a price K has a price $P_t^{call}(s^t)$ at node s^t

$$P_t^{call}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{P_T^S(d; s^T) - K, 0\right\}$$

- Put option

$$P_t^{put}(s^t) = \sum_{s^T | s^t} \frac{p_T(s^T)}{p_t(s^t)} \max\left\{K - P_T^S(d; s^T), 0\right\}.$$

Asset Returns and Stochastic Discount Factor

Definition

A stochastic discount factor (pricing kernel) is a stochastic process $\{m_{t+1}(s^{t+1})\}$ that satisfies

$$E\left(m_{t+1}(s^{t+1})R^j_{t+1}(s^{t+1})|s^t\right) = 1 \text{ for all } t, s^t$$

for all assets j in the economy.

Derivation of Unique SDF for Complete Markets Model

- Generalized household budget constraint

$$\begin{aligned} & c_t(s^t) + \sum_j P_t^j(d;s^t)a_{t+1}^j(s^t) \\ \leq \quad & y_t^j(s^t) + \sum_j a_t^j(s^{t-1})\left[d_t^j(s^t) + P_t^j(d;s^t)\right] \end{aligned}$$

- First order conditions

$$\begin{aligned} \beta^t \pi_t(s^t) c_t(s^t)^{-\sigma} &= \lambda_t(s^t) \\ \beta^{t+1}\pi_{t+1}(s^{t+1})c_{t+1}(s^{t+1})^{-\sigma} &= \lambda_{t+1}(s^{t+1}) \end{aligned}$$

$$\lambda_t(s^t)P_t^j(d;s^t) = \sum_{s_{t+1}} \lambda_{t+1}(s^{t+1}) * \left[d_{t+1}^j(s^{t+1}) + P_{t+1}^j(d;s^{t+1})\right]$$

- Substituting and re-arranging yields

$$1 = \sum_{s_{t+1}} \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})c_{t+1}(s^{t+1})^{-\sigma}}{\beta^t\pi_t(s^t)c_t(s^t)^{-\sigma}} \star \frac{d_{t+1}^j(s^{t+1}) + P_{t+1}^j(d;s^{t+1})}{P_t^j(d;s^t)}$$

Derivation of Unique SDF for Complete Markets Model

$$
\begin{aligned}
1 &= \sum_{s_{t+1}} \frac{\beta^{t+1}\pi_{t+1}(s^{t+1})c_{t+1}(s^{t+1})^{-\sigma}}{\beta^t \pi_t(s^t)c_t(s^t)^{-\sigma}} * \frac{\left[d^j_{t+1}(s^{t+1}) + P^j_{t+1}(d;s^{t+1})\right]}{P^j_t(d;s^t)} \\
&= E\left(\beta\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\sigma} R^j_{t+1}(s^{t+1})|s^t\right) \\
&= E\left(m_{t+1}(s^{t+1})R^j_{t+1}(s^{t+1})|s^t\right)
\end{aligned}
$$

- Unique stochastic discount factor is given by

$$
m_{t+1}(s^{t+1}) = \beta\left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\sigma}
$$

- For the risk-free bond

$$
\begin{aligned}
E\left(m_{t+1}(s^{t+1})R^B_{t+1}(s^t)|s^t\right) &= 1 \\
R^B_{t+1}(s^t) &= \frac{1}{E\left(m_{t+1}(s^{t+1})|s^t\right)}
\end{aligned}
$$

Using the Stochastic Discount Factor

- Let $E_t = E(.|s^t)$ denote conditional expectation. For all assets j

$$E_t(m_{t+1}R^j_{t+1}) = 1$$

- Rewrite as

$$\begin{aligned} 1 &= E_t(m_{t+1})E_t(R^j_{t+1}) + Cov_t(m_{t+1}, R^j_{t+1}) \\ E_t(R^j_{t+1}) &= \frac{1 - Cov_t(m_{t+1}, R^j_{t+1})}{E_t(m_{t+1})} \end{aligned}$$

- Using the fact that $E_t(m_{t+1}) = 1/R^B_{t+1}$ (remember R^B_{t+1} is risk-free from the perspective if time t) we obtain

$$\frac{E_t(R^j_{t+1})}{R^B_{t+1}} = 1 - Cov_t(m_{t+1}, R^j_{t+1})$$

Using the Stochastic Discount Factor

- Note that

$$\begin{aligned}\frac{E_t(R^j_{t+1})}{R^B_{t+1}} &= \frac{1+E_t(r^j_{t+1})}{1+r^B_{t+1}} = \frac{1+r^B_{t+1}+E_t(r^j_{t+1})-r^B_{t+1}}{1+r^B_{t+1}} \\ &= 1+\frac{E_t(r^j_{t+1})-r^B_{t+1}}{1+r^B_{t+1}} \approx 1+E_t(r^j_{t+1})-r^B_{t+1}\end{aligned}$$

- Combining with

$$\frac{E_t(R^j_{t+1})}{R^B_{t+1}} = 1 - Cov_t(m_{t+1}, R^j_{t+1})$$

yields

$$\begin{aligned}E_t(r^j_{t+1}) - r^B_{t+1} &\approx -Cov_t(m_{t+1}, R^j_{t+1}) \\ &= -\rho_t(m_{t+1}, R^j_{t+1}) std_t(m_{t+1}) std_t(R^j_{t+1})\end{aligned}$$

Equity Premium Puzzle

- Assume CRRA utility. Equity premium becomes

$$\begin{aligned} E_t(r^j_{t+1}) - r^B_{t+1} &= -\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R^{stock}_{t+1} \right] \\ &\quad * std_t \left(\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \right) * std_t \left(R^{stock}_{t+1} \right) \end{aligned}$$

- If $\rho_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R^{stock}_{t+1} \right] > 0$, that is, if (loosely) consumption growth and stock returns are negatively correlated no hope to generate any equity premium.
- Positive *correlation* between consumption growth and asset returns generates premium.

Equity Premium Puzzle

$$E_t(r^j_{t+1}) - r^B_{t+1} = -\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}, R^{stock}_{t+1}\right]$$
$$* std_t \left(\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right) * std_t \left(R^{stock}_{t+1}\right)$$

- Conditional on positive correlation between c_{t+1}/c_t and R^{stock}_{t+1}, the size of explained equity premium is increasing in $std\left(R^{stock}_{t+1}\right)$ and volatility of consumption growth.
- For given dispersion of c_{t+1}/c_t, high σ helps to explain a larger share of the equity premium.
- Given empirically weak, but positive correlation between c_{t+1}/c_t and R^{stock}_{t+1} and low volatility of c_{t+1}/c_t, empirically observed equity premium can only be rationalized with implausibly large σ.

Equity Premium Puzzle

- Problem with large σ?

$$R_{t+1}^{B} = \frac{1}{E_t(m_{t+1})} = \frac{1}{E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)} = \frac{1}{\beta E_t\left(\frac{1}{\left(\frac{c_{t+1}}{c_t}\right)^{\sigma}}\right)}$$

- With growth c_{t+1}/c_t tends to be positive. Making σ large makes $E_t(m_{t+1})$ small, thus R_{t+1}^{B} large.
- Why? Large σ implies small $IES = 1/\sigma$, strong desire for flat consumption. But in data c grows. Need large R_{t+1}^{B} to induce hh's to postpone consumption: Weil's (1989) risk free rate puzzle.
- Resolution 1: make β large ($\beta > 1$). As long as $\beta(1+g_c)^{1-\sigma} < 1$ (on average), $u < \infty$. But is $\beta > 1$ plausible?
- Resolution 2: Epstein-Zin utility with IES γ, risk aversion σ :

$$u(c, s^t) = \left\{ c_t(s^t)^{1-\frac{1}{\gamma}} + \beta\left[\sum_{s^{t+1}} \pi_t(s^{t+1}|s^t)u(c, s^{t+1})^{1-\sigma}\right]^{\frac{1-\frac{1}{\gamma}}{1-\sigma}} \right\}^{\frac{\gamma}{\gamma-1}}$$

Hansen-Jagannathan Bounds

- Note that $-\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R^S_{t+1} \right] \leq 1$, since correlations are bounded between -1 and 1.
- Thus

$$
\begin{aligned}
\frac{E_t(r^S_{t+1}) - r^B_{t+1}}{std_t\left(R^S_{t+1}\right)} &= -\rho_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}, R^S_{t+1} \right] \\
&\quad * std_t \left(\beta \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\sigma} \right) \\
\frac{E_t(r^S_{t+1}) - r^B_{t+1}}{std_t\left(r^S_{t+1}\right)} &\leq std_t\left(m_{t+1}\right)
\end{aligned}
$$

- Left hand side is so-called Sharpe ratio (excess return per unit of risk). Expression provides upper bound for Sharpe ratio that can be rationalized by the model (where content of model is summarized by m_{t+1}).

Empirical Tests of the Complete Consumption Insurance Hypothesis

- Altug and Miller (1990): Risk sharing tests with endogenous labor supply using U.S. data.
- Mace (1991): U.S. household data from CEX.
- Cochrane (1991): U.S. household data from PSID.
- Townsend (1994): Indian Village Data.
- Attanasio and Davis (1996): Wage and consumption data for different education group. Full insurance against movements in college wage premium?
- Hayashi, Altonji and Kotlikoff (1996): Full consumption insurance among members of extended families.
- Schulhofer-Wohl (2011): Complete risk sharing tests in presence of heterogeneity in risk tolerance, using U.S. data.
- Mazzocco and Saini (2012): Complete risk sharing tests in presence of heterogeneity in risk tolerance, using Indian village data.

Derivation of Mace's Empirical Specifications

- Under assumptions 1 and 2

$$\Delta \ln c_t^i = \Delta \ln c_t$$

that is, the growth rate of individual consumption, but be independent of individual income growth or other individual-specific shocks.

- In the regression

$$\Delta \ln c_t^i = \alpha_1 \Delta \ln c_t + \alpha_2 \Delta \ln y_t^i + \epsilon_t^i$$

under the null hypothesis of complete consumption insurance $\alpha_1 = 1$ and $\alpha_2 = 0$.

- Error term ϵ_t^i captures preference shocks as well as measurement error in consumption.

Generalization: Presence of Preference Shocks

- Assume

$$U^i(c_t^i(s^t), s^t) = e^{b^i(s^t)} \cdot \frac{c_t^i(s^t)^{1-\sigma} - 1}{1-\sigma}$$

$b^i(s^t)$ captures preference shocks (e.g. changes in family size).

- FOC in planners problem, then taking logs yields

$$\ln(c_t^i(s^t)) = \frac{1}{\sigma} b^i(s^t) + \frac{1}{\sigma}\ln(\alpha^i) - \frac{1}{\sigma}\ln\left(\frac{\lambda(s^t)}{\beta^t \pi_t(s^t)}\right)$$

- Taking averages

$$\begin{aligned} \frac{1}{N}\sum_j \ln(c_t^j(s^t)) &= \frac{1}{\sigma N}\sum_j b^j(s^t) + \frac{1}{\sigma N}\sum_j \ln(\alpha^j) \\ &\quad -\frac{1}{\sigma}\ln\left(\frac{\lambda(s^t)}{\beta^t \pi_t(s^t)}\right) \end{aligned}$$

Generalization: Presence of Preference Shocks

- Substituting back for $-\frac{1}{\sigma}\ln\left(\frac{\lambda(s^t)}{\beta^t\pi_t(s^t)}\right)$ yields

$$
\begin{aligned}
\ln(c_t^i(s^t)) &= \frac{1}{\sigma}\left(b^i(s^t) - \frac{1}{N}\sum_j b^j(s^t)\right) \\
&+ \frac{1}{\sigma}\left(\ln(\alpha^i) - \frac{1}{N}\sum_j \ln(\alpha^j)\right) + \frac{1}{N}\sum_j \ln(c_t^j(s^t)) \\
&= \frac{1}{\sigma}\left(b^i(s^t) - b^a(s^t)\right) + \frac{1}{\sigma}\left(\ln(\alpha^i) - \ln(\alpha^a)\right) + \ln(c_t^a(s^t))
\end{aligned}
$$

$$
\begin{aligned}
b^a(s^t) &= \frac{1}{N}\sum_j b^j(s^t) \\
\ln(\alpha^a) &= \frac{1}{N}\sum_j \ln(\alpha^j) \\
\ln(c_t^a(s^t)) &= \frac{1}{N}\sum_j \ln(c_t^j(s^t))
\end{aligned}
$$

Generalization: Presence of Preference Shocks

- Taking first differences to remove fixed effect $\frac{1}{\sigma}\left(\ln(\alpha^i) - \ln(\alpha^a)\right)$ yields

$$\Delta \ln(c_t^i) = \Delta \ln(c_t^a) + \frac{1}{\sigma}\left(\Delta b_t^i - \Delta b_t^a\right)$$

- Test the complete insurance hypothesis

$$\Delta \ln(c_t^i) = \alpha_1 \Delta \ln(c_t^a) + \alpha_2 \Delta \ln(y_t^i) + \epsilon_t^i$$

- Under null hypothesis $\alpha_1 = 1$ and $\alpha_2 = 0$.

With CARA Utility

- Assume

$$U^i(c_t^i(s^t), s^t) = -\frac{1}{\gamma} e^{-\gamma\left(c_t^i(s^t) - b^i(s^t)\right)}$$

- Then

$$\Delta c_t^i = \Delta c_t^m + \left(\Delta b_t^i - \Delta b_t^m\right)$$

where

$$\begin{aligned} c_t^m &= \frac{1}{N} \sum_j c_t^i \\ b_t^m &= \frac{1}{N} \sum_j b_t^i \end{aligned}$$

- Test complete insurance by running

$$\Delta c_t^i = \alpha_1 \Delta c_t^m + \alpha_2 \Delta y_t^i + \epsilon_t^i$$

- Under the null hypothesis $\alpha_1 = 1$ and $\alpha_2 = 0$.

Implementing Mace's Risk Sharing Tests: Data

- Needed: at least two income and consumption observations per household
- CEX data from 1980 to 1983
- Total number of households in sample is $10,695$.
- Each household contributes one (in first difference) observation
- Several measures of consumption
- Disposable income, defined as before-tax income minus income taxes, deductions for social security etc.

Table 1: Growth Rate Specification

$$\Delta \ln(c_t^i) = \alpha_0 + \alpha_1 \Delta \ln(c_t^a) + \alpha_2 \Delta \ln(y_t^i) + \epsilon_t^i$$

Cons.	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	F-test	R^2
Tot. Cons.	$-.04$ (.01)	1.06 (.08)	.04 (.007)	14.12*	0.021
Nondur.	$-.02$ (.01)	.97 (.07)	.04 (.006)	22.69*	0.027
Services	$-.02$ (.01)	.93 (.10)	.04 (.01)	12.44*	0.011
Food	$-.02$ (.01)	.91 (.07)	.04 (.006)	18.67*	0.020

Table 2: First Difference Specification

$$\Delta c_t^i = \alpha_0 + \alpha_1 \Delta c_t^m + \alpha_2 \Delta y_t^i + \epsilon_t^i$$

Cons.	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	F-test	R^2
Tot. Cons.	-77.87 (19.32)	1.06 (.11)	.03 (.02)	1.27	.008
Nondur.	-13.97 (3.33)	0.99 (.06)	.01 (.003)	7.71*	.023
Services	-30.47 (16.47)	1.01 (.10)	.01 (.007)	1.14	.009
Food	-7.46 (2.12)	1.01 (.08)	.005 (.002)	2.52	.020

Summary of Results

- Mixed, more towards rejecting perfect risk sharing
- Supportive results for CARA utility
- But Nelson (1994): if use correct consumption data (last three months rather than just last month) and throw out incomplete income reporters, then perfect risk sharing rejected even for CARA utility
- Measurement error in income potentially severe.

The Problem of Measurement Error

- Suppose want to estimate

$$\Delta c_t^i = \alpha_2 \Delta y_t^i + \epsilon_t^i$$

  Perfect risk sharing implies $\hat{\alpha}_2 = 0$.
- Income variable is measured with error

$$\Delta z_t^i = \Delta y_t^i + \Delta v_t^i$$

  where Δz_t^i is measured income change, Δy_t^i is true income change and Δv_t^i is measurement error.
- Assume $E\left(\Delta v_t^i\right) = 0$, $Var\left(\Delta v_t^i\right) = \sigma_v^2$, $Var\left(\Delta y_t^i\right) = \sigma_y^2$ and $\Delta v_t^i, \epsilon_t^i$ and Δy_t^i are mutually independent
- We estimate

$$\Delta c_t^i = \alpha_2 \Delta z_t^i + \epsilon_t^i$$

- OLS estimate for α_2 is

$$\hat{\alpha}_2 = \frac{N^{-1} \sum_{j=1}^N \Delta c_t^j \Delta z_t^j}{N^{-1} \sum_{j=1}^N \left(\Delta z_t^j\right)^2}$$

The Problem of Measurement Error

$$
\begin{aligned}
\text{plim}_{N\to\infty}\hat{\alpha}_2 &= \text{plim}_{N\to\infty}\frac{N^{-1}\sum_{j=1}^{N}\Delta c_t^j \Delta z_t^j}{N^{-1}\sum_{j=1}^{N}\left(\Delta z_t^j\right)^2} \\
&= \text{plim}_{N\to\infty}\frac{N^{-1}\sum_{j=1}^{N}\left(\alpha_2 \Delta y_t^i + \epsilon_t^i\right)\left(\Delta y_t^j + \Delta v_t^j\right)}{N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^j + \Delta v_t^j\right)^2} \\
&= \alpha_2 \text{plim}_{N\to\infty}\frac{N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^i\right)^2}{N^{-1}\sum_{j=1}^{N}\left(\Delta y_t^j\right)^2 + N^{-1}\sum_{j=1}^{N}\left(\Delta v_t^j\right)^2} \\
&= \alpha_2 \cdot \frac{\sigma_y^2}{\sigma_y^2 + \sigma_v^2} = \alpha_2 \cdot \frac{1}{1 + \frac{\sigma_v^2}{\sigma_y^2}}
\end{aligned}
$$

- Without measurement error $(\sigma_v^2 = 0)$ $\hat{\alpha}_2$ estimates α_2 consistently.
- The higher the ratio between measurement noise and signal, σ_v^2/σ_y^2 the more is the estimator $\hat{\alpha}_2$ asymptotically biased towards zero.
- One resolution: use measures of idiosyncratic shocks that are more precisely measured: Cochrane (1991).
- Adding classical measurement error in Δc_t^i so that measured $\Delta \tilde{c}_t^i = \Delta c_t^i + \Delta \gamma_t^i$ does not change this argument at all.

Tests of Efficient Risk Sharing with Heterogeneity in Risk Preferences: (Mazzocco and Saini, 2012, AER)

- Social planner problem

$$\max_{\{c_t^i(s^t), l_t^i(s^t)\}} \sum_{i=1}^{2} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \alpha^i \beta^t \pi_t(s^t) U^i(c_t^i(s^t))$$

$$s.t.$$

$$\begin{aligned} \sum_{i=1}^{N} c_t^i(s^t) &= \bar{Y}_t(s^t) \\ c_t^i(s^t) &\geq 0 \end{aligned}$$

- Standard efficiency condition

$$\alpha^1 U_c^1(c_t^1(s^t)) = \alpha^2 U_c^2(c_t^2(s^t))$$

- Together with resource constraint determines split of $\bar{Y}_t(s^t)$.

Tests of Efficient Risk Sharing with Heterogeneity in Risk Preferences

- Let U^1, U^2 be CRRA with $\sigma_1 < \sigma_2$.
- Plotting $\alpha^i U_c^i(c_t^i(s^t))$ against $c_t^i(s^t)$, the curves for $i = 1$ and $i = 2$ intersect once (and only once). If $\sigma_1 = \sigma_2$, they are parallel to each other (and on top of each other if $\alpha^1 = \alpha^2$).
- Plotting the efficient $c_t^i(s^t)$ against $\bar{Y}_t(s^t)$, the curves (expenditure functions) intersect once (and only once) if $\sigma_1 < \sigma_2$. Let intersection point be denoted by $\bar{Y}^*$.
- Consumption of household 1 efficiently varies more with $\bar{Y}_t(s^t)$ than consumption of household 2. Less risk averse household bears more of the consumption risk.
- Test of whether expenditure functions of households intersect forms basis of test of preference heterogeneity.

Tests of Efficient Risk Sharing with Heterogeneity in Risk Preferences

- Absent preference shocks, with identical CRRA preferences

$$\Delta \log c_{t+1}^{i} = \Delta \log c_{t+1}^{a}$$

where

$$\log c_{t+1}^{a} = \frac{1}{2} \sum_{i=1}^{2} \log c_{t+1}^{i}$$

- Now suppose $\bar{Y}_t < \bar{Y}^* < \bar{Y}_{t+1}$, then with efficient risk sharing and $\sigma_1 < \sigma_2$ we have

$$\Delta \log c_{t+1}^{2} < \Delta \log c_{t+1}^{a} < \Delta \log c_{t+1}^{1}$$

which a researcher that presupposes $\sigma_1 = \sigma_2$ has to interpret as rejection of efficient risk sharing.

- This shows potential problem of risk sharing regressions with preference heterogeneity.

Signing the Bias

- Suppose households share risk efficiently
- Suppose households have CRRA utility with $\sigma_i \neq \sigma_j$.
- Suppose that
  - if $\Delta \bar{Y}_{t+1} \geq 0$, then $\Delta \log y^i_{t+1}$ is decreasing in σ_i (in good aggregate times the least risk averse have particularly high income growth)
  - if $\Delta \bar{Y}_{t+1} < 0$, then $\Delta \log y^i_{t+1}$ is increasing in σ_i (in bad aggregate times the most risk averse have particularly little income losses)
- Then in the regression

$$\Delta \log c^i_{t+1} - \Delta \log c^a_{t+1} = \xi \Delta \log y^i_{t+1} + \varepsilon^i_{t+1}$$

we have $\hat{\xi} > 0$, despite the fact that households share risk efficiently.

Intuition

- Estimated $\hat{\xi}$

$$\hat{\xi} = \xi + \frac{Cov(\Delta \log y^i_{t+1}, \varepsilon^i_{t+1})}{Var(\Delta \log y^i_{t+1})}$$

- Since households share risk efficiently, the truth is $\xi = 0$.
- But $Cov(\Delta \log y^i_{t+1}, \varepsilon^i_{t+1}) > 0$. Why?
- In bad times $\Delta \bar{Y}_{t+1} < 0$, $\Delta \log c^i_{t+1}$ is particularly low for low σ households (thus $\varepsilon^i_{t+1} < 0$) and $\Delta \log y^i_{t+1}$ is low for these households. Reverse logic for high σ households.
- In good times reverse is true. Thus $Cov(\Delta \log y^i_{t+1}, \varepsilon^i_{t+1}) > 0$.
- Problem: unobserved σ_i is an omitted variable; leads to correlation of $\Delta \log y^i_{t+1}$ with error term ε^i_{t+1}.
- Biased estimate $\hat{\xi}$ erroneously indicates inefficient risk sharing.
- Hard to find a good IV for $\Delta \log y^i_{t+1}$ because needs to be uncorrelated with σ_i, correlated with $\Delta \log y^i_{t+1}$.
- Note: Schulhofer-Wohl (2011) argues that in the data households with low σ indeed self-select into jobs for which income growth more correlated with aggregate income growth.

Tests of Efficient Risk Sharing with Heterogeneity in Risk Preferences

- Social Planner Problem

$$\max_{\{c_t^i(s^t), l_t^i(s^t)\}} \sum_{i=1}^{N} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \alpha^i \beta^t \pi_t(s^t) U^i(c_t^i(s^t), l_t^i(s^t))$$

$$s.t.$$

$$\begin{aligned} \sum_{i=1}^{N} c_t^i(s^t) &= \sum_{i=1}^{N} w_t^i(s^t) l_t^i(s^t) + Y_t(s^t) \\ c_t^i(s^t) &\geq 0, l_t^i(s^t) \in [0,1] \end{aligned}$$

where $Y_t(s^t)$ is nonlabor endowment in economy. Note: relative to Mazzocco and Saini (2012) we suppress preference shocks, change notation between labor and leisure.

- Rewrite resource constraint as

$$\sum_{i=1}^{N} \left(c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t)) \right) = \sum_{i=1}^{N} w_t^i(s^t) + Y_t(s^t) \equiv \bar{Y}_t(s^t)$$

- Define "expenditure" of household i on consumption and leisure as

$$\rho_t^i(s^t) = c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t))$$

- Split social planner problem into three steps

- Step 3: Conditional on a given "expenditure" $\rho_t^i(s^t)$, what is the efficient allocation between consumption and leisure for agent i?

$$\begin{aligned} V^i(\rho_t^i(s^t), w_t^i(s^t)) &= \max_{c_t^i(s^t), l_t^i(s^t)} U^i(c_t^i(s^t), l_t^i(s^t)) \\ & \quad s.t. \\ \rho_t^i(s^t) &= c_t^i(s^t) + w_t^i(s^t)(1 - l_t^i(s^t)) \\ c_t^i(s^t) &\geq 0, l_t^i(s^t) \in [0,1] \end{aligned}$$

- Familiar optimality condition

$$\frac{U_l^i(c_t^i(s^t), l_t^i(s^t))}{U_c^i(c_t^i(s^t), l_t^i(s^t))} = -w_t^i(s^t)$$

together with "expenditure" constraint determines indirect utility function $V^i(\rho_t^i(s^t), w_t^i(s^t))$.

- Note: if labor supply is exogenous, this step is trivial and $V^i(\rho_t^i(s^t), w_t^i(s^t)) = U^i(c_t^i(s^t))$. Nothing new here.

- Step 2: Optimal allocation of expenditures across two arbitrary households i, j at node s^t. This is the crucial step for the tests.

$$\begin{aligned} & V^{i,j}(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) \\ = & \max_{\rho_t^i(s^t),\rho_t^j(s^t)} \alpha^i V^i(\rho_t^i(s^t), w_t^i(s^t)) \\ & +\alpha^j V^j(\rho_t^j(s^t), w_t^j(s^t)) \\ & s.t. \\ \rho_t^i(s^t) + \rho_t^j(s^t) = & \rho_t^{i,j}(s^t) \end{aligned}$$

- Solution $\rho^i(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)), \rho^j(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t))$ as a function of labor productivities $(w_t^i(s^t), w_t^j(s^t))$ and total expenditure $\rho_t^{i,j}(s^t)$ on the (i, j) pair.
- Note: if labor is exogenous, we have executed this step 2 before.

- Step 1: Optimal split of society-wide resources $\bar{Y}_t(s^t)$ between groups of households (if $N = 2$, this point is mute).
- This step allows execution of tests even if not all households in society observed.
- Assume that number of households is even (if not, need to group three households into one group).

$$\max_{\{\rho_t^{2i-1,2i}(s^t)\}} \sum_{i=1}^{N/2} V^{2i-1,2i}(\rho_t^{2i-1,2i}(s^t), w_t^{2i-1}(s^t), w_t^{2i}(s^t))$$

$$\text{s.t. } \sum_{i=1}^{N/2} \rho_t^{2i-1,2i}(s^t) = \bar{Y}_t(s^t).$$

Theorem

Take an arbitrary partition of households (say the one proposed above). A consumption-leisure allocation that solves step 1-3 solves the social planner problem.

Test of Heterogeneity in Risk Tolerance

- Subproblem 1 deals with efficient allocation within a household.
- Subproblem 3 studies the efficient allocation of resources across groups of households of size 2.
- Now suppose two households i, j share risk efficiently, i.e. solve subproblem 2.
- Suppose there are two histories $s^t, \hat{s}^\tau$ such that labor productivities are constant for each i, j (although they can vary across i, j).

$$\begin{aligned} w^i(s^t) &= w^i(\hat{s}^\tau) \\ w^j(s^t) &= w^j(\hat{s}^\tau) \end{aligned}$$

- Suppose we vary total expenditures from $\rho_t^{i,j}(s^t)$ to $\rho_t^{i,j}(\hat{s}^t)$ and the expenditure functions (ρ^i, ρ^j) cross:

$$\begin{aligned} \rho^i(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) &> \rho^j(\rho_t^{i,j}(s^t), w_t^i(s^t), w_t^j(s^t)) \\ \rho^i(\rho_t^{i,j}(\hat{s}^t), w_t^i(s^t), w_t^j(s^t)) &< \rho^j(\rho_t^{i,j}(\hat{s}^t), w_t^i(s^t), w_t^j(s^t)) \end{aligned}$$

- Then $U^i \neq U^j$. If U^i, U^j are CRRA, then $\sigma^i \neq \sigma^j$

Remarks

- One maintained assumption: efficient risk sharing. Thus tests only useful in context of risk sharing tests (or if efficient risk sharing can be ascertained independently).
  - What if tests of efficient risk sharing reject that hypothesis later on?
- Second maintained assumption: wages held constant across $s^t, \hat{s}^\tau$. Need to find a way to implement this in the data.
- Consequence of the test for heterogeneity:
  - If test not rejected, can proceed as before.
  - If homogeneity of risk preferences rejected, need to device tests that are robust to preference heterogeneity

Tests of Necessary Conditions for Efficient Risk Sharing

- Suppose the U^i are strictly concave and i, j share risk efficiently. Then
  1. The expenditure functions ρ^i, ρ^j are strictly increasing in $\rho^{i,j}$.
  2. Apart from w^i, w^j and $\rho^{i,j}$, no idiosyncratic variable should enter the expenditure functions ρ^i, ρ^j.
- Note: if period utility affected by observable heterogeneity (e.g. family size and composition) or unobservable heterogeneity, this can enter ρ^i, ρ^j as well. But not nonlabor income y^i or other idiosyncratic shocks.
- Note: if consumption and leisure are separable or wages are constant over time and across states, the above two conditions are also sufficient for efficient risk sharing.

Implementing the Tests

- For each pair i, j, of households, estimate the expenditure functions $\rho^i(\rho_t^{i,j}, w_t^i, w_t^j), \rho^j(\rho_t^{i,j}, w_t^i, w_t^j)$.
- Test 1 (Preference Heterogeneity): Holding wages fixed, compute

$$g_t^{i,j} = \rho_t^i - \rho_t^j$$

  and test whether always positive/negative. If not, evidence for preference heterogeneity.
- Test 2 (Risk Sharing Test 1): Test whether slope of ρ^i, ρ^j with respect to $\rho^{i,j}$ is positive. If not, evidence against efficient risk sharing.
- Test 3 (Risk Sharing Test 2): Test whether nonlabor income y^i of household i enters significantly in ρ^i. If yes evidence against efficient risk sharing.

Data

- INCRISAT (International Crops Research Institute for the Semi-Arid Tropics) VLS (Village Level Studies) on Indian villages.
- From 1975 6 villages, from 1981 10 villages in rural India.
- From each village 40 households (10 landless laborers, 10 small farmers, 10 medium size farmers, 10 large farmers).
- Key: weather is very important for these villages, and has lots of annual and seasonal variation. Life is risky there.
- 3 villages selected (Aurepale, Shirapur, Kanzara)
- Monthly data from 1975 - 1985. About 120 observations for about 30 households in each village.
- Observed variables: labor supply, labor income (thus labor productivity), assets (used to construct nonlabor income), price of goods, monetary and nonmonetary transactions (from which consumption is constructed), demographics of household (including caste).

Results (Selected Summary)

- Standard risk sharing tests (adding income changes to regression of consumption change) strongly reject efficient risk sharing.
- Tests of preference heterogeneity: homogeneous preferences rejected for 25% of pairs in Aurepale, 16% in Shirapur, 35% in Kanzara.
- Efficient risk sharing tests: results fairly uniform across the two tests. For few (less than 5%) of pairs efficient risk sharing rejected. But enough to still formally reject efficient risk sharing at the village level.
- Very few rejections across pairs of efficient risk sharing at the village-caste level.

Permanent Income-Life Cycle (PILCH) Models

- Budget constraint for Complete Markets Model

$$c_t(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \leq y_t(s^t) + a_t(s^t)$$

- PILCH Model

$$c_t(s^t) + q_t(s^t) a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1})$$

  where $q_t(s^t)$ is the price at date t, event history s^t, of one unit of consumption delivered in period $t+1$ regardless of what event s_{t+1} is realized.

- Only one-period risk-free IOU's only: no explicit insurance allowed, only self-insurance via asset accumulation.

Income Fluctuation Problem

$$\max_{\{c_t(s^t),a_{t+1}(s^t)\}} \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U(c_t(s^t), s^t)$$

s.t.

$$\begin{aligned} c_t(s^t) + q_t(s^t)a_{t+1}(s^t) &= y_t(s^t) + a_t(s^{t-1}) \\ c_t(s^t) &\geq 0, \text{ No Ponzi condition} \\ a_0(s_{-1}) &= a_0 \text{ given} \end{aligned}$$

A Simple 2 Period Toy Model

$$\max_{c_1,c_2,a_1} U(c_1) + \beta U(c_2)$$

s.t.

$$\begin{aligned} c_1 + qa_1 &= y_1 \\ c_2 &= y_2 + a_1 \\ c_1, c_2 &\geq 0 \end{aligned}$$

- Consolidate the budget constraints to

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

- Here $1 + r = \frac{1}{q}$ is the one-period risk-free (gross) interest rate.

Euler equation

$$U'(c_1) = \beta(1+r)U'(c_2)$$

1. Increase in r makes period 2 consumption relatively less expensive compared to period 1 consumption. This is the Substitution effect.
2. Increase in r reduces absolute price of period 2 consumption, acting like an increase in income. This is the income effect.
3. Provided that household has income in the second period, an increase in r reduces the present value of lifetime income. This is the "human capital" effect.

With CRRA Utility

$$c_1 = \left(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right)^{-1}\left(y_1 + \frac{y_2}{1+r}\right)$$

$$\begin{aligned}\frac{\partial c_1}{\partial(1+r)} &= -\frac{y_2}{(1+r)^2}\left(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right)^{-1} \\ &\quad -\left(y_1 + \frac{y_2}{1+r}\right)\left(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}\right)^{-1} * \\ &\quad \left(\beta^{\frac{1}{\sigma}}\left(\frac{1}{\sigma} - 1\right)(1+r)^{\frac{1}{\sigma}-2}\right)\end{aligned}$$

1. If $\sigma = 1$, income and substitution effect cancel out and current consumption only declines because of the human capital effect. If $y_2 = 0$, then a change in r has no effect on c_1, a_1.
2. If $0 < \sigma < 1$ (high IES), substitution effect dominates income effect and current consumption declines with an increase in interest rate
3. If $\sigma > 1$ (low IES), income effect dominates substitution effect and total effect of a change of r on c_1 is ambiguous.

Digression: Properties of CRRA Utility

- Member of the general class of HARA (Hyperbolic absolute risk aversion) utility functions

$$U(c) = \frac{1-\mu}{\mu}\left(\frac{\alpha c}{1-\mu} + \omega\right)^{\mu}$$

- Constant relative risk aversion

$$\sigma(c) = -\frac{U''(c)c}{U'(c)} = \sigma$$

Digression: Properties of CRRA Utility

Define intertemporal elasticity of substitution (IES) $ies_t(c_{t+1}, c_t)$ as inverse of the percentage change in the marginal rate of substitution between consumption at t and $t+1$ in response to a percentage change in the consumption ratio $\frac{c_{t+1}}{c_t}$]:

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{d\left(\frac{c_{t+1}}{c_t}\right)}}{\frac{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}{\frac{c_{t+1}}{c_t}}}\right]^{-1}$$

IES for the CRRA Utility Function

- For the CRRA utility function note that

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}$$

- Thus

$$ies_t(c_{t+1}, c_t) = - \left[\frac{-\sigma\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma-1}}{\frac{\beta\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}}{\frac{c_{t+1}}{c_t}}} \right]^{-1} = \frac{1}{\sigma}$$

Alternative Definition of IES

- From the first order conditions of the household problem

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = \frac{1}{1+r}$$

- Thus the IES can alternatively be written as

$$ies_t(c_{t+1}, c_t) = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\frac{\left[\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}\right]}{\left[\frac{d\left(\frac{1}{1+r}\right)}{\frac{1}{1+r}}\right]}$$

- That is, the IES measures the percentage change in the consumption growth rate in response to a percentage change in the gross real interest rate, the intertemporal price of consumption.

- For the CRRA utility function (with time-varying interest rate) the Euler equation reads as

$$(1 + r_{t+1})\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = 1.$$

Taking logs on both sides and rearranging one obtains

$$\ln(1 + r_{t+1}) + \log(\beta) = \sigma \left[\ln(c_{t+1}) - \ln(c_t) \right]$$

$$\ln(c_{t+1}) - \ln(c_t) = \frac{1}{\sigma} \ln(\beta) + \frac{1}{\sigma} \ln(1 + r_{t+1}).$$

- Use this equation to estimate $\frac{1}{\sigma}$.
- Parameter σ controls both risk aversion and desire for consumption smoothing
- Problem for asset pricing: equity premium puzzle and risk-free rate puzzle.

- Potential resolution: recursive utility (Epstein and Zin, 1989)

$$U(c_1, c_2(s)) = \left\{ [c_1]^{1-\frac{1}{\gamma}} + \beta \left[\sum_s \pi(s) [c_2(s)]^{1-\sigma} \right]^{\frac{1-\frac{1}{\gamma}}{1-\sigma}} \right\}^{\frac{1}{1-\frac{1}{\gamma}}}$$

- If $\sigma = \gamma$ then this coincides with $CRRA$ utility.
- γ measures the intertemporal elasticity of substitution
- σ measures risk aversion
- For infinite horizon and consumption allocation $c = \{c_t(s^t)\}$, define $V(c, s^t)$ as expected lifetime utility from consumption allocation c, from node s^t onwards. Then

$$V(c, s^t) = \left\{ \begin{array}{c} \left[c_t(s^t)\right]^{1-\frac{1}{\gamma}} + \\ \beta \left[\sum_{s^{t+1}} \pi(s^{t+1}|s^t) \left[V(c, s^{t+1})\right]^{1-\sigma} \right]^{\frac{1-\frac{1}{\gamma}}{1-\sigma}} \end{array} \right\}^{\frac{1}{1-\frac{1}{\gamma}}}$$

- Note: these preferences violate expected utility

Many Periods and Certainty Equivalence

Assumption 3: *The price of the one period bond* $q = \frac{1}{1+r}$ *is nonstochastic and constant over time.*
Assumption 4: *The income process of households is non-stochastic.*

- Define

$$W_0 = a_{-1} + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} < \infty$$

- Standard Euler equation

$$U_c(c_t, s^t) = \beta(1+r)U_c(c_{t+1}, s^{t+1})$$

and hence

$$U_c(c_t, s^t) = \left(\frac{1+\rho}{1+r}\right)^t U_c(c_0, s_0)$$

Special Cases: $\rho = r$.

- If U is separable between consumption and s^t then

$$c_t = c_{t+1}$$

- In periods in which the preference shock s^t makes marginal utility high, consumption also has to be high, since marginal utility is decreasing in consumption.
- Example I: Changes in family size: With larger household size given consumption expenditures yields higher marginal utility.
  - Let s_t represent number of people in household
  - Assume period utility function takes form (with $\sigma > 1$)

$$U(c_t, s^t) = \frac{\left(\frac{c_t}{s_t}\right)^{1-\sigma} - 1}{1-\sigma}$$

  - Euler equation

$$\frac{c_t}{c_0} = \left(\frac{s_t}{s_0}\right)^{1-\frac{1}{\sigma}}$$

  - With $\sigma > 1$: As family size increases $s_t > s_0$ so will optimal household consumption.

- Example 2: Consumption and leisure: If marginal utility of consumption is high when agents work a lot, then consumption is high when labor supply is high.
  - Let s_t be hours worked (exogenous for now).
  - Assume period utility function takes form (with $\sigma > 1$ and $\gamma \in (0,1)$):

$$U(c_t, s_t) = \frac{\left(c_t^{\gamma}(1-s_t)^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma}$$

  - Euler equation

$$\frac{c_t}{c_0} = \left(\frac{1-s_t}{1-s_0}\right)^{\frac{(1-\gamma)(1-\sigma)}{1-\gamma+\gamma\sigma}}$$

  - In periods where labor supply s_t is high, so should consumption be.
- If $\rho \neq r$ (and in absence of preference shocks) consumption monotonically trends upwards (if $r > \rho$) or monotonically downwards (if $\rho > r$) but is not hump-shaped. Euler equation:

$$\frac{c_t}{c_0} = \left(\frac{1+\rho}{1+r}\right)^{-\frac{t}{\sigma}}$$

Separable CRRA Utility

- Define $\gamma = \frac{1}{1+r}\left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\sigma}} < 1$. Then

$$\begin{aligned} c_t &= \left(\frac{1+r}{1+\rho}\right)^{\frac{t}{\sigma}}\left(\frac{1-\gamma}{1-\gamma^{T+1}}\right) W_0 \\ &= MPC(t,T) * W_0 \end{aligned}$$

- Increase in lifetime horizon T reduces marginal propensities to consume out of lifetime wealth $MPC(t,T)$ at all ages t
- If $\rho = r$, then $MPC(t,T)$ is constant across age, as is consumption.

$$MPC(t,T) = \frac{r}{1 + r - \left(\frac{1}{1+r}\right)^{T}}$$

- If $T = \infty$ we obtain the classical Permanent Income Hypothesis:

$$\begin{aligned} MPC(t,\infty) &= \frac{r}{1+r} \\ c_t &= \frac{rW_0}{1+r} \end{aligned}$$

Stochastic Income and Quadratic Preferences

- No Ponzi Condition

$$a_{t+1} \geq -\sup_t \sum_{\tau=t+1}^{\infty} \frac{y_\tau}{(1+r)^{\tau-t}}$$

- Attaching Lagrange multiplier $\lambda(s^t)$ to the event history s^t budget constraint and taking FOC

$$\begin{aligned} \beta^t \pi_t(s^t) U_c(c_t(s^t), s^t) &= \lambda(s^t) \\ \beta^{t+1} \pi_{t+1}(s^{t+1}) U_c(c_{t+1}(s^{t+1}), s^{t+1}) &= \lambda(s^{t+1}) \\ \lambda(s^t) q &= \sum_{s^{t+1}|s^t} \lambda(s^{t+1}) \end{aligned}$$

Incomplete Markets Euler Equation

- Combining yields

$$\begin{aligned} U_c(c_t(s^t), s^t) &= \left(\frac{1+r}{1+\rho}\right) \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t) U_c(c_{t+1}(s^{t+1}), s^{t+1}) \\ &= \left(\frac{1+r}{1+\rho}\right) E\left(U_c(c_{t+1}(s^{t+1}), s^{t+1})|s^t\right) \end{aligned}$$

or, more compactly,

$$U_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t\left(U_c(c_{t+1}, s^{t+1})\right)$$

- Complete Markets Euler equation

$$U_c(c_t(s^t), s^t) = \left(\frac{1+r}{1+\rho}\right) U_c(c_{t+1}(s^{t+1}), s^{t+1})$$

Martingale (Random Walk) Hypothesis (Hall 1978)

- **Assumption 5:** *The period utility function is given by*

$$U(c_t(s^t), s^t) = -\frac{1}{2}\left(c_t(s^t) - \bar{c}\right)^2 + v(s^t)$$

- Euler equation becomes

$$E_t c_{t+1} = \alpha_1 + \alpha_2 c_t$$

with $\alpha_1 = \bar{c}\left(1 - \frac{1+\rho}{1+r}\right)$ and $\alpha_2 = \frac{1+\rho}{1+r}$. With $\rho = r$:

$$E_t c_{t+1} = c_t$$

- What about the solution $c_t = \bar{c}$ for all t, s^t?
- Implications
  - Certainty equivalence
  - Consumption should obey the regression

$$c_{t+1} = \alpha_1 + \alpha_2 c_t + u_{t+1}$$

    where u_{t+1} is a random variable satisfying $E_t u_{t+1} = 0$.
  - Only unexpected part of income y_{t+1} affects c_{t+1}. The part of y_{t+1} already predicable in t should not affect c_{t+1}.

Empirical Test in Hall (1978)

- Aggregate Consumption and Income Data, Quarterly from 1948-1977
- Regression 1

$$c_{t+1} = \alpha_1 + \alpha_2 c_t + \alpha_3 c_{t-1} + \alpha_4 c_{t-2} + \alpha_5 c_{t-3} + u_{t+1}$$

Parameter Estimates			
$\alpha_1 = 8.2$ (8.3)	$\alpha_2 = 1.13$ (.092)	$\alpha_3 = -.04$ (.142)	$\alpha_4 = .03$ (.142)

- Regression 2

$$c_{t+1} = \beta_1 + \beta_2 c_t + \beta_3 y_t + v_{t+1}$$

Parameter Estimates			R^2
$\beta_1 = -16$ (11)	$\beta_2 = 1.024$ (.044)	$\beta_3 = -.01$ (.032)	.9988

Explicit Solution for Consumption Profile

- Consolidate budget constraints to obtain

$$E_t \sum_{s=0}^{T-t} \frac{c_{t+s}}{(1+r)^s} = E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} + a_t \equiv W_t$$

- Using the Euler equation and the law of iterated expectations we determine that$E_t c_{t+s} = c_t$ for all $s \geq 0$. Thus:

$$c_t = \begin{cases} \theta_{T-t}^{-1} \frac{rW_t}{1+r} & \text{if } T < \infty \\ \frac{r}{1+r} W_t & \text{if } T = \infty \end{cases}$$

- Here $\theta_{T-t} = \left(1 - \frac{1}{(1+r)^{T-t+1}}\right)$, where $T - t$ is the remaining lifetime. For $T - t = \infty$ we have $\theta_t = \theta = 1$.

Consumption Changes in Response to Income Changes

- Define $\Delta c_t = c_t - c_{t-1}$ as the realized change in consumption between $t-1$ and t.
- After some tedious algebra

$$\theta_{T-t}\Delta c_t = \frac{r}{1+r}\sum_{s=0}^{T-t}\frac{(E_t - E_{t-1})y_{t+s}}{(1+r)^s} = \eta_t$$

- η_t is the revision in expectations from $t-1$ to t in the present discounted value of income from period t onward.
- The specific value of η_t depends on the stochastic process for $\{y_t\}$

Examples

- Income process with transitory and permanent shocks:

$$\begin{aligned} y_t &= y_t^p + u_t \\ y_t^p &= y_{t-1}^p + v_t \end{aligned}$$

- We have that

$$\begin{aligned} y_{t+s} &= y_{t-1} + u_{t+s} - u_{t-1} + \sum_{\tau=t}^{t+s} v_\tau \\ E_t y_{t+s} &= y_{t-1} - u_{t-1} + v_t + \begin{cases} u_t \text{ if } s = 0 \\ 0 \text{ if } s > 0 \end{cases} \\ E_{t-1} y_{t+s} &= y_{t-1} - u_{t-1} + 0 + 0 \\ (E_t - E_{t-1})\, y_{t+s} &= \begin{cases} u_t + v_t & \text{if } s = 0 \\ v_t & \text{if } s > 0 \end{cases} \end{aligned}$$

- Therefore

$$\begin{aligned} \theta_{T-t} \Delta c_t = \eta_t &= \frac{r}{1+r} u_t + \theta_{T-t} v_t \\ \Delta c_t &= \frac{r \theta_{T-t}^{-1}}{1+r} u_t + v_t \end{aligned}$$

Alternative Income Processes

- Simple AR(1)

$$y_t = \delta y_{t-1} + \varepsilon_t$$

  with $0 < \delta < 1$

- We find that

$$\theta_t \Delta c_t = \frac{r}{1+r} \sum_{s=0}^{T-t} \left(\frac{\delta}{1+r} \right)^s \varepsilon_t$$

- Change in consumption in reaction to a persistent (but not permanent) shock ε_t lies in between that induced by a transitory shock and that of a permanent shock

$$\frac{r}{1+r} < \frac{r}{1+r} \sum_{s=0}^{T-t} \left(\frac{\delta}{1+r} \right)^s < 1$$

Partial Equilibrium Extensions of PILCH Models

- So far: certainty equivalence

$$\begin{aligned} c_t &= E_t c_{t+1} \\ E_t c_{t+1} &= \alpha_1 + \alpha_2 c_t \end{aligned}$$

  with $\alpha_1 = \bar{c}\left(1 - \frac{1+\rho}{1+r}\right)$ and $\alpha_2 = \frac{1+\rho}{1+r}$

- Now: relax assumption of linear marginal utility $\rightarrow$ precautionary savings motive
- Also: relax assumption that agents' borrowing constraints never binding
- Interesting implications, but models typically have to be solved numerically

Precautionary Saving

- With certainty equivalence variance of future labor income does not matter

$$
\begin{aligned}
c_t &= \frac{r}{1+r} W_t \\
W_t &= E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} + a_t
\end{aligned}
$$

- Agents are precautionary savers if an increase in risk of future labor income increases current saving (and thus reduces current consumption).
- Certainty equivalence rules out precautionary saving

A Simple Model and a General Result

- General stochastic Euler equation
$$U_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t U_c(c_{t+1}, s^{t+1})$$
- Simple two-period model
  - Assume $\rho = r = 0$. Period 0 income y_0 is nonstochastic. Period 1 income y_1 stochastic
  $$y_1 = \bar{y}_1 + \tilde{y}_1$$
  where $\bar{y}_1 = E_0 y_1$ and $\tilde{y}_1$ is a random variable with $E_0(\tilde{y}_1) = 0$
  - Human wealth, consumption in $t = 1$ and saving s out of wealth:
  $$\begin{aligned} W &= y_0 + \bar{y}_1 = E_0\left[y_0 + \frac{y_1}{1+r}\right] \\ a_1 &= y_0 - c_0 \\ c_1 &= y_1 + a_1 = W + \tilde{y}_1 - c_0 \end{aligned}$$
- Euler equation becomes
$$U_c(c_0) = E_0 U_c(W - c_0 + \tilde{y}_1)$$
- Question: how do c_0, a_1 vary with $\sigma_y^2 = Var(\tilde{y}_1)$

- Example

$$\tilde{y}_1 = \begin{cases} -\varepsilon & \text{with prob. } \frac{1}{2} \\ \varepsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

with $0 < \varepsilon < \bar{y}_1$. Thus $\sigma_y^2 = \varepsilon^2$

- Then, using IFT

$$\begin{aligned} U_{cc}(c_0)\frac{dc_0}{d\varepsilon} &= \frac{1}{2}U_{cc}\left(W - c_0 + \varepsilon\right))\left(-\frac{dc_0}{d\varepsilon} + 1\right) \\ &\quad + \frac{1}{2}U_{cc}\left(W - c_0 - \varepsilon\right))\left(-\frac{dc_0}{d\varepsilon} - 1\right) \end{aligned}$$

$$\frac{dc_0}{d\varepsilon} = \frac{\frac{1}{2}\left(U_{cc}(W - c_0 + \varepsilon) - U_{cc}(W - c_0 - \varepsilon)\right)}{U_{cc}(c_0) + \frac{1}{2}\left[U_{cc}(W - c_0 + \varepsilon) + U_{cc}W - c_0 - \varepsilon)\right]}$$

- Denominator negative, nominator is positive if and only if

$$U_{cc}\left(W - c_0 + \varepsilon\right)) - U_{cc}\left(W - c_0 - \varepsilon\right)) > 0$$

- This is true if and only if $U_{ccc}(c) > 0$. Hence $\frac{dc_0}{d\varepsilon} < 0$ and $\frac{ds}{d\varepsilon} > 0$ if and only if *marginal* utility is strictly convex.

An Application: Ricardian Equivalence

- Government gives subsidy t in period 0.
- Issues debt, repays in period 1 by proportional (nondistortionary) income tax τ in the second period.
- By law of large numbers and government budget balance

$$t = \tau \left[\bar{y}_1 + E(\tilde{y}_1) \right] = \tau \bar{y}_1.$$

- Households' new incomes and expected wealth

$$\begin{aligned} \hat{y}_0 &= y_0 + t \\ \hat{y}_1 &= (1-\tau) y_1 \\ \hat{W} &= y_0 + t + (1-\tau) E y_1 = y_0 + \bar{y}_1 = W \end{aligned}$$

- Variance of households' income in period 1 is now given by $(1-\tau)^2 \sigma_y^2 < \sigma_y^2$
- Due to previous result marginal increase in τ, t induces households to save more, consumes less in period 0 as long as $U_{ccc} > 0$.
- Ricardian equivalence fails.

Prudence

- Kimball (1990) has defined the term "prudence" to mean the "propensity to prepare and forearm oneself in the face of uncertainty" and hence the intensity of the precautionary saving motive (although the concept of prudence applies to other decisions in uncertain environments).
- Note: risk aversion is controlled by the concavity of the utility function whereas prudence is controlled by the convexity of the marginal utility function.
- Prudence characterizes preferences, precautionary saving characterizes behavior.

The General Result (Kimball 1990)

- Very clever application of Pratt's theory of risk aversion.
- Equivalent risk premium $\theta(c, \tilde{y}_1)$ for zero mean risk $\tilde{y}_1$ defined by:

$$U(c - \theta(c, \tilde{y}_1)) = EU(c + \tilde{y}_1)$$

- Compensating risk premium $\theta^*(c, \tilde{y}_1)$ defined by:

$$U(c) = EU(c + \tilde{y}_1 + \theta^*(c, \tilde{y}_1))$$

- Pratt's theorem for small risks:

$$\theta(c, \tilde{y}_1) = -\frac{1}{2}\sigma_y^2 \frac{U_{cc}(c)}{U_c(c)} + o(\sigma_y^2)$$

$$\theta^*(c, \tilde{y}_1) = -\frac{1}{2}\sigma_y^2 \frac{U_{cc}(c)}{U_c(c)} + o^*(\sigma_y^2)$$

- Justifies the coefficient of absolute risk aversion $-\frac{U_{cc}(c)}{U_c(c)}$ as quantitative measure of willingness to pay to avoid (small) risk.

Translation to Precautionary Saving

- Optimal consumption-saving choice $s, c_0 = W - s$ given by

$$U_c(c_0) = EU_c(W - c_0 + \tilde{y}_1)$$

- Without risk

$$U_c(c_0) = U_c(W - c_0)$$

- Equivalent precautionary saving premium $\psi(c_0, \tilde{y}_1)$ defined by

$$U_c(W - c_0 - \psi(c_0, \tilde{y}_1)) = EU_c(W - c_0 + \tilde{y}_1)$$

  Reduction in wealth such that optimal consumption c_0 is the same in world with risk, and without risk and reduced wealth $W - \psi(c_0, \tilde{y}_1)$. Leftward shift of consumption function $c(W, \tilde{y}_1)$.

- Similarly, compensating precautionary saving premium $\psi^*(c_0, \tilde{y}_1)$

$$U_c(W - c_0) = EU_c(W - c_0 + \tilde{y}_1 + \psi^*(c_0, \tilde{y}_1))$$

  Rightward shift of consumption function, starting from $c(W, 0)$.

The General Result (Kimball 1990)

Proposition

The equivalent and compensating precautionary saving premia are given by

$$\begin{aligned}
\psi(c_0, \tilde{y}_1) &= -\frac{1}{2}\sigma_y^2 \frac{U_{ccc}(c_0)}{U_{cc}(c_0)} + o(\sigma_y^2) \\
\psi^*(c_0, \tilde{y}_1) &= -\frac{1}{2}\sigma_y^2 \frac{U_{ccc}(c_0)}{U_{cc}(c_0)} + o^*(\sigma_y^2)
\end{aligned}$$

The residual terms $o(\sigma_y^2)$ satisfies

$$\lim_{\sigma_y^2 \to 0} \frac{o(\sigma_y^2)}{\sigma_y^2} = \lim_{\sigma_y^2 \to 0} \frac{o^*(\sigma_y^2)}{\sigma_y^2} = 0.$$

The General Result: Proof

- Risk premium

$$U(c - \theta(c, \tilde{y}_1)) = EU(c + \tilde{y}_1)$$

- Precautionary Saving Premium

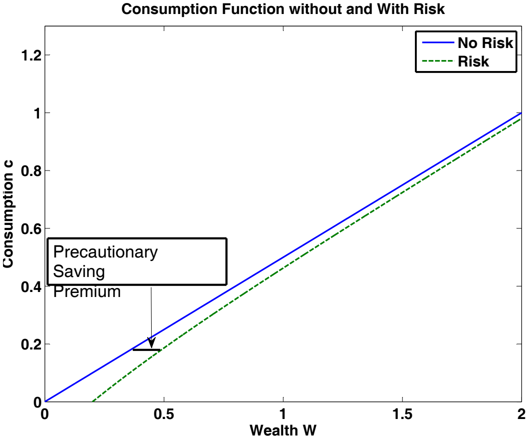
$$U_c(W - c_0 - \psi(c_0, \tilde{y}_1)) = EU_c(W - c_0 + \tilde{y}_1)$$

- Redefine choice variable $s := W - c_0 = y_0 + \bar{y}_1 = a_1 + \bar{y}_1$. Then precautionary saving premium defined by

$$U_c(s - \psi(s, \tilde{y}_1)) = EU_c(s + \tilde{y}_1)$$

- Exactly defined as the risk premium, just with one more derivative of the utility function.

Precautionary Premium



Measures of Prudence

- Result says that $\psi^*(s, \tilde{y}_1) > 0$ iff $U_{ccc} > 0$.
- *Magnitude* of shift in consumption or saving function due to risk is proportional to the index of absolute prudence:

$$p(s) = -\frac{U_{ccc}(s)}{U_{cc}(s)}$$

  which justifies $p(s)$ as a quantitative measure of prudence. Remember that $E(c_1) = s$.
- For CRRA and CARA utility

$$\begin{aligned} p(s) &= \frac{\sigma + 1}{s} \\ p(s) &= \gamma \end{aligned}$$

- In both cases risk aversion and prudence is controlled by a single parameter (σ and γ).

Remarks I: Global Result

- For two utility functions U, V, if

$$-\frac{U_{ccc}(s)}{U_{cc}(s)} > -\frac{V_{ccc}(s)}{V_{cc}(s)}$$

  for all s then

$$\psi_U^*(s, \tilde{y}_1) > \psi_V^*(s, \tilde{y}_1)$$

  for all s and all nondegenerate $\tilde{y}_1$.
- Suppose both U, V that yield same consumption and savings functions in the absence of risk (e.g. both utility functions are CRRA with $\sigma_U > \sigma_V$).
- Then result implies that for same introduction of risk $\tilde{y}_1$ households with U shift consumption function to the right uniformly more than households with V.

Remarks II: Additional Risk

- So far: introduction of risk from risk-free status quo.
- Suppose we introduce new risk in addition to already existing one. If both risks independent, results go through unchanged.
- But if not (e.g. if the new risk is a mean-preserving spread of existing risk) then the theoretical results do not apply.

Remarks III: Risk and Marginal Propensity to Consume

- Remember
$$W(c,\tilde{y}_1) = W(c,0) + \psi^*(s(c,0),\tilde{y}_1).$$
where $s(c,0) = W(c,0) - c$.
- Thus
$$\frac{\partial W(c,\tilde{y}_1)}{\partial c} - \frac{\partial W(c,0)}{\partial c} = \frac{\partial \psi^*(s(c,0),\tilde{y}_1)}{\partial s}\frac{\partial s(c,0)}{\partial c}$$
Because of time separability, $\frac{\partial s(c,0)}{\partial c} > 0$.
- Sign of the left hand side is determined by the sign of $\partial\psi^*/\partial s$, which in turn is determined by sign of $p'(s)$
- Since inverse of $\frac{\partial W(c,.)}{\partial c}$ is marginal propensity to consume out of wealth, MPC out of wealth in the first period (that is, the slope of the consumption function), at a given consumption level c is strictly declining (increasing) with income risk if absolute prudence is strictly increasing (decreasing) at c.

Remarks III: An Example

- Let $U(c) = \ln(c)$ (strictly decreasing absolute prudence) and $\tilde{Y}_1 = \pm\varepsilon$ with equal probability.
- Consumption function

$$c(W;\varepsilon) = \frac{3}{4}W - \frac{1}{4}\left(8\varepsilon^2 + W^2\right)^{\frac{1}{2}} \text{ for } \varepsilon \leq W$$

- Note: $c(W;\varepsilon)$ is strictly increasing and strictly concave, with $\lim_{W\to\infty}(c(W;0) - c(W;\varepsilon)) = 0$ and

$$c(W;\varepsilon) < c(W;0) = \frac{1}{2}W \text{ for all } W \geq \varepsilon$$

- Finally

$$\frac{1}{2} = \frac{\partial c(W;0)}{\partial W} < \frac{\partial c(W;\varepsilon)}{\partial W} = \frac{3}{4} - \frac{1}{16\left(\left(\frac{\varepsilon}{W}\right)^2 + \frac{1}{16}\right)^{\frac{1}{2}}}$$

and $\lim_{W\to\infty} \frac{\partial c(W;\varepsilon)}{\partial W} = \frac{1}{2} = \frac{\partial c(W;0)}{\partial W}$.

Linear Approximation

- General model

$$U_c(c_t, s^t) = \left(\frac{1+r}{1+\rho}\right) E_t \left(U_c(c_{t+1}, s^{t+1})\right)$$

- With separability and CRRA utility

$$\begin{aligned}\left(\frac{1+r}{1+\rho}\right) E_t \left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right] &= 1 \\ e^{\sigma \ln(c_t) + \ln(1+r) - \ln(1+\rho)} E_t \left(e^{-\sigma \ln(c_{t+1})}\right) &= 1\end{aligned}$$

- Take a first order approximation of the term $c_{t+1}^{-\sigma}$ around $c_{t+1} = c_t$.

$$\begin{aligned}c_{t+1}^{-\sigma} &\approx c_t^{-\sigma} - (c_{t+1} - c_t)\sigma c_t^{-\sigma-1} \\ &= c_t^{-\sigma}\left(1 - \sigma \frac{c_{t+1} - c_t}{c_t}\right).\end{aligned}$$

- Using this

$$\left(\frac{1+r}{1+\rho}\right) E_t \left[1 - \sigma \frac{c_{t+1} - c_t}{c_t}\right] = 1$$

Linear Approximation

- Again...

$$\left(\frac{1+r}{1+\rho}\right) E_t \left[1 - \sigma \frac{c_{t+1} - c_t}{c_t}\right] = 1$$

- Approximating $\frac{c_{t+1}-c_t}{c_t} = \Delta \ln c_{t+1}$ and $1 - \frac{1+r}{1+\rho} = \frac{r-\rho}{1+r} \approx \ln(1+r) - \ln(1+\rho)$ we obtain

$$E_t \Delta \ln c_{t+1} = \frac{1}{\sigma} \left[\ln(1+r) - \ln(1+\rho)\right]$$

- Euler equation estimation: use OLS regression

$$\Delta \ln c_{t+1} = \alpha_0 + \alpha_1 \ln(1 + r_{t+1}) + \beta X_t + \varepsilon_{t+1}$$

to estimate IES $\alpha_1 = \frac{1}{\sigma}$, and to test theory by testing $\beta = 0$.

A Parametric Example for the General Model

- Assume that $\ln(c_{t+1}) \sim N(\mu, \sigma_c^2)$:
- Note that

$$
\begin{aligned}
E_t\left(e^{-\sigma \ln(c_{t+1})}\right) &= \int_{-\infty}^{\infty} e^{-\sigma u} \frac{e^{-\frac{(u-\mu)^2}{2\sigma_c^2}}}{(2\pi)^{0.5}\sigma_c} du \\
&= e^{\frac{(\mu-\sigma\sigma_c^2)^2-\mu^2}{2\sigma_c^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{\left[u-(\mu-\sigma\sigma_c^2)\right]^2}{2\sigma_c^2}}}{(2\pi)^{0.5}\sigma_c} du \\
&= e^{\frac{(\mu-\sigma\sigma_c^2)^2-\mu^2}{2\sigma_c^2}} = e^{\frac{1}{2}\sigma^2\sigma_c^2-\mu\sigma} \\
&= e^{\frac{1}{2}\sigma^2\sigma_c^2-\sigma E_t \ln(c_{t+1})}
\end{aligned}
$$

- Then Euler equation becomes

$$
e^{-\sigma E_t \Delta \ln(c_{t+1}) + \ln(1+r) - \ln(1+\rho) + \frac{1}{2}\sigma^2\sigma_c^2} = 1
$$

- Taking logs

$$
E_t \Delta \ln(c_{t+1}) = \frac{1}{\sigma}\left[\ln(1+r) - \ln(1+\rho\right] + \frac{1}{2}\sigma\sigma_c^2
$$

A Parametric Example for the General Model

- True solution

$$E_t \Delta \ln(c_{t+1}) = \frac{1}{\sigma} \left[\ln(1+r) - \ln(1+\rho) \right] + \frac{1}{2} \sigma \sigma_c^2$$

- With linear approximation or without uncertainty (and CRRA utility) the Euler equation is

$$E_t \Delta \ln(c_{t+1}) = \frac{1}{\sigma} \left[\ln(1+r) - \ln(1+\rho) \right]$$

Observations

1. No certainty equivalence.
2. Uncertainty about future consumption tilts consumption profile upward in expectation; households postpone consumption for precautionary motives.
3. Degree to which consumption is postponed is determined by σ, controlling prudence for the CRRA utility function
4. Variables v_t that, at period t, help to predict variability of future consumption σ_c^2, help predict expected consumption growth. Consider regression

$$\ln c_{t+1} = \alpha_1 + \alpha_2 \ln c_t + \alpha_3 v_t + \varepsilon_t$$

   where v_t may be current wealth, can't interpret a significant $\hat{\alpha}_3$ as evidence against the hypothesis of the PILCH model with CRRA utility

Liquidity Constraints: Zeldes (1989)

- Constraint $a_{t+1} \geq \bar{A}$
- Consider quadratic utility: Optimal consumption rule

$$c_t = \frac{r}{1+r}\left[E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} + a_t\right]$$

- Budget constraint

$$\begin{aligned} a_{t+1} - (1+r)a_t &= (1+r)(y_t - c_t) \\ a_{t+1} - a_t &= \Delta a_{t+1} = (1+r)y_t - rE_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \\ &= y_t - rE_t \sum_{s=1}^{\infty} \frac{y_{t+s}}{(1+r)^s} = -E_t \sum_{s=1}^{\infty} \frac{\Delta y_{t+s}}{(1+r)^{s-1}} \end{aligned}$$

- If income shocks are *iid*: $y_t = \varepsilon_t$ with $E_t(\varepsilon_t) = 0$ then since $E_t \Delta y_{t+s} = E_t(\varepsilon_{t+1} - \varepsilon_t) = -\varepsilon_t$ for $s = 1$ and $E_t \Delta y_{t+s} = 0$ for $s > 1$:

$$\Delta a_{t+1} = \varepsilon_t$$

- Assets follow a random walk

The Euler Equation with Liquidity Constraints

- Assume $a_{t+1}(s^t) \geq 0$ for all s^t. Attach Lagrange multiplier $\mu(s^t)$ to the borrowing constraint $a_{t+1}(s^t) \geq 0$ at event history s^t. FOCs

$$\begin{aligned}
\beta^t \pi_t(s^t) U_c(c_t(s^t), s^t) &= \lambda(s^t) \\
\beta^{t+1} \pi_{t+1}(s^{t+1}) U_c(c_{t+1}(s^{t+1}), s^{t+1}) &= \lambda(s^{t+1}) \\
\frac{\lambda(s^t)}{1+r} - \mu(s^t) &= \sum_{s^{t+1}|s^t} \lambda(s^{t+1})
\end{aligned}$$

- Complementary slackness conditions: $a_{t+1}(s^t), \mu(s^t) \geq 0$ and $a_{t+1}(s^t)\mu(s^t) = 0$.
- Combining the first order condition yields

$$U_c(c_t(s^t)) - \frac{\mu(s^t)(1+r)}{\beta^t \pi_t(s^t)} = (1+r)\beta \sum_{s^{t+1}|s^t} \beta \pi_{t+1}(s^{t+1}|s^t) U_c(c_{t+1}(s^{t+1}), s^{t+1})$$

$$\begin{aligned}
U_c(c_t) &\geq \frac{1+r}{1+\rho} E_t U_c(c_{t+1}) \\
&= \frac{1+r}{1+\rho} E_t U_c(c_{t+1}) \text{ if } a_{t+1} > 0
\end{aligned}$$

The Euler Equation with Liquidity Constraints

- Can also write Euler equation as:

$$U_c(c_t) = \max\left\{U_c(y_t + a_t), \frac{1+r}{1+\rho}E_t U_c(c_{t+1})\right\}$$

- Why? From budget constraint

$$\begin{aligned} c_t &= y_t + a_t - \frac{a_{t+1}}{1+r} \\ &\leq y_t + a_t \end{aligned}$$

where $c_t = y_t + a_t$ is the consumption level if the liquidity constraint is binding.

- Either liquidity constraint is biding and

$$U_c(c_t) = U_c(y_t + a_t)$$

or it is slack, in which case $a_{t+1} > 0$ and $c_t < y_t + a_t$ and

$$U_c(y_t + a_t) < U_c(c_t) = \frac{1+r}{1+\rho}E_t U_c(c_{t+1})$$

Empirical Tests of Liquidity Constraints (Zeldes 1989)

- Divide sample of households into two groups, depending on current wealth positions. One group composed of households whose liquidity constraint is likely not binding (high wealth households), other group of households whose constraint is most likely binding (low wealth households).
- Empirical test consists of testing whether current income helps forecast consumption growth for both groups separately in a Hall (1978) regression.
- Evidence for the presence of liquidity constraints: violations of over-identifying restriction for low-wealth group, but not for the high-wealth group.
- Remember: with prudent preferences current income may help explain consumption growth (because it helps predict future income risk).

Liquidity Constraints with Quadratic Preferences: Precautionary Saving without Prudence

- With quadratic preferences Euler equation becomes

$$c_t = \min\{y_t + a_t, \frac{1+r}{1+\rho} E_t c_{t+1} + \frac{\rho - r}{1+\rho} \bar{c}\}$$

- For simplicity assume $\rho = r$, so that

$$\begin{aligned} c_t &= \min\{y_t + a_t, E_t c_{t+1}\} \\ &= \min\{y_t + a_t, E_t \min\{y_{t+1} + a_{t+1}, E_{t+1} c_{t+2}\}\} \end{aligned}$$

Implications

1. If the certainty equivalence solution has asset holdings satisfying $a_t \geq 0$ with probability 1, it is the optimal consumption allocation even in the presence of borrowing constraints.
2. Even if the liquidity constraint is not binding in period t, future potential borrowing constraints affect current consumption choices. If there is any $t + s$ and contingency in which the borrowing constraint is binding, then consumption c_t today is reduced below $E_t c_{t+s}$.
3. Thus lack of binding borrowing constraints today does not mean they are not there and do not affect current consumption choices.
4. Suppose variance of future income increases. More low realizations of y_{t+1} possible. If the set of y_{t+1} for which borrowing constraint binds, increases, $E_t c_{t+1} = E_t \min\{y_{t+1} + a_{t+1}, E_t c_{t+2}\}$ declines and so does c_t. Saving increases in reaction to rise in risk, because agents, afraid of future contingencies of low consumption and aware of their inability to smooth low income shocks via borrowing, increase their precautionary savings.

Implications

- When observing increases in saving as reaction to increased income uncertainty, this may have:

  1. Preference-based interpretation (agents are prudent: $U_{ccc} > 0$)
  2. Institution-based interpretation (credit markets prevent or limit uncontingent borrowing).

Combining Prudence and Liquidity Constraints - Theoretical Results: Case $T = \infty$ and $\rho < r$

- Main results

$$\begin{aligned} \lim_{t\to\infty} c_t &= \infty \text{ almost surely} \\ \lim_{t\to\infty} a_t &= \infty \text{ almost surely} \end{aligned}$$

- Why?

$$\begin{aligned} U_c(c_t) &\geq \frac{1+r}{1+\rho} E_t U_c(c_{t+1}) \\ &> E_t U_c(c_{t+1}) \end{aligned}$$

- Marginal utility (strictly positive) follows supermartingale. By martingale convergence theorem sequence of random variables $\{U_c(c_t)\}$ converges almost surely to limit random variable $U_c(c)$.
- Since

$$E_0 U_c(c_{t+1}) \leq \left(\frac{1+\rho}{1+r}\right)^{t+1} U_c(c_0)$$

and U_c is strictly positive $\{U_c(c_t)\}$ converges a.s. to $U_c(c) = 0$.

Combining Prudence and Liquidity Constraints - Theoretical Results: Case $T = \infty$ and $\rho < r$

- $\{U_c(c_t)\}$ converges a.s. to $U_c(c) = 0$.
- Hence

$$
\begin{aligned}
\lim_{t\to\infty} c_t &= \infty \text{ almost surely} \\
\lim_{t\to\infty} a_t &= \infty \text{ almost surely}
\end{aligned}
$$

- Implications
  - State space not bounded for assets
  - No stationary general equilibrium can exist with $\rho < r$

Digression: Martingale Convergence Theorem

- Probability space $(F, \mathcal{F}, \pi)$
- F is an arbitrary set, for our purposes the set of infinite histories $s^\infty = (s_0, s_1, \ldots, s_t, \ldots)$
- $\mathcal{F}$ is a σ-algebra on F, i.e. a set of subsets of F satisfying
    - $F \in \mathcal{F}$
    - $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$
    - $A_1, A_2, \ldots \in \mathcal{F}$ implies $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$
- π is a probability measure on F, i.e. a real-valued function $\pi : \mathcal{F} \to \mathbf{R}$ satisfying
    - $\pi(\emptyset) = 0$ and $\pi(F) = 1$
    - $\pi(A) \geq 0$ for all $A \in \mathcal{F}$
    - If $\{A_i\}_{i=1}^\infty$ is a countable sequence of disjoint sets $A_i \in \mathcal{F}$, then

$$\pi\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \pi(A_i)$$

Digression: Martingale Convergence Theorem

- A function $X : F \to \mathbf{R}$ is called measurable with respect to a sigma algebra $\mathcal{F}$ if for all $a \in \mathbf{R}$

$$\{s \in F : X(s) \leq a\} \in \mathcal{F}$$

  A measurable function on a probability space is called a random variable

- Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be a sequence of σ-algebras satisfying $\mathcal{F}_i \subset \mathcal{F}$. If

$$\mathcal{F}_t \subset \mathcal{F}_{t+1}$$

  for all t, then $\{\mathcal{F}_t\}_{t=0}^{\infty}$ is called a filtration.

- Let $X_0, X_1, \ldots$ be a sequence of random variables such that X_t is measurable with respect to $\mathcal{F}_t$. Then the sequence of $\{X_t\}_{t=0}^{\infty}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and the sequence $\{\mathcal{F}_t, X_t\}_{t=0}^{\infty}$ is called a stochastic process.

Digression: Martingale Convergence Theorem

- A stochastic process $\{\mathcal{F}_t, X_t\}_{t=0}^{\infty}$ is a martingale if for all t we have $E(X_t) < \infty$ and
$$E(X_{t+1}|\mathcal{F}_t) = X_t$$
with probability 1
- It is a submartingale if instead
$$E(X_{t+1}|\mathcal{F}_t) \geq X_t$$
with probability 1 and a supermartingale if
$$E(X_{t+1}|\mathcal{F}_t) \leq X_t$$
with probability 1.
- Martingale Convergence Theorem: Let $\{\mathcal{F}_t, X_t\}_{t=0}^{\infty}$ be a supermartingale satisfying
$$K = \sup_t E(|X_t|) < \infty$$
Then X_t converges, with probability one, to a random variable X that satisfies $E(|X|) \leq K$.

Interpretation of all this

- $\mathcal{F}_t$ can be understood to contain the "information" an agent has at time t.
- Let $\{\mathcal{F}_t\}_{t=0}^{\infty}$ be constructed as follows: if $s \in F$ and $\tilde{s} \in F$ satisfy

$$s^t = (s_0, \ldots s_t) = \tilde{s}^t$$

  and $s \in A \in \mathcal{F}_t$, then $\tilde{s} \in A$.
- $\pi_t(s^t)$ is meant to be defined as

$$\pi_t(s^t) = \int 1_{A(s^t)} d\pi$$

  where $A(s^t) = \{\tilde{s} \in F : \tilde{s}^t = s^t\}$. Note that π_t is a probability measure on the measurable space $(S^t, \mathcal{F}_t)$.
- Consumption functions $\{c_t\}_{t=0}^{\infty}$ as a stochastic process. Since c_t is required to be measurable with respect to $\mathcal{F}_t$, for all events $\tilde{s} \in F$ with $\tilde{s}^t = s^t$, the agent consumes the same, $c_t(s^t)$
- Can start with sets $\left\{S^t\right\}_{t=0}^{\infty}$, filtrations $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and probability measures $\pi_t(s^t)$, which then induce $(F, \mathcal{F}, \pi)$.

Case $T = \infty$ and $\rho = r$

Proposition

Suppose $\{y_t\}$ is a deterministic sequence. Define

$$x_t = \frac{r}{1+r} \sum_{\tau=t}^{\infty} \frac{y_\tau}{(1+r)^{\tau-t}}$$

Then

$$\bar{c} = \lim_{t \to \infty} c_t = \sup_t x_t = \bar{x}$$

- Interpretation: Suppose an agent arrives at date t with $a_t = 0$. If borrowing constraint is never binding again he should consume

$$c_t = x_t$$

as a standard permanent income hypothesis consumer would do. Proposition says that the effects of potentially binding borrowing constraints never vanish until the period with the highest annuity value of future labor income, x_t, is reached.

Case $T = \infty$ *and* $\rho = r$

Proposition

Suppose U *is bounded,* $\{y_t\}$ *is stochastic and satisfies the following condition: there exists* $\varepsilon > 0$ *such that for all* $\alpha \in \mathbf{R}$ *and all* s^t

$$prob\left\{\alpha \leq \sum_{\tau=t}^{\infty} \frac{y(s^\tau)}{(1+r)^{\tau-t}} \leq \alpha + \varepsilon \,\middle|\, s^t \right\} < 1 - \varepsilon$$

Then

$$\begin{aligned} \lim_{t \to \infty} c_t &= \infty \ a.s. \\ \lim_{t \to \infty} a_t &= \infty \ a.s. \end{aligned}$$

Case $T = \infty$ *and* $\rho = r$

- Interpretation of the condition: Income process must be "sufficiently" stochastic so that present discounted value of future income leaves set $[\alpha, \alpha + \varepsilon]$ with probability of at least ε.

- Finite-state Markov chain with $\Pi(y) > 0$ for all y and $\pi(y'|y) > 0$ for all y, y' satisfies this condition.

- Implications
    - State space not bounded for assets
    - No stationary general equilibrium can exist for $\rho = r$ and income process sufficiently stochastic

Case $T = \infty$ *and* $\rho > r$

- Recursive formulation

$$
\begin{aligned}
& v(a,y) \\
= \quad & \max_{a',c \geq 0} \left\{ u(c) + \frac{1}{1+\rho} \sum_{y'} \pi(y'|y) v(a',y') \right\} \\
& s.t.
\end{aligned}
$$

$$
c + \frac{a'}{1+r} = y + a
$$

- Euler Equation

$$
U_c(c) = \max \left\{ U_c(y+a), \frac{1+r}{1+\rho} \sum_{y'} \pi(y'|y) U_c(c') \right\}
$$

Income IID

- a and y enter as sum in the DP
- Define"cash at hand" $x = a + y$
- Rewrite the functional equation as

$$v(x) = \max_{c,a' \geq 0} \left\{ u(c) + \frac{1}{1+\rho} \sum_{y'} \pi(y') v(x') \right\}$$

$$\begin{aligned} \text{s.t. } c + \frac{a'}{1+r} &= x \\ x' &= a' + y' \end{aligned}$$

with Euler equation

$$U_c(x - \frac{a'(x)}{1+r}) = \max \left\{ U_c(c(x)), \frac{1+r}{1+\rho} \sum_{y'} \pi(y') U_c(c'(x')) \right\}$$

Income IID

Proposition

(Schechtman and Escudero) Let $T = \infty$ *and* $\Pi(y) = 1$ *(no income risk). Then there exists an* $\tilde{x}$ *such that if* $x \geq \tilde{x}$, *then*

$$x' = a'(x) + y < x$$

Also $a'(y) = 0$ *and* $c(y) = y$. *Consumption is strictly increasing in cash at hand, or*

$$\frac{dc(x)}{dx} > 0$$

There exists an $\bar{x} > y$ *such that* $a'(x) = 0$ *for all* $x \leq \bar{x}$ *and* $a'(x) > 0$ *for all* $x > \bar{x}$. *Finally* $\frac{dc(x)}{dx} \leq 1$ *and* $\frac{da'(x)}{dx} < 1 + r$.

Stochastic Income

Proposition

(Schechtman and Escudero, Deaton): Consumption is strictly increasing in cash at hand, i.e.

$$\frac{dc(x)}{dx} \in (0, 1].$$

Optimal asset holdings are either constant at the borrowing limit or strictly increasing in cash at hand, i.e.

$$a'(x) = 0 \quad or \quad \frac{da'(x)}{dx} \in (0, 1 + r)$$

There exists $\bar{x} > y_1$ *such that for all* $x \leq \bar{x}$ *we have* $c(x) = x$ *and* $a'(x) = 0$.

Upper Bound of State Space

Proposition

(Schechtman and Escudero): Suppose the period utility function is of constant absolute risk aversion form $u(c) = -e^{-c}$*, then for the infinite life income fluctuation problem, if* $\Pi(y = 0) > 0$ *we have*

$$x_t \to +\infty$$

almost surely.

Suppose that the marginal utility function has the property that there exist finite $e_{u'}$ *such that*

$$\lim_{c \to \infty} \left(\log_c u'(c) \right) = e_{u'}$$

Then there exists a $\tilde{x}$ *such that*

$$x' = a'(x) + y_N \leq x$$

for all $x \geq \tilde{x}$*.*

Upper Bound of State Space

- If the utility function is of CRRA form with risk aversion parameter σ, then since

$$\log_c c^{-\sigma} = -\sigma \log_c c = -\sigma$$

  we have $e_{u'} = -\sigma$ and hence for these utility function the previous proposition applies.

- For CARA utility function

$$\begin{aligned} \log_c e^{-c} &= -c \log_c e = -\frac{c}{\ln(c)} \\ -\lim_{c\to\infty} \frac{c}{\ln(c)} &= -\infty \end{aligned}$$

  and hence the proposition does not apply.

Serially Correlated Income

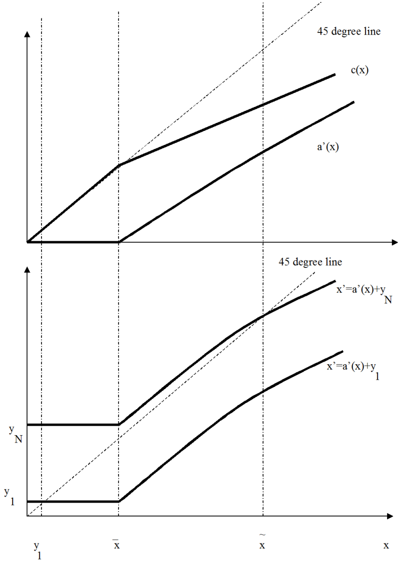
- Need current income in the state space separately because it contains information about future income. Recursive formulation

$$v(a,y) = \max_{c,a' \geq 0} \left\{ u(c) + \frac{1}{1+\rho} \sum_{y' \in Y} \pi(y'|y) v(a',y') \right\}$$

$$\text{s.t. } c + \frac{a'}{1+r} = y + a$$

Proposition

(Huggett 1993): $c(a,y)$ is strictly increasing in a and $a'(a,y)$ is constant at the borrowing limit or strictly increasing. For $N = 2$, assumptions on the Markov transition function and CRRA utility there exists an $\tilde{a}$ such that $a'(a,y) \leq a$ for all $a \geq \tilde{a}$



45 degree line
c(x)
a'(x)
45 degree line
x'=a'(x)+y
N
x'=a'(x)+y
1
y
N
y
1
y
1
x
~
x
x

Numerical Solutions of Models with Precautionary Saving

- Assume that stochastic labor income process is finite state, stationary Markov process with domain $Y = \{y_1, \ldots, y_N\}$ and transition probabilities $\pi(y'|y)$. Assume $y_1 \geq 0$ and $y_{i+1} > y_i$.
- Define borrowing constraint $\bar{A}$ as

$$\bar{A} = \frac{1+r}{r} y_1$$

or

$$\bar{A}_{t+1} = y_1 \sum_{s=0}^{T-t-1} (1+r)^{-s}$$

- Idea: if borrow to the maximum and roll over debt, after interest payments can guarantee non-negative consumption even with worst income realization (natural debt limit, Aiyagari (1994)).
- Alternatively, $\bar{A}_{t+1} = 0$.

Numerical Solutions of Precautionary Savings Models

- Recursive formulation: state variables (a, y)
- Bellman equation:

$$v_t(a,y) = \max \left\{ U\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v_{t+1}(a', y') \right\}$$

$$\text{s.t. } -\bar{A} \leq a' \leq (1+r)(a+y)$$

- First order condition

$$\frac{1}{1+r} U'\left(y + a - \frac{a'}{1+r}\right) = \beta \sum_{y'} \pi(y'|y) v'_{t+1}(a', y')$$

- Envelope condition

$$v'_t(a,y) = U'\left(y + a - \frac{a'}{1+r}\right)$$

- Combining yields

$$U'\left(y + a - \frac{a'}{1+r}\right) = \beta(1+r) \sum_{y'} \pi(y'|y) U'\left(y' + a' - \frac{a''}{1+r}\right)$$

Income Shocks IID

- Shocks have density $\pi(y)$
- Can reduce state space to cash at hand $x = a + y$
- Budget constraint

$$\begin{aligned} c + \frac{a'}{1+r} &= x \\ x' &= a' + y' \end{aligned}$$

- Bellman equation

$$v_t(x) = \max_{a'} \left\{ U\left(x - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y') v_{t+1}(a' + y') \right\}$$

$$\text{s.t.} \;\; -\bar{A} \leq a' \leq (1+r)(a+y)$$

2 Methods for 2 Models

- Finite vs. Infinite Time Horizon
- Value Function (use Bellman equation) vs. Policy Function Iteration (use Euler equation)

1. Finite Horizon, Value Function Iteration

- Want: sequences of value functions $\{v_t(a,y)\}_{t=0}^{T}$ and associated policy functions $\{c_t(a,y), a_t'(a,y)\}_{t=0}^{T}$
- Agent dies at period T, can normalize $v_{T+1}(a,y) \equiv 0$.
- Iterate backwards on

$$v_t(a,y) =$$

$$\max \left\{ U\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v_{t+1}(a',y') \right\}$$

$$\text{s.t. } -\bar{A} \leq a' \leq (1+r)(a+y)$$

2. Finite Horizon, Policy Function Iteration

- Want: sequences of policy functions $\{c_t(a,y), a'_t(a,y)\}$.
- Given that the agent dies at period $T+1$ we know

$$\begin{aligned} c_T(a,y) &= a+y \\ a'_T(a,y) &= 0 \end{aligned}$$

- The Euler equation between period t and $t+1$

$$U'\left(y+a-\frac{a'_t(a,y)}{1+r}\right)=\beta(1+r)*$$

$$\sum_{y'}\pi(y'|y)U'\left(y'+a'_t(a,y)-\frac{a'_{t+1}(a_t(a,y),y')}{1+r}\right)$$

where $a'_{t+1}(.,.)$ is known from the previous step.

3. Infinite Horizon, Value Function Iteration

- Want: time invariant value function $v(a, y)$ and associated policy functions $a'(a, y)$ and $c(a, y)$.
- Need to find a fixed point to Bellman's equation, since there is no final period to start from.
- Initial guess $v^0(a, y)$
- Iterate on the functional equation

$$v^n(a, y) = \max_{a'} \left\{ U\left(y + a - \frac{a'}{1+r}\right) + \beta \sum_{y'} \pi(y'|y) v^{n-1}(a', y') \right\}$$

$$\text{s.t. } -\bar{A} \leq a' \leq (1+r)(a+y)$$

until convergence, i.e. until

$$\left\| v^n - v^{n-1} \right\| \leq \varepsilon$$

- Under appropriate assumptions operator defined by Bellman's equation is a contraction mapping and hence convergence of the iterative procedure to a unique fixed point is guaranteed

4. Infinite Horizon, Policy Function Iteration

- Want: time invariant policy functions $a'(a,y)$ and $c(a,y)$
- No final time period, so we guess an initial policy $a'^0(a,y)$ or $c^0(a,y)$ and then iterate on

$$U'\left(y + a - \frac{a'^n(a,y)}{1+r}\right) = \beta(1+r) *$$

$$\sum_{y'} \pi(y'|y) U'\left(y' + a'^n(a,y) - \frac{a'^{n-1}(a'^n(a,y), y')}{1+r}\right)$$

until

$$\left\| a'^n - a'^{n-1} \right\| \leq \varepsilon_a$$

- Deaton and Laroque (1992) show that under the assumption $\beta(1+r) < 1$ the operator defined by the Euler equation is a contraction mapping, so that convergence to a unique fixed point is guaranteed

Nonstationary Income: Deaton (1991), Carroll (1992)

- Problem: state space unbounded
- Define $z_{t+1} = \frac{y_{t+1}}{y_t}$ to be the stochastic growth rate of economy, assumed to be *iid* (log-income is a random walk). Assume CRRA utility
- Trick: normalize variables

$$\begin{aligned} \theta_t &= \frac{c_t}{y_t} \\ w_t &= \frac{x_t}{y_t} = \frac{a_t + y_t}{y_t} \end{aligned}$$

- Budget constraint

$$\begin{aligned} c_t + \frac{a_{t+1}}{1+r} &= y_t + a_t \\ c_t + \frac{x_{t+1} - y_{t+1}}{1+r} &= x_t \\ \theta_t + \frac{w_{t+1} - 1}{1+r} z_{t+1} &= w_t \\ w_{t+1} &= \frac{(1+r)(w_t - \theta_t)}{z_{t+1}} + 1 \end{aligned}$$

Nonstationary Income

- Stochastic Euler equation

$$\left(\frac{1+r}{1+\rho}\right) E_t\left[c_{t+1}^{-\sigma}\right] = c_t^{-\sigma}$$

- Dividing both sides by $y_t^{-\sigma}$ yields

$$\theta_t^{-\sigma} = \left(\frac{1+r}{1+\rho}\right) E_t\left[\left(\frac{c_{t+1}}{y_t}\right)^{-\sigma}\right] = \left(\frac{1+r}{1+\rho}\right) E_t\left[(\theta_{t+1} z_{t+1})^{-\sigma}\right]$$

- Assume that $z' = z_{t+1}$ can take on only a finite number of values.
- State variable w.
- Want: policy functions $\theta(w)$ and $w'(w)$

$$\theta(w)^{-\sigma} = \left(\frac{1+r}{1+\rho}\right) \sum_{z'} \pi(z') \left(\theta(w')z'\right)^{-\sigma} =$$

$$\left(\frac{1+r}{1+\rho}\right) \sum_{z'} \pi(z') \left(\theta\left(\frac{(1+r)(w-\theta(w))}{z'} + 1\right) z'\right)^{-\sigma}$$

Nonstationary Income

- Work on the Euler equation

$$\theta(w)^{-\sigma} = \left(\frac{1+r}{1+\rho}\right)\sum_{z'}\pi(z')\left(\theta(w')z'\right)^{-\sigma} =$$

$$\left(\frac{1+r}{1+\rho}\right)\sum_{z'}\pi(z')\left(\theta\left(\frac{(1+r)(w-\theta(w))}{z'}+1\right)z'\right)^{-\sigma}$$

- Again policy function iteration: guess $\theta^0(w)$ and then iterate on

$$\theta^n(w)^{-\sigma} = \left(\frac{1+r}{1+\rho}\right) *$$

$$\sum_{z'}\pi(z')\left(\theta^{n-1}\left(\frac{(1+r)(w-\theta^n(w))}{z'}+1\right)z'\right)^{-\sigma}$$

Other Remarks

- For all approaches either need maximization or finding roots to nonlinear equations for given point on state space. Approximation either on discrete grid or entire state space. Feel free to choose
- Simulation: all one needs are the policy functions and some random numbers to generate a sequence of labor income $\{y_i\}_{i=0}^{M}$. Then start from a_0 and find time series of consumption, asset holdings, by

$$\begin{aligned} c_0 &= c(a_0, y_0) \\ a_1 &= a'(a_0, y_0) \end{aligned}$$

and recursively

$$\begin{aligned} c_i &= c(a_i, y_i) \\ a_{i+1} &= a'(a_i, y_i) \end{aligned}$$

Consumption Response to Income Shocks

- Absent binding borrowing constraints, consumption at t should not respond to expected income changes in t. Or anything else in information set at t
- Huge empirical literature tests this (excess sensitivity tests). See Parker (1999) or Souleles (1999) on consumption response to tax rebates, to hitting the social security cap.
- How about income shocks?
  - Theory: depends on persistence of income shocks
  - Data? Blundell, Pistaferri and Preston (2008): How much consumption insurance
  - Extension to wage shocks by Blundell, Pistaferri and Saporta-Eksten (2016)

Consumption Insurance Coefficients

- Suppose income follows the process

$$\begin{aligned} \log(y_{it}) &= p_{it} + \varepsilon_{it} \\ p_{it} &= p_{it-1} + \eta_{it} \end{aligned}$$

- The consumption insurance coefficients for transitory and permanent shocks are defined as

$$\begin{aligned} \phi_t^{\varepsilon} &= 1 - \frac{Cov_i(\Delta \log(c_{it}), \varepsilon_{it})}{Var_i(\varepsilon_{it})} \\ \phi_t^{\eta} &= 1 - \frac{Cov_i(\Delta \log(c_{it}), \eta_{it})}{Var_i(\eta_{it})} \end{aligned}$$

- Note: if $\Delta \log(c_{it}) = \eta_{it}$ then consumption responds 1:1 to permanent income shocks. No insurance against η shocks:

$$\phi_t^{\eta} = 1 - \frac{Cov_i(\Delta \log(c_{it}), \eta_{it})}{Var_i(\eta_{it})} = 1 - \frac{Cov_i(\eta_{it}, \eta_{it})}{Var_i(\eta_{it})} = 0$$

- Similarly, if $Cov_i(\Delta \log(c_{it}), \varepsilon_{it}) = 0$, consumption does not respond to transitory shocks and $\phi_t^{\varepsilon} = 1$.

How Much Consumption Insurance Is There?

- PSID data: Blundell et al. (2008)
- Precautionary savings model: Kaplan and Violante (2010)
- Note: from model-simulated data can calculate insurance coefficients directly (since we know the shocks). In data only observe income, but not permanent and transitory shocks. BPP (2008) provide a method to estimate the coefficients.

Table 1—Results from the Benchmark Models with NBC and ZBC

	Permanent shock			Transitory shock		
	Data BPP	Model BPP	Model TRUE	Data BPP	Model BPP	Model TRUE
Natural BC	0.36 (0.09)	0.22	0.23	0.95 (0.04)	0.94	0.94
Zero BC	0.36 (0.09)	0.07	0.23	0.95 (0.04)	0.82	0.82

Stochastic Earnings or Wage Processes

- Key quantitative ingredient into the model
- Key papers: Liliard and Weiss (1979), MaCurdy (1982), Abowd and Card (1989), Baker (1997), Haider (2001), Meghir and Pistaferri (2004), Guvenen (2007) and many others.
- Labor earnings or income e_{iht} of individual or household i with actual or potential labor market experience (or age) h at time t is assumed to follow process of the form

$$\log(e_{iht}) = g(\theta_t, X_{it}) + f(\gamma_i, X_{it}) + \log(y_{it})$$

Three Components of Earnings Process

1. Deterministic determinant $g(\theta_t, X_{it})$ of earnings:

$$g(\theta_t, X_{it}) = \theta_{0t} + \theta_{1t} h_{it} + \theta_{2t} h_{it}^2 + \theta_{3t} h_{it}^3.$$

   Parameter vector θ allowed to vary with t, but not with i.

2. Household-specific deterministic life cycle earnings determinant $f(\gamma_i, X_{it})$

$$f(\gamma_i, X_{it}) = \alpha_i + \beta_i h_{it}$$

   (α_i, β_i) are household-specific parameters, drawn from normal population distribution with zero mean, variances $\sigma_\alpha^2, \sigma_\beta^2$, covariance $\sigma_{\alpha\beta}$.

   - No disagreement that should allow $\sigma_\alpha^2 > 0$ (initial and permanent differences in the *levels* of earnings).
   - Key question I: allow for heterogeneity in slope of earnings profile, $\sigma_\beta^2 > 0$, or not? Key paper: MaCurdy (1982).

3. Stochastic part of the earnings process $\log(y_{it})$:

$$\begin{aligned} \log(y_{it}) &= z_{it} + \phi_t \varepsilon_{it} \\ z_{it} &= \rho z_{it-1} + \pi_t \eta_{it} \end{aligned}$$

   $\varepsilon_{it}, \eta_{it}$ iid, uncorrelated with (α_i, β_i). Key question II: How big is ρ?

Estimation

- Parameters to be estimated: θ_t's, the variances and the covariance of (α_i, β_i) and the (ϕ_t^2, π_t^2) as well as ρ.
- Stage 1: estimate θ_t's and obtain residuals $\log(\hat{e}_{iht})$
- Stage 2: estimate variances and the covariance of (α_i, β_i), (ϕ_t^2, π_t^2), ρ by minimum distance (by choice of parameters minimize weighted distance between cross-sectional moments in the data and model).
- Moments used:

$$E(\log(e_{iht})^2)$$
$$E(\log(e_{iht})\log(e_{ih+n,t+n}))$$

- For data, $E(.)$ denotes cross-sectional sample averages.
- For model, can compute moments explicitly:

$$E(\log(e_{iht})^2) = \sigma_\alpha^2 + 2\sigma_{\alpha\beta}h + \sigma_\beta^2 h^2 + \phi_t^2 + E(z_{iht}^2)$$

$$E(\log(e_{iht})\log(e_{ih+n,t+n})) = \sigma_\alpha^2 + 2\sigma_{\alpha\beta}(2h+n) + \sigma_\beta^2 h(h+n) + \rho^n E(z_{iht}^2)$$

where $E(z_{iht}^2)$ in turn is a function of ρ and the π_t^2.

Results

Sample	ρ	σ_α^2	σ_β^2	$corr_{\alpha\beta}$
All	0.988 (.024)	0.058 (.011)	–	–
Coll.	0.979 (.055)	0.031 (.021)	–	–
HS	0.972 (.023)	0.053 (.015)	–	–
All	0.821 (.030	0.022 (.074)	0.00038 (.00008)	−0.23 (.43)
Coll	0.805 (.061)	0.023 (.112)	0.00049 (.00014)	−0.70 (1.22)
HS	0.829 (.029)	0.038 (.081)	0.00020 (.00009)	−0.25 (.59)

RIP vs. HIP

- Why did MaCurdy (and many others) advocate RIP?
- Suppose $\phi_t = 0$, $\pi_t = \pi$ so that stochastic part of the earnings process becomes

$$\begin{aligned} \log(\hat{e}_{iht}) &= \alpha_i + \beta_i h_{it} + z_{it} \\ z_{it} &= \rho z_{it-1} + \pi \eta_{it}. \end{aligned}$$

- Earnings growth rates

$$\Delta \log(\hat{e}_{iht}) = \log(\hat{e}_{iht}) - \log(\hat{e}_{ih-1t-1}) = \beta_i + \Delta z_{it}$$

- Note that

$$z_{it} = \pi \sum_{\tau=0}^{\infty} \rho^{t-\tau} \eta_{i\tau}$$

- Hence

$$\Delta z_{it} = z_{it} - z_{it-1} = \pi \left[(\rho - 1) \sum_{\tau=0}^{\infty} \rho^{t-1-\tau} \eta_{i\tau} + \eta_{it} \right]$$

RIP vs. HIP

- Autocovariance function

$$
\begin{aligned}
Cov\left[\Delta \log(\hat{e}_{iht}), \Delta \log(\hat{e}_{ih+nt+n})\right] &= E\left[\Delta \log(\hat{e}_{iht}) \cdot \Delta \log(\hat{e}_{ih+nt+n})\right] \\
&= \sigma_\beta^2 - \frac{\pi^2 \rho^{n-1}(1-\rho)}{1+\rho}.
\end{aligned}
$$

- For large enough leads n we should see empirically $Cov\left[\Delta \log(\hat{e}_{it}), \Delta \log(\hat{e}_{it+n})\right] > 0$ if $\sigma_\beta^2 > 0$.
- MaCurdy (1982): autocovariance function in data negative for $n \leq 10$, and insignificantly different from zero for $n \geq 2$.
- Interpretation: inconsistent with $\sigma_\beta^2 > 0$ and $\rho = 1$, because for $n \geq 2$ the autocovariances are statistically zero
- Note that in the presence of transitory shocks the autocovariance function of the model is negative rather than zero at $n = 1$ even if $\rho = 1$.
- Guvenen (2007): with empirical estimates for $\sigma_\beta^2 > 0$ autocovariances negative or zero for $n \leq 12$. Model-simulated autocovariances (with the estimated parameters) are not inconsistent with the data.

Size of ρ

- Claim: if truth is HIP but we estimate under the restriction $\sigma_\beta^2 = 0$, estimate of ρ may be biased upward.
- Consider 2 households with different earnings growth, with $\beta_1 < \beta_2$.

$$\log(e_{it}) = \alpha + \beta_i t$$

- Econometrician estimates

$$\log(e_{it}) = \rho \log(e_{it-1}) + \eta_{it}. \tag{2}$$

- In the first period the econometrician observes

$$\begin{aligned} \log(e_{11}) &= \alpha + \beta_1 \\ \log(e_{21}) &= \alpha + \beta_2 \end{aligned}$$

  Has to interpret this as negative shock for $i = 1$, positive shock for $i = 2$.
- Under truth earnings of both households deviate more and more from common average. Econometrician has to interpret this as repeated, persistent negative shocks for $i = 1$ and positive shocks for $i = 2$.
- Difference between $\rho = 0.98$ (RIP), $\rho = 0.82$ (HIP) large. RIP: 20 years later 2/3 of shock present. HIP: 2%. Big difference for self insurance.

Why General Equilibrium?

1. It imposes theoretical discipline: relation between ρ and r is determined endogenously
2. It gives rise to an endogenously determined consumption and wealth distribution and hence provides a theory of consumption and wealth inequality.
3. It enables meaningful policy experiments

A Model Without Aggregate Risk

- Continuum of measure 1 of individuals, each facing an income fluctuation problem
- Same stochastic labor endowment process $\{y_t\}_{t=0}^{\infty}$ where $y_t \in Y = \{y_1, y_2, \ldots y_N\}$. Labor income: $w_t y_t$
- Labor endowment: Markov process with transitions $\pi(y'|y) > 0$.
- Law of large numbers: $\pi(y'|y)$ is also the deterministic fraction of the population that has this particular transition.
- Unique stationary distribution associated with π, denoted by Π.
- At period 0 income y_0 of all agents is given. Distribution Π.
- Total labor endowment in the economy at each point of time

$$\bar{L} = \sum_y y \Pi(y)$$

- Probability of event history y^t, given initial event y_0

$$\pi_t(y^t|y_0) = \pi(y_t|y_{t-1}) * \ldots * \pi(y_1|y_0)$$

- Substantial idiosyncratic risk, but no aggregate risk. Thus there is hope for stationary equilibrium with constant w and r.

A Model Without Aggregate Risk

- Preferences (with $\beta = \frac{1}{1+\rho}$)

$$u(c) = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Budget constraint

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

- Borrowing constraint $a_{t+1} \geq 0$
- Initial condition of agent (a_0, y_0), population measure $\Phi_0(a_0, y_0)$
- Allocation: $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}$
- Technology

$$Y_t = F(K_t, L_t)$$

- Capital depreciates at rate $0 < \delta < 1$.
- Aggregate resource constraint

$$C_t + K_{t+1} - (1 - \delta) K_t = F(K_t, L_t)$$

- The only asset in economy is the physical capital stock. No state-contingent claims (i.e. incomplete markets).

Competitive Equilibrium

Definition

Given Φ_0, a SME is allocations for households $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}$, allocations for the representative firm $\{K_t, L_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that

1. Given prices, allocations maximize utility subject to the budget constraints and nonnegativity constraints.
2. Factor prices

$$\begin{aligned} r_t &= F_k(K_t, L_t) - \delta \\ w_t &= F_L(K_t, L_t) \end{aligned}$$

3. For all t, $L_t = \bar{L}$ and

$$\begin{aligned} K_{t+1} &= \int \sum_{y^t \in Y^t} a_{t+1}(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) \\ F(K_t, L_t) + (1-\delta)K_t &= \int \sum_{y^t \in Y^t} c_t(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) + K_{t+1} \end{aligned}$$

Recursive Equilibrium

- Individual state (a, y)
- Aggregate state variable $\Phi(a, y)$
- $A = [0, \infty)$: set of possible asset holdings
- Y : set of possible labor endowment realizations
- $\mathcal{P}(Y)$ is power set of Y
- $\mathcal{B}(A)$ is Borel σ-algebra of A
- $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$.
- $\mathcal{M}$ the set of all probability measures on the measurable space $M = (Z, \mathcal{B}(Z))$

Recursive Equilibrium

- Household problem in recursive formulation

$$v(a,y;\Phi) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v(a', y'; \Phi')$$

$$\begin{aligned} \text{s.t. } c + a' &= w(\Phi)y + (1 + r(\Phi))a \\ \Phi' &= H(\Phi) \end{aligned}$$

- Function $H : \mathcal{M} \rightarrow \mathcal{M}$ is called the aggregate “law of motion”

Definition

A RCE is value function $v : Z \times M \to R$, household policy functions $a', c : Z \times M \to R$, firm policy functions $K, L : M \to R$, pricing functions $r, w : M \to R$ and law of motion $H : M \to M$ s.t.

1. v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies Bellman equation and a', c are the policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$

$$\begin{aligned} r(\Phi) &= F_K(K(\Phi), L(\Phi)) - \delta \\ w(\Phi) &= F_L(K(\Phi), L(\Phi)) \end{aligned}$$

3. For all $\Phi \in \mathcal{M}$, $L(\Phi) = \int y d\Phi$ and

$$\begin{aligned} K'(\Phi') &= K(H(\Phi)) = \int a'(a, y; \Phi) d\Phi \\ \int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi &= F(K(\Phi), L(\Phi)) + (1 - \delta) K(\Phi) \end{aligned}$$

4. Aggregate law of motion H is generated by π and a'

Transition Functions

- Define transition function $Q_\Phi : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ by

$$Q_\Phi((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \left\{ \begin{array}{c} \pi(y'|y) \text{ if } a'(a,y;\Phi) \in \mathcal{A} \\ 0 \text{ else} \end{array} \right.$$

for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$

- $Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))$ is the probability that an agent with current assets a and current income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow.
- Hence

$$\begin{aligned} \Phi'(\mathcal{A},\mathcal{Y}) &= (H(\Phi))(\mathcal{A},\mathcal{Y}) \\ &= \int Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy) \end{aligned}$$

Definition

A stationary RCE is value function $v : Z \to R$, household policy functions $a', c : Z \to R$, firm policies K, L, prices r, w and a measure $\Phi \in M$ such that

1. v, a', c are measurable with respect to $B(Z)$, v satisfies the household's Bellman equation and a', c are associated policy functions, given r, w.
2. K, L satisfy, given r, w
$$\begin{aligned} r &= F_k(K, L) - \delta \\ w &= F_L(K, L) \end{aligned}$$
3. $L = \int y d\Phi$ and $K = \int a'(a, y) d\Phi$ and
$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta)K$$
4. Let Q be transition function induced by π and a'. $\forall (\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$
$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

Example: Discrete State Space

- Suppose $A = \{a_1, \ldots, a_M\}$. Then Φ is $M * N \times 1$ column vector and $Q = (q_{ij,kl})$ is $M * N \times M * N$ matrix with

$$q_{ij,kl} = \Pr\big((a', y') = (a_k, y_l) | (a, y) = (a_i, y_j)\big)$$

- Stationary measure Φ satisfies matrix equation

$$\Phi = Q^T \Phi.$$

- Φ is (rescaled) eigenvector associated with eigenvalue $\lambda = 1$ of Q^T.
- Note: Q^T is a stochastic matrix and thus has at least one unit eigenvalue. If it has more than one unit eigenvalue, continuum of stationary measures.

Theoretical Results: Existence, Uniqueness and Stability

- Existence (and Uniqueness) of Stationary RCE boils down to one equation in one unknown
- By Walras' law forget about goods market
- Labor market equilibrium $L = \bar{L}$ and $\bar{L}$ is exogenously given
- Asset market clearing condition

$$K = K(r) = \int a'(a, y) d\Phi \equiv Ea(r)$$

- Capital demand of firm $K(r)$ is defined implicitly as

$$r = F_k(K(r), \bar{L}) - \delta$$

- From assumptions on production function $K(r)$ is continuous, strictly decreasing function on $r \in (-\delta, \infty)$ with

$$\begin{aligned} \lim_{r \to -\delta} K(r) &= \infty \\ \lim_{r \to \infty} K(r) &= 0 \end{aligned}$$

Theoretical Results: Existence, Uniqueness and Stability

- Now characterization of capital supply (asset demand) $Ea(r)$
- $Ea(r) \in [0, \infty]$ for all r in $(-\delta, \infty)$
- Wage rate w is a function of r via

$$w(r) = F_L(K(r), \bar{L})$$

with

$$w'(r) < 0$$

- Already problem with uniqueness

Household Problem as Function of r

Proposition

(Huggett 1993) For $\rho > 0, r > -1$ and $y_1 > 0$ and $CRRA$ utility with $\sigma > 1$ the functional equation has a unique solution v which is strictly increasing, strictly concave and continuously differentiable in its first argument. The optimal policies are continuous functions that are strictly increasing (for $c(a, y)$) or increasing or constant at zero (for $a'(a, y)$)

- Similar results can be proved for the *iid* case and arbitrary bounded U with $\rho > r$ and $\rho > 0$, see Aiyagari (1994).
- Boundedness of the state space: requires $\rho > r$ and additional assumptions (*iid* and limiting exponent of U_c or Huggett's assumptions). Let $\bar{a}$ denote upper bound

Existence of Unique Invariant Measure

- From now on assume $\exists \bar{a}$ s.t. $a'(\bar{a}, y_N) = \bar{a}$ and $a'(a, y) \leq \bar{a}$ for all $y \in Y$ and all $a \in [0, \bar{a}]$. State space $Z = [0, \bar{a}] \times Y$ and optimal policy $a_r'(a, y)$ defined on Z, indexed by r
- Asset demand

$$Ea(r) = \int a_r'(a, y) d\Phi_r$$

- Need Φ_r that satisfies

$$\Phi_r(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi_r$$

where Q_r is the Markov transition function defined by a_r as

$$Q_r((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a_r'(a, y) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

- Need to establish that operator $T_r^* : \mathcal{M} \to \mathcal{M}$ defined by

$$\left(T_r^*(\Phi)\right)(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

has a unique fixed point

Existence of Unique Invariant Measure

Proposition

(Hopenhayn and Prescott) If the state space Z is compact and

1. Q_r *is a transition function*
2. Q_r *is increasing*
3. *There exists* $z^* \in Z$, $\varepsilon > 0$ *and* N *such that*

$$P^N(d, \{z : z \leq z^*\}) > \varepsilon \text{ and } P^N(c, \{z : z \geq z^*\}) > \varepsilon$$

where d *is maximal element of* Z *and* c *is minimal element of* Z.

Then the operator T_r^* *has a unique fixed point* Φ_r *and for all* $\Phi_0 \in M$ *the sequence of measures defined by*

$$\Phi_n = (T^*)^n \Phi_0$$

converges weakly to Φ_r

Existence of Unique Invariant Measure

- Assumption 1 requires that Q_r is transition function, i.e. $Q_r(z, .)$ is probability measure on $(Z, \mathcal{B}(Z))$ for all $z \in Z$ and $Q_r(., \mathcal{Z})$ is $\mathcal{B}(Z)$-measurable $\forall \mathcal{Z} \in \mathcal{B}(Z)$. Use that $a'(a, y)$ is continuous.
- The assumption that Q_r is increasing requires that for any nondecreasing function $f : Z \to \mathbf{R}$ we have that

$$(Tf)(z) = \int f(z')Q_r(z, dz')$$

  is also nondecreasing. Note that $a'(a, y)$ is increasing in (a, y).
- Monotone mixing condition 3. satisfied? Pick $z^* = (\frac{1}{2}(a'(0, y_N) + \bar{a}), y_1)$. Start at d with a sequence of bad shocks y_1 and from c with a sequence of good shocks y_N.
- Conclusion of the theorem assures existence of a unique invariant measure Φ_r which can be found by iterating on the operator T^*
- Convergence is in the weak sense: for every continuous and bounded real-valued function f on Z we have

$$\lim_{n\to\infty} \int f(z)d\Phi_n = \int f(z)d\Phi_r$$

Existence of Equilibrium

- Hence function $Ea(r)$ is well-defined on $r \in [-\delta, \rho)$
- Since $a_r'(a, y)$ is continuous jointly in (r, a) and Φ_r is continuous in r (in the sense of weak convergence) the function $Ea(r)$ is a continuous function of r on $[-\delta, \rho)$
- $\lim_{r \to -\delta} Ea(r) < \infty$ is fine, but what about

$$\lim_{r \to \rho} Ea(r) > K(\rho)$$

- If both satisfied, then there exists r^* such that

$$K(r^*) = Ea(r^*)$$

  and a stationary RCE
- Uniqueness? Not necessarily
- Stability? Nobody knows

Computation of the General Equilibrium

1. Fix an $r \in (-\delta, \rho)$. For a fixed r solve household's recursive problem. This yields a value function v_r and decision rules a'_r, c_r

2. The policy function a'_r and π induce Markov transition function Q_r. Compute the unique stationary measure Φ_r associated with this transition function

3. Compute excess demand for capital

$$d(r) = K(r) - Ea(r)$$

   If zero, stop, if not, adjust r.

Qualitative Results

- Complete markets model: $r^{CM} = \rho$
- This model: $r^* < \rho$
- Overaccumulation of capital and oversaving (because of precautionary reasons: liquidity constraints, prudence, or both)
- Question: How big a difference does it make?

Calibration

- $CRRA$ with values $\sigma = \{1, 3, 5\}$
- $\rho = r^{CM} = 0.0416$ ($\beta = 0.96$).
- Cobb-Douglas production function with $\alpha = 0.36$
- Depreciation rate $\delta = 8\%$

Earnings Profile

- Assume process

$$\log(y_{t+1}) = \theta \log(y_t) + \sigma_\varepsilon \left(1 - \theta^2\right)^{\frac{1}{2}} \varepsilon_{t+1}$$

- Note

$$\begin{aligned} corr(\log(y_{t+1}), \log(y_t)) &= \theta \\ Var(\log(y_{t+1})) &= \sigma_\varepsilon^2 \end{aligned}$$

- Discretize, using Tauchen's method: Take $N = 7$.

Earnings Profile

- Since $\log(y_t) \in (-\infty, \infty)$, subdivide in intervals

$$
\begin{aligned}
I_1 &= (-\infty, -\frac{5}{2}\sigma_\varepsilon) \\
I_2 &= [-\frac{5}{2}\sigma_\varepsilon, -\frac{3}{2}\sigma_\varepsilon) \\
I_3 &= [-\frac{3}{2}\sigma_\varepsilon, -\frac{1}{2}\sigma_\varepsilon) \\
I_4 &= [-\frac{1}{2}\sigma_\varepsilon, \frac{1}{2}\sigma_\varepsilon) \\
I_5 &= [\frac{1}{2}\sigma_\varepsilon, \frac{3}{2}\sigma_\varepsilon) \\
I_6 &= [\frac{3}{2}\sigma_\varepsilon, \frac{5}{2}\sigma_\varepsilon) \\
I_7 &= [\frac{5}{2}\sigma_\varepsilon, \infty)
\end{aligned}
$$

Earnings Profile

- State space for log-income: "midpoints"

$$Y^{\log} = \{-3\sigma_\varepsilon, -2\sigma_\varepsilon, -\sigma_\varepsilon, 0, \sigma_\varepsilon, 2\sigma_\varepsilon, 3\sigma_\varepsilon\}$$

- Matrix of transition probabilities π: Fix a state $s_i = \log(y) \in Y^{\log}$ today. The conditional probability of a particular state $s_j = \log(y') \in Y^{\log}$ tomorrow is

$$\pi(\log(y') = s_j | \log(y) = s_i) = \int_{I_j} \frac{e^{-\frac{(x-\theta s_i)^2}{2\sigma_y}}}{(2\pi)^{0.5}\sigma_y} dx$$

  where $\sigma_y = \sigma_\varepsilon \left(1 - \theta^2\right)^{\frac{1}{2}}$

- Find the stationary distribution of π, hopefully unique, by solving the matrix equation

$$\Pi = \pi^T \Pi$$

- Take $\tilde{Y} = e^{Y^{\log}}$

$$\tilde{Y} = \{e^{-3\sigma_\varepsilon}, e^{-2\sigma_\varepsilon}, e^{-\sigma_\varepsilon}, 1, e^{\sigma_\varepsilon}, e^{2\sigma_\varepsilon}, e^{3\sigma_\varepsilon}\}$$

Earnings Profile

- Compute average labor endowment

$$\bar{y} = \sum_{y \in \tilde{Y}} y\Pi(y)$$

- Normalize all states by $\bar{y}$

$$\begin{aligned} Y &= \{y_1, \ldots, y_7\} \\ &= \{\frac{e^{-3\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-\sigma_\varepsilon}}{\bar{y}}, \frac{1}{\bar{y}}, \frac{e^{\sigma_\varepsilon}}{\bar{y}}, \frac{e^{2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{3\sigma_\varepsilon}}{\bar{y}}\} \end{aligned}$$

- Note: average labor endowment

$$\sum_{y \in Y} y\Pi(y) = 1$$

- Parameter values considered

$$\begin{aligned} \theta &\in \{0, 0.3, 0.6, 0.9\} \\ \sigma_\varepsilon &\in \{0.2, 0.4\} \end{aligned}$$

Results

- With Cobb-Douglas production function, $\bar{L} = 1$ we have $Y = K^{\alpha}$,

$$r + \delta = \alpha K^{\alpha - 1} = \frac{\alpha Y}{K} = \frac{\alpha \delta}{s}$$

where s is the aggregate saving rate. Thus

$$s = \frac{\alpha \delta}{r + \delta}$$

- Benchmark: complete markets: $r^{CM} = \rho = 4.16\%, s = 23.7\%$.
- Keeping prudence and dispersion fixed, increase in persistence of income shock leads to increased precautionary saving, bigger overaccumulation of capital, compared to complete markets.
- Keeping fixed persistence and dispersion in income, an increase in prudence σ leads to more precautionary saving and more severe overaccumulation of capital
- Keeping prudence and income persistence constant, an increase in the dispersion of the income process leads to more precautionary saving and more severe overaccumulation of capital.

Wealth Distribution

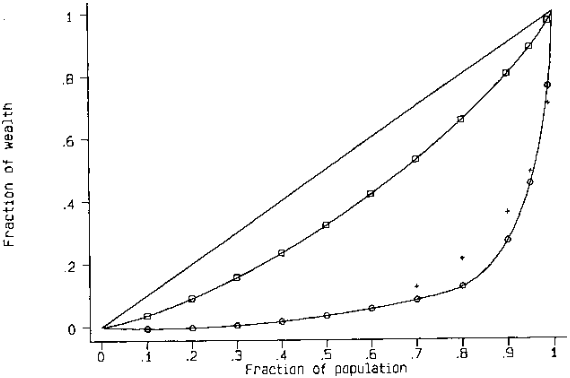


FIG. 3.—Lorenz curves for wealth holdings (+ refers to the data, □ to the benchmark model, and ○ to the stochastic-β model).

Savings and Interest Rates

TABLE II

A. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.2$)

$\rho \backslash \mu$	1	3	5
0	4.1666/23.67	4.1456/23.71	4.0858/23.83
0.3	4.1365/23.73	4.0432/23.91	3.9054/24.19
0.6	4.0912/23.82	3.8767/24.25	3.5857/24.86
0.9	3.9305/24.14	3.2903/25.51	2.5260/27.36

B. Net return to capital in %/aggregate saving rate in % ($\sigma = 0.4$)

$\rho \backslash \mu$	1	3	5
0	4.0649/23.87	3.7816/24.44	3.4177/25.22
0.3	3.9554/24.09	3.4188/25.22	2.8032/26.66
0.6	3.7567/24.50	2.7835/26.71	1.8070/29.37
0.9	3.3054/25.47	1.2894/31.00	−0.3456/37.63

The Aiyagari Model in Continuous Time

- Household problem in the representative agent neoclassical growth model in continuous time
- Assume constant factor prices (w, r)
- Household maximization problem in sequence form

$$\max_{(c(t),a(t))} \int_0^\infty e^{-\rho t} U(c(t))dt$$

$$\text{s.t.} \qquad \dot{a} = ra(t) + w - c(t)$$

  with $a(0)$ given.
- Typical Approach: Hamiltonian ($\approx$ Lagrangian). Denote co-state variable associated with a as λ, form current value Hamiltonian

$$\mathcal{H}(a, c, \lambda) = U(c) + \lambda(ra + w - c)$$

- Taking first order conditions and differentiating with time yields standard Euler equation (with CRRA utility)

$$\frac{\dot{c(t)}}{c(t)} = \frac{r - \rho}{\sigma}$$

Detour: Recursive Formulation in Continuous Time

- Hamilton-Jacobi-Bellman Equation with value function $V(a)$

$$\begin{aligned} \rho V(a) &= \max_{c,\dot{a}} \left\{ U(c) + V'(a)\dot{a} \right\} \\ \text{s.t.} &\quad \dot{a} = ra + w - c \end{aligned}$$

- ... or plugging in the budget constraint to eliminate x

$$\rho V(a) = \max_{c} \left\{ U(c) + V'(a)(ra + w - c) \right\}$$

- Note that defining $\lambda = V'(a)$ we obtain

$$\rho V(a) = \max_{c,\lambda} \{U(c) + \lambda(ra + w - c)\} = \max_{c,\lambda} \{\mathcal{H}(a, c, \lambda)\}$$

where $\mathcal{H}(a, c, \lambda) = U(c) + \lambda(ra + w - c)$ is again the current value Hamiltonian

Deriving Euler Equation from Recursive Formulation

- Basic logic is same as in discrete time: use FOC and envelope condition

$$
\begin{aligned}
U'(c) &= V'(a) && (3)\\
\rho V'(a) &= V''(a)(ra+w-c) + rV'(a) && (4)
\end{aligned}
$$

- Using budget constraint $\dot{a} = ra + w - c$ rewrite envelope condition as

$$(\rho - r)V'(a) = V''(a)\dot{a}$$

- Define the new variable $\lambda = V'(a)$ and note that

$$V''(a)\dot{a} = \frac{dV'(a)}{da} \times \frac{da}{dt} = \frac{d\lambda}{dt} = \dot{\lambda}.$$

- Then we can write both equations as

$$
\begin{aligned}
U'(c) &= \lambda\\
(\rho - r)\lambda &= \dot{\lambda}
\end{aligned}
$$

- Differentiate FOC with respect to t, combine with Envelope condition to obtain Euler equation (with CRRA utility as special case):

$$\frac{\dot{c}}{c} = (\rho - r)\left(\frac{U'(c)}{U''(c)c}\right) = \frac{r-\rho}{\sigma}$$

Dynamic Programming: Discrete vs. Continuous Time

- Period length Δ, discrete time DP, discount fac. $\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \Delta\rho$

$$\begin{aligned} V(a) &= \max_{c, a_\Delta - a} \left\{ \Delta U(c) + e^{-\rho\Delta} V(a_\Delta) \right\} \\ \text{s.t.} & \quad a_\Delta - a = \Delta(ra + w - c) \end{aligned}$$

- Use $e^{-\rho\Delta} \approx 1 - \Delta\rho$, subtract $(1 - \Delta\rho)V(a)$ from both sides:

$$\begin{aligned} \rho\Delta V(a) &= \max_{c, a_\Delta - a} \left\{ \Delta U(c) + (1 - \Delta\rho) \frac{V(a_\Delta) - V(a)}{a_\Delta - a} (a_\Delta - a) \right\} \\ \text{s.t.} & \quad a_\Delta - a = \Delta(ra + w - c) \end{aligned}$$

- Now divide both sides by Δ

$$\begin{aligned} \rho V(a) &= \max_{c, \frac{a_\Delta - a}{\Delta}} \left\{ U(c) + (1 - \Delta\rho) \frac{V(a_\Delta) - V(a)}{a_\Delta - a} \frac{(a_\Delta - a)}{\Delta} \right\} \\ \text{s.t.} & \quad \frac{a_\Delta - a}{\Delta} = ra + w - c \end{aligned}$$

- Now take $\Delta \to 0$

$$\rho V(a) = \max_{c, \dot{a}} \left\{ U(c) + V'(a)\dot{a} \right\} \text{ s.t. } \dot{a} = ra + w - c$$

Introduction of Idiosyncratic Risk: Poisson Processes

- Set of labor productivity (income) states $y \in Y = \{y_1, \cdots, y_N\}$
- Denote by $\pi_{i,j} > 0$ the Poisson intensity of switching from state i to state $j \neq i$.
- Interpretation: Probability of switching from state i to j in small time interval Δ is $\Delta \pi_{i,j}$
- Define $\pi_{i,i} = -\sum_{j \neq i} \pi_{i,j}$. Collect in the $N \times N$ matrix $\pi = (\pi_{i,j})$.
- Invariant distribution Π over states is an $N \times 1$ vector satisfying

$$\pi^T \Pi = 0 \qquad \text{and} \qquad \sum_{y \in Y} \Pi(y) = 1$$

- Example $Y = \{y_1, y_2\}$ and $\pi = \begin{pmatrix} -\pi_{1,2} & \pi_{1,2} \\ \pi_{2,1} & -\pi_{2,1} \end{pmatrix}$. Then

$$\begin{aligned} -\Pi(y_1)\pi_{1,2} + \pi_{2,1}(1 - \Pi(y_1)) &= 0 \\ \Pi(y_1) &= \frac{\pi_{2,1}}{\pi_{2,1} + \pi_{1,2}} \end{aligned}$$

- As before, normalize states such that $\sum_{y \in Y} y\Pi(y) = 1$.

Hamilton-Jacobi-Bellman Equation: Idiosyncratic Risk

- Recall without risk

$$\rho V(a) = \max_c \left\{ U(c) + V'(a)(ra + w - c) \right\}$$

- Introduction of idiosyncratic risk into HJB:

$$\rho V(a, y_i) = \max_c \left\{ U(c) + V'(a, y_i)(ra + w y_i - c) + \sum_{j \neq i} \pi_{i,j} [V(a, y_j) - V(a, y_i)] \right\}$$

- Also assume potentially tight borrowing constraint $a \geq \underline{a}$.
- Borrowing limit might be $\underline{a} = 0$ or $\underline{a} = \frac{y_1}{r}$ (natural borrowing limit) or something in between. Domain of HJB is $[\underline{a}, \infty)$.
- Solution: value function $V(a, y_i)$ and policy functions $(c(a, y_i), s(a, y_i))$ where the saving function (i.e., $\dot{a}$) is defined by $s(a, y_i) = ra + w y_i - c(a, y_i)$.
- Achdou et al. (2022) focus on $N = 2$, but nothing in the model set-up (but in the characterization) requires $N = 2$.

Aggregation

- Denote by $\phi(a, y_i)$ the joint population density (pdf) over assets and labor productivity and by $\Phi(a, y_i)$ the corresponding cdf.
- Note that the joint distribution might have a mass point at $\underline{a}$, thus aggregation will use Φ.
- Asset market clearing condition

$$K = \sum_{y_i} \int_{\underline{a}}^{\infty} a d\Phi(a, y_i)$$

- Production side of economy completely unchanged:

$$\begin{aligned} r &= F_K(K, 1) - \delta \\ w &= F_L(K, 1) \end{aligned}$$

- As before, can express $(K(r), w(r))$ as functions of r. Thus

$$K(r) = \sum_{y_i} \int_{\underline{a}}^{\infty} a d\Phi_r(a, y_i)$$

Theoretical Characterization of HJB

$$\rho V(a, y_i) = \max_c \left\{ U(c) + V'(a, y_i)(ra + wy_i - c) + \sum_{j \neq i} \pi_{i,j}[V(a, y_j) - V(a, y_i)] \right\}$$

- First order condition for consumption reads as

$$\begin{aligned} U'(c) &= V'(a, y_i) \\ c(a, y_i) &= (U')^{-1}(V'(a, y_i)) \end{aligned}$$

- Plugging into the HJB delivers a (nonlinear first order ordinary) differential equation in the unknown function $V(.,.)$

$$\begin{aligned} \rho V(a, y_i) &= U((U')^{-1}(V'(a, y_i))) + V'(a, y_i)(ra + wy_i - (U')^{-1}(V'(a, y_i))) \\ &+ \sum_{j \neq i} \pi_{i,j}[V(a, y_j) - V(a, y_i)] \end{aligned}$$

- Binding borrowing constraint? Never for $a > \underline{a}$.
- Potentially binding constraint reflected in boundary condition: $\forall y_i$

$$V'(\underline{a}, y_i) \geq U'(r\underline{a} + wy_i)$$

- Since FOC $U'(c) = V'(a, y_i)$ still holds at $\underline{a}$, boundary condition guarantees $s(\underline{a}, y_i) = r\underline{a} + wy_i - c(\underline{a}, y_i) \geq 0$.

Theory: Saving Behavior of the Wealth-Poor

Proposition (Achdou et al., Proposition 1)

Let $N = 2$ and assume that $-\frac{U''(r\underline{a}+wy_1)}{U'(r\underline{a}+wy_1)} < \infty$ and $r < \rho$. Then the consumption- and saving functions for $y = y_1$ satisfy

1. *Borrowing constraint only binding at $a = \underline{a}$:*

$$\begin{aligned} s(\underline{a}, y_1) &= 0 \\ s(a, y_1) &< 0 \quad \forall a > \underline{a} \end{aligned}$$

2. *Behavior close to the borrowing constraint:*

$$\begin{aligned} s(a, y_1) &\sim -\sqrt{2v_1}\sqrt{a-\underline{a}} \\ c(a, y_1) &\sim wy_1 + ra + \sqrt{2v_1}\sqrt{a-\underline{a}} \\ c'(a, y_1) &\sim r + \sqrt{\frac{v_1}{2(a-\underline{a})}} \quad \textit{where} \\ v_1 &= \frac{(\rho - r)U''(c(\underline{a}, y_1) + \pi_{1,2}(U''(c(\underline{a}, y_1) - U''(c(\underline{a}, y_2))}{-U''(c(\underline{a}, y_1))} \end{aligned}$$

and where $f(a) \sim g(a)$ as $a \to \underline{a}$ means $\lim_{a \to \underline{a}} f(a)/g(a) = 1$

Theory: Consumption-Saving Behavior of the Wealth-Rich

Proposition (Achdou et al., Proposition 2)

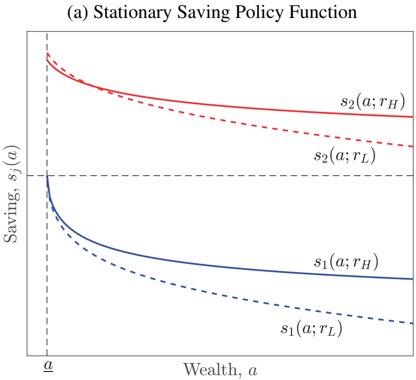
Let $N = 2$ and assume that $U(c)$ is CRRA with risk aversion σ and $r < \rho$. Then as $a \to \infty$ and $\forall y$

$$\begin{aligned} s(a, y) &\sim \frac{r - \rho}{\sigma} a \\ c(a, y) &\sim \frac{\rho - (1 - \sigma) r}{\sigma} a \end{aligned}$$

Thus there exists an $a_{\max}$ such that $\forall y$ and $\forall a > a_{\max}$ we have $s(a, y) < 0$.

- Asymptotically, the consumption function approaches that of the permanent income hypothesis.

Illustration: Optimal Savings Policies



- Note: this plots the *change* in assets, that is, $\dot{a}$. Individuals with $y = y_1$ dis-save, individuals with $y = y_2$ save.
- Increase in r increases saving in most parts of the state space (strong substitution effect). But see high income people ($y = y_2$) with low wealth $a \approx \underline{a}$.

Stationary Distribution: Kolmogorov Forward Equation

- On $a \in (\underline{a}, \infty)$, stationary pdf $\phi_r(a, y_i)$ satisfies $\forall i = 1, \cdots, N$

$$\frac{d[s(a, y_i; r)\phi_r(a, y_i)]}{da} + \sum_{j \neq i} \pi_{i,j}\phi_r(a, y_i) = \sum_{j \neq i} \pi_{j,i}\phi_r(a, y_j)$$

- Intuition: outflow out of (a, y_i) (LHS) must equal inflow into (a, y_i) (RHS) for stationary distribution. Note that the $\phi_r(a, y_i)$'s are connected across (i, j).
- Furthermore, $\forall i = 1, \cdots, N$

$$\Phi_r(\underline{a}, y_i) + \int_{\underline{a}}^{\infty} \phi_r(a, y_i) da = \Pi(y_i)$$

- Intuition: potential Dirac mass point $\Phi_r(\underline{a}, y_i)$ at the borrowing constraint $\underline{a}$ plus probability mass above mass point must equal to mass of people with labor productivity y_i for all y-states.

Kolmogorov Forward Equation: Attempt at Intuition

- Without Poisson transitions ($\pi_{i,j} = 0$): not interesting but instructive.

$$\frac{d[s(a, y_i; r)\phi_r(a, y_i)]}{da} = 0$$

- Denote by $\tilde{a}(t)$ the stochastic asset level and by a a specific value. The cdf at time t is defined by $\Phi(a, y_i; t) = \Pr(\tilde{a}(t) \leq a, \tilde{y}(t) = y_i)$
- Now want to derive how $\Phi(a, y_i; t)$ evolves over time. Consider again small time interval Δ. Then

$$\begin{aligned} \tilde{a}(t+\Delta) &= \tilde{a}(t) + \Delta s(\tilde{a}(t+\Delta), y_i) \\ \tilde{a}(t) &= \tilde{a}(t+\Delta) - \Delta s(\tilde{a}(t+\Delta), y_i) \end{aligned}$$

- For small Δ, change in assets is given by

$$\Delta s(\tilde{a}(t+\Delta), y_i) = \Delta s(\tilde{a}(t), y_i)$$

If $s(\tilde{a}(t+\Delta), y_i) < 0$, then $\tilde{a}(t+\Delta) < \tilde{a}(t)$; assets are drifting $\downarrow$.

Kolmogorov Forward Equation: Attempt at Intuition

- Then (since we are ignoring income transitions for now)

$$\begin{aligned}\Pr(\tilde{a}(t+\Delta) \leq a) &= \Pr(\tilde{a}(t) \leq a) + \Pr(a \leq \tilde{a}(t) \leq a - \Delta s(a, y_i)) \\ &= \Pr(\tilde{a}(t) \leq a - \Delta s(a, y_i))\end{aligned}$$

- In terms of the cdf:

$$\Phi(a, y_i; t+\Delta) = \Phi(a - \Delta s(a, y_i), y_i; t)$$

- Subtracting $\Phi(a, y_i; t)$ from both sides and dividing by Δ yields

$$\frac{\Phi(a, y_i; t+\Delta) - \Phi(a, y_i; t)}{\Delta} = \frac{\Phi(a - \Delta s(a, y_i), y_i; t) - \Phi(a, y_i; t)}{\Delta}$$

- Taking the limit as $\Delta \to 0$ yields (and making a change of variables $x = \Delta s(a, y_i)$, and imposing stationarity

$$\begin{aligned}\lim_{\Delta \to 0} \frac{\Phi(a, y_i; t+\Delta) - \Phi(a, y_i; t)}{\Delta} &= \lim_{x \to 0} \frac{\Phi(a - x; t) - \Phi(a, y_i; t)}{x} s(a, y_i) \\ 0 &= -\phi(a, y_i) s(a, y_i)\end{aligned}$$

- ...and differentiating with respect to a yields

$$0 = -\frac{ds(a, y_i)\phi(a, y_i)}{da}$$

Kolmogorov Forward Equation with Poisson Income Transitions

- With income transitions $\Phi(a, y_i; t+\Delta) = \Phi(a - \Delta s(a, y_i), y_i; t)$ becomes

$$\Phi(a, y_i; t+\Delta) = (1-\Delta \sum_{j \neq i} \pi_{i,j})\Phi(a-\Delta s(a, y_i), y_i; t)+\Delta \sum_{j \neq i} \pi_{j,i}\Phi(a-\Delta s(a, y_j), y_j; t)$$

- Again subtracting $\Phi(a, y_i; t)$ from both sides and dividing by Δ, taking limit as $\Delta \to 0$ and imposing stationarity yields

$$0 = -\phi(a, y_i)s(a, y_i) - \sum_{j \neq i} \pi_{i,j}\Phi(a, y_i) + \sum_{j \neq i} \pi_{j,i}\Phi(a, y_j)$$

...and differentiating with respect to a yields the Kolmogorov forward equation characterizing the stationary pdf on $(\underline{a}, \infty)$

$$0 = -\frac{d[s(a, y_i)\phi(a, y_i)]}{da} - \sum_{j \neq i} \pi_{i,j}\phi(a, y_i) + \sum_{j \neq i} \pi_{j,i}\phi(a, y_j)$$

Theoretical Characterization: Stationary Distribution

- Recall that on $(\underline{a}, \infty)$ the stationary pdf (making dependence on r explicit now) $\phi_r(a, y_i)$ satisfies $\forall i = 1, \cdots, N$

$$\frac{d[s(a, y_i; r)\phi_r(a, y_i)]}{da} + \sum_{j \neq i} \pi_{i,j}\phi_r(a, y_i) = \sum_{j \neq i} \pi_{j,i}\phi_r(a, y_j)$$

- System of ordinary differential equations. Paper shows that they can be decoupled and solved separately for each y.
- Boundary condition at $a = \underline{a}$ for each y_i

$$m_i + \int_{\underline{a}}^{\infty} \phi_r(a, y_i)da = \Pi(y_i)$$

where m_i is the Dirac mass point at the borrowing limit $\underline{a}$ for $y = y_i$.

Theoretical Characterization: Stationary Distribution

Proposition (Achdou et al., Proposition 3)

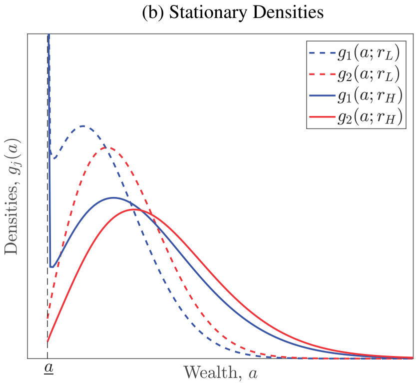
Let $N = 2$ and assume that $U(c)$ is CRRA with risk aversion σ and $r < \rho$. Then the stationary distribution $\Phi(a, y)$ has the following properties

1. *Properties close to the Borrowing Constraint:* $\Phi(\underline{a}, y_1) = m_1 > 0$, *i.e., there is a mass point at the borrowing constraint for low-income types* $y = y_1$ *and*

$$\Phi(a, y_1) \sim m_1 e^{\pi_{1,2}\sqrt{2(a-\underline{a})/v_1}}$$

   $\Phi(\underline{a}, y_2) = m_2 = 0$, *that is, there is no mass point at the borrowing constraint for* $y = y_2$. *The density is finite at* $\underline{a}$: $\phi(\underline{a}, y_2) < \infty$.
2. *Behavior at the right tail:* $\Phi(a, y)$ *has bounded support* $[\underline{a}, a_{\max}]$ *and has no mass point at* $a_{\max}$.
3. *The density* $\phi(a, y)$ *is continuous and differentiable for all* $a > \underline{a}$.

Illustration: Invariant Distribution



- Mass point at borrowing limit. Becomes smaller as $r \uparrow$.
- Continuous density to the right of the constraint. Mass shifts to the right as $r \uparrow$.

Computation: HJB Equation

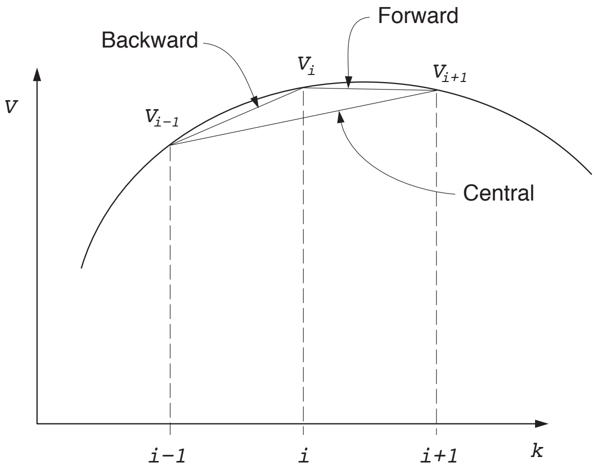
- Recall the HJB equation

$$\begin{aligned} \rho V(a, y_j) &= U(c) + V'(a, y_j)(ra + wy_j - c) + \sum_{l \neq j} \pi_{j,l}[V(a, y_l) - V(a, y_j)] \\ c &= (U')^{-1}(V'(a, y_j)) \end{aligned}$$

- Discretized asset state space with equispaced grid $A = \{a_1, \cdots, a_M\}$ with $a_1 = \underline{a}$ and distance between grid points given by Δa.
- Index income states $y = y_j$ today by j and the "other" income states y_l by $l \neq j$.
- Value function defined on the grid: $V_{i,j} = V(a_i, y_j)$. Matrix of dimension $(M \times N)$.
- Approximate the derivative of value function by a forward or backward difference approximation

$$\begin{aligned} V'(a_i, y_j) &\approx \frac{V_{i+1,j} - V_{i,j}}{\Delta a} \equiv V'_{i,j,F} \\ V'(a_i, y_j) &\approx \frac{V_{i,j} - V_{i-1,j}}{\Delta a} \equiv V'_{i,j,B} \end{aligned}$$

Forward and Backward Difference Approximation



Finite Difference Approximation of HJB Equation

- HJB on a discretized state space

$$
\begin{aligned}
\rho V_{i,j} &= U(c_{i,j}) + V'_{i,j}(ra_i + wy_j - c_{i,j}) + \sum_{l \neq j} \pi_{j,l}[V_{i,l} - V_{i,j}] \\
c_{i,j} &= (U')^{-1}(V'_{i,j})
\end{aligned}
$$

- "Only" unknowns are the $M \times N$ values $V_{i,j}$. Two complications:

  1. Use forward or backward difference approximation for derivatives of the value function? Use upwind scheme.
  2. HJB is a (highly) nonlinear system of $M \times N$ equations. Can in principle give to nonlinear equation solver and hope for the best. In practice: iterative procedure. Either explicit or implicit method.

Explicit Method: Upwind Scheme

- Explicit method more intuitive, less efficient and stable
- Start with initial guess for value function, e.g., $V_{i,j}^0 = \frac{U(ra_i+wy_j)}{\rho}$.
- For a step size parameter Δ update the value function according to

$$\begin{aligned} \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta} + \rho V_{i,j}^n &= U(c_{i,j}^n) + (V_{i,j}^n)'(ra_i + wy_j - c_{i,j}^n) \\ &+ \sum_{l \neq j} \pi_{j,l}[V_{i,l}^n - V_{i,j}^n] \\ c_{i,j}^n &= (U')^{-1}((V_{i,j}^n)') \end{aligned}$$

- For each iteration n, need to calculate the derivative of the value function $(V_{i,j}^n)'$. Use upwind scheme.
- Broad idea: If at given (a_i, y_j) saving $s_{i,j} > 0$, use forward difference $(V_{i,j,F}^n)'$. Otherwise use backward difference.

Upwind Scheme: Details

- For a given step n and associated $V_{i,j}^n$, define

$$\begin{aligned}
s_{i,j,F}^n &= ra_i + wy_j + (U')^{-1}((V_{i,j,F}^n)') \\
s_{i,j,B}^n &= ra_i + wy_j + (U')^{-1}((V_{i,j,B}^n)') \\
(V_{i,j}^n)' &= (V_{i,j,F}^n)' \mathbf{1}_{\{s_{i,j,F}^n > 0\}} + (V_{i,j,B}^n)' \mathbf{1}_{\{s_{i,j,B}^n < 0\}} \\
&+ (\bar{V}_{i,j}^n)' \mathbf{1}_{\{s_{i,j,F}^n \leq 0 \leq s_{i,j,B}^n\}} \\
(\bar{V}_{i,j}^n)' &= U'(ra_i + wy_j)
\end{aligned}$$

- Impose state constraint by setting

$$(V_{1,j}^n)' = U'(ra_1 + wy_j)$$

Implicit Method: Updating

- Implicit method less intuitive, but efficient and stable
- Start with initial guess for value function, e.g., $V^0_{i,j} = \frac{U(ra_i + wy_j)}{\rho}$.
- For a step size parameter Δ update the value function according to

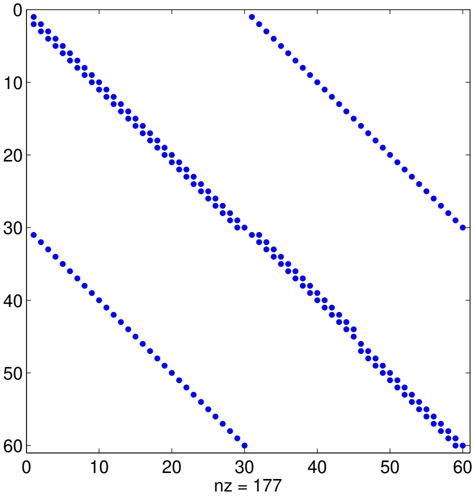
$$\begin{aligned} \frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta} + \rho V^{n+1}_{i,j} &= U(c^n_{i,j}) + (V^{n+1}_{i,j})'(ra_i + wy_j - c^n_{i,j}) \\ &+ \sum_{l \neq j} \pi_{j,l}[V^{n+1}_{i,l} - V^{n+1}_{i,j}] \\ c^n_{i,j} &= (U')^{-1}((V^n_{i,j})') \end{aligned}$$

- Upward scheme more complicated.
- Updating involves iterating on a matrix equation.

$$\frac{V^{n+1} - V^n}{\Delta} + \rho V^{n+1} = U^n + \mathbf{A^n} V^{n+1}$$

- Matrix $\mathbf{A} = \lim_{n\rightarrow} \mathbf{A^n}$ is sparse, resurfaces in Kolmogorov forward equation.

Poisson Intensity Matrix **A**



Discretization of Kolmogorov Forward Equation

- Distance between grid points Δa.
- Discretize: $\phi_{i,j}$ is height of the histogram. Probability mass of point (a_i, y_j) is $\phi_{i,j}\Delta a$.
- Kolmogorov forward equation in discretized form

$$
\begin{aligned}
0 &= -[s_{i,j}\phi_{i,j}]' - \sum_{l\neq j} \pi_{j,l}\phi_{i,j} + \sum_{l\neq j} \pi_{l,j}\phi_{i,l} \\
1 &= \sum_i \sum_j \phi_{i,j}\Delta a
\end{aligned}
$$

- Note: as before, the derivative $[s_{i,j}\phi_{i,j}]'$ has to be calculated as finite difference.
- KFE can be written compactly as $\mathbf{A}^T\phi = 0$, where $\mathbf{A}$ is the matrix from the implicit scheme of the HJB equation.
- Thus the HJB and the KFE "like each other".
- In contrast to HJB, the KFE is a linear system (eigenvector-eigenvalue problem) and can be solved in one shot.

Aggregate Asset Supply

- This step (and the associated figure) is completely unchanged from the discrete time economy.
- Market clearing

$$K(r) = Ea(r)$$

- Here

$$Ea(r) = \sum_i \sum_j a_i \phi_{i,j} \Delta a$$

Unexpected Aggregate Shocks and Transition Dynamics

- Hypothetical thought experiment:
  - Economy is in stationary equilibrium, with a given government policy
  - Unexpectedly government policy changes. Exogenous change may be either transitory or permanent
  - Want to compute transition path induced by the exogenous change, from the old stationary equilibrium to a new stationary equilibrium
- Example: permanent introduction of a capital income tax at rate τ. Receipts are rebated lump-sum to households as government transfers T.

Definition of Equilibrium

- State space: $Z = Y \times \mathbf{R}_+$, the set of all possible (y, a).
- Let $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(\mathbf{R}_+)$ and $\mathbf{M}$ be the set of all finite measures on the measurable space $(Z, \mathcal{B}(Z))$.
- Household problem

$$\begin{aligned} v_t(a, y) &= \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v_{t+1}(a', y') \\ \text{s.t. } c + a' &= w_t y + (1 + (1 - \tau_t) r_t) a + T_t \end{aligned}$$

Definition of Equilibrium

Definition

Given initial distribution Φ_0, fiscal legislation $\{\tau_t\}_{t=0}^{\infty}$ a competitive equilibrium is sequence of functions for household $\{v_t, c_t, a_{t+1} : Z \to \mathbf{R}\}_{t=0}^{\infty}$, sequence of firm production plans $\{L_t, K_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$, government transfers $\{T_t\}_{t=0}^{\infty}$, and sequence of measures $\{\Phi\}_{t=1}^{\infty}$ s.t. $\forall t$,

- Given $\{w_t, r_t\}$ and $\{T_t, \tau_t\}$ the functions $\{v_t\}$ solve Bellman equation in t and $\{c_t, a_{t+1}\}$ are associated policy functions
- Prices w_t and r_t satisfy

$$w_t = F_L(K_t, L_t)$$

$$r_t = F_K(K_t, L_t) - \delta.$$

- Government Budget Constraint: for all $t \geq 0$.

$$T_t = \tau_t r_t K_t$$

Definition

- Market Clearing:

$$\begin{aligned}\int c_t(a,y)d\Phi_t + K_{t+1} &= F(K_t, L_t) + (1-\delta)K_t \\ L_t &= \int y_t d\Phi_t \\ K_{t+1} &= \int a_t(a,y)d\Phi_t\end{aligned}$$

- Aggregate Law of Motion: Define Markov transition functions $Q_t : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ induced by the transition probabilities π and optimal policy $a_{t+1}(y,a)$ as

$$Q_t((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a_t(a,y) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$. Then for all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi_{t+1}(\mathcal{A},\mathcal{Y}) = [\Gamma_t(\Phi_t)](\mathcal{A},\mathcal{Y}) = \int Q_t((a,y),(\mathcal{A},\mathcal{Y}))d\Phi_t$$

Definition of Stationary Equilibrium

Definition

A stationary equilibrium is an equilibrium such that all elements of the equilibrium that are indexed by t are constant over time.

Computation of the Transition Path

- Key: *assume* that after T periods the transition from old to new stationary equilibrium is completed.

- Note that under the assumption $v_T = v_\infty$, then for a given sequence of prices $\{r_t, w_t\}_{t=1}^{T}$ household problem can be solved backwards

- Suggests the following algorithm.

Algorithm

- Fix T
- Compute stationary equilibrium $\Phi_0, v_{0,} r_0, w_0, K_0$ associated with $\tau = \tau_0 = 0$
- Compute stationary equilibrium $\Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$ associated with $\tau_\infty = \tau$. Assume that

$$\Phi_T, v_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$$

- Guess sequence of capital stocks $\{\hat{K}_t\}_{t=1}^{T-1}$ Note that capital stock at time $t = 1$ is determined by decisions at time 0, $\hat{K}_1 = K_0$. Note that $L_t = L_0 = \bar{L}$ is fixed. We also obtain

$$\begin{aligned} \hat{w}_t &= F_L(\hat{K}_t, \bar{L}) \\ \hat{r}_t &= F_K(\hat{K}_t, \bar{L}) - \delta \\ \hat{T}_t &= \tau_t \hat{r}_t \hat{K}_t. \end{aligned}$$

- Since we know $v_T(a, y)$ and $\{\hat{r}_t, \hat{w}_t, \hat{T}_t\}_{t=1}^{T-1}$ we can solve for $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$ backwards.

Algorithm

- With policy functions $\{\hat{a}_{t+1}\}$ define transition laws $\{\hat{\Gamma}_t\}_{t=1}^{T-1}$. We know $\Phi_0 = \Phi_1$ from the initial stationary equilibrium. Iterate the distributions forward
$$\hat{\Phi}_{t+1} = \hat{\Gamma}_t(\hat{\Phi}_t)$$
for $t = 1, \ldots, T-1$.
- With $\{\hat{\Phi}_t\}_{t=1}^{T}$ we can compute, for $t = 1, \ldots, T$.
$$\hat{A}_t = \int a d\hat{\Phi}_t$$
- Check whether
$$\max_{1 \leq t < T} \left| \hat{A}_t - \hat{K}_t \right| < \varepsilon$$
If yes, go to next step. If not, adjust your guesses for $\{\hat{K}_t\}_{t=1}^{T-1}$
- Check whether $\left\| \hat{\Phi}_T - \Phi_T \right\| < \varepsilon$. If yes, the transition converges smoothly into the new steady state and we are done and should save $\{\hat{v}_t, \hat{a}_{t+1}, \hat{c}_t, \hat{\Phi}_t, \hat{r}_t, \hat{w}_t, \hat{K}_t\}$. If not, increase T.

Welfare Consequences of the Policy Reform

- This procedure determines aggregate variables such as r_t, w_t, Φ_t, K_t and individual decision rules c_r, a_{t+1}.
- The value functions enable us to make statements about the welfare consequences of the tax reform.
- We have value functions $\{v_t\}_{t=0}^T$.
- Interpretation of the value functions: $v_0(a, y)$, $v_1(a, y)$ and $v_T(a, y) = v_\infty(a, y)$.
- In principle we can use v_0, v_1 and v_T to determine the welfare consequences from the reform. But: utility is an ordinal concept.

Consumption Equivalent Variation

- Suppose that
$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$
- Optimal consumption allocation in initial stationary equilibrium, in sequential formulation, $\{c_s\}_{s=0}^{\infty}$.
$$v_0(a, y) = E_0 \sum_{s=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
- If increase consumption in each date, in each state, in the old stationary equilibrium, by a fraction g. Then $\{(1+g)c_s\}_{s=0}^{\infty}$ and
$$\begin{aligned} v_0(a, y; g) &= E_0 \sum_{s=0}^{\infty} \beta^t \frac{[(1+g)c_t]^{1-\sigma}}{1-\sigma} = (1+g)^{1-\sigma} E_0 \sum_{s=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ &= (1+g)^{1-\sigma} v_0(a, y) \end{aligned}$$
- By what percent g do we have to increase consumption in the old stationary equilibrium for agent to be indifferent between old stationary equilibrium and transition induced by policy reform.

Consumption Equivalent Variation

- This percent g solves

$$\begin{aligned} v_0(a, y; g) &= v_1(a, y) \\ g(a, y) &= \left[\frac{v_1(a, y)}{v_0(a, y)}\right]^{\frac{1}{1-\sigma}} - 1 \end{aligned}$$

- $g(a, y) > 0$ iff $v_1(a, y) > v_0(a, y)$. $g(a, y)$ varies by (a, y).
- Steady state welfare gain (of agent being born with (a, y) into new as opposed to old stationary equilibrium.):

$$g_{ss}(a, y) = \left[\frac{v_T(a, y)}{v_0(a, y)}\right]^{\frac{1}{1-\sigma}} - 1$$

- Define as expected steady state welfare gain

$$g_{ss} = \left[\frac{\int v_T(a, y) d\Phi_T}{\int v_0(a, y) d\Phi_0}\right]^{\frac{1}{1-\sigma}} - 1$$

- For these measures need not compute transition path. But: welfare measures based on steady state comparisons may be misleading.

Examples of Policy-Induced Transition Analysis

- Transition from PAYGO to fully funded social security system. Conesa and Krueger, RED 1999.
- Transition from economy with positive capital income taxes to economy with $\tau_k = 0$ (and corresponding increase in τ_l or τ_c). Domeij and Heathcote, IER 2004.

Life Cycle Model with Social Security

- Recursive formulation of household problem in transition

$$
\begin{aligned}
v_t(a, y, j) &= \max_{c, a' \geq 0, \ell \in [0,1]} u(c, \ell) + \beta \psi_j \sum_{y' \in Y} \pi(y'|y) v_{t+1}(a', y', j+1) \\
\text{s.t. } c + a' &= I(j)(1-\tau_t) w_t \epsilon_j y \ell + (1+r_t)(a + Tr_t) + (1 - I(j)) SS_t
\end{aligned}
$$

- Age indicator $I(j) = 1$ for all $20 \leq j < j_r$ and $I(j) = 0$ for $j \geq j_r$.
- Life cycle elements:
  - Mortality risk $1 - \psi_j$
  - Deterministic life cycle productivity profile ϵ_j
  - Pay payroll taxes τ_t if $I(j) = 1$; receiving benefits SS_t if $I(j) = 0$.

Information Needed for Household Problem

- Factor prices

$$
\begin{aligned}
r_t &= F_K(K_t, L_t) - \delta \\
w_t &= F_L(K_t, L_t)
\end{aligned}
$$

- Market clearing

$$
\begin{aligned}
L_t &= \int \epsilon_j \ell_t(a, y, j) d\Phi_t(a, y, j) \\
K_{t+1} &= \int a_t'(a, y, j) d\Phi_t(a, y, j)
\end{aligned}
$$

- Transfers

$$
Tr_{t+1} = \frac{\int (1 - \psi_j) a_t'(a, y, j) d\Phi_t(a, y, j)}{\int d\Phi_{t+1}(a, y, j)}
$$

Social Security System

- Social security: size parameterized by replacement rate $b_t \in \{0, 0.5\}$.
- Benefits equal to b_t times average earnings

$$SS_t = b_t \frac{w_t L_t}{\int_{\{j<j_r\}} d\Phi_t(a,y,j)}$$

- Budget balance of the system

$$\tau_t = b_t \frac{\int_{\{j \geq j_r\}} d\Phi_t(a,y,j)}{\int_{\{j<j_r\}} d\Phi_t(a,y,j)} = \frac{SS_t \int_{\{j \geq j_r\}} d\Phi_t(a,y,j)}{w_t L_t}$$

- Tax rate equal to replacement rate times old-age dependency ratio.

Steady State Analysis: Results for Social Security Reform

TABLE V
Steady-State Results

Var.	No heterogeneity		Het. (sym. case)		Het. (asym. case)	
	In. St.St.	Fi. St.St.	In. St.St.	Fi. St.St.	In. St.St.	Fi. St.St.
b	50%	0%	50%	0%	50%	0%
r	6.0%	4.9%	5.5%	4.3%	3.4%	2.0%
w	1.18	1.25	1.21	1.30	1.36	1.49
h	32.8%	34.5%	31.3%	33.2%	29.4%	31.0%
K/Y	2.98	3.30	3.12	3.51	3.84	4.49
y	1.04	1.17	1.08	1.22	1.31	1.51
SS/y	38.9%	0	38.9%	0	38.9%	0
$cv(lab)$	0.52	0.51	0.71	0.68	1.39	1.38
$cv(weal)$	0.81	0.93	0.92	0.94	1.17	1.58
EV^{SS}	—	12.7%	—	12.8%	—	11.2%

Transition Analysis: Results for Social Security Reform

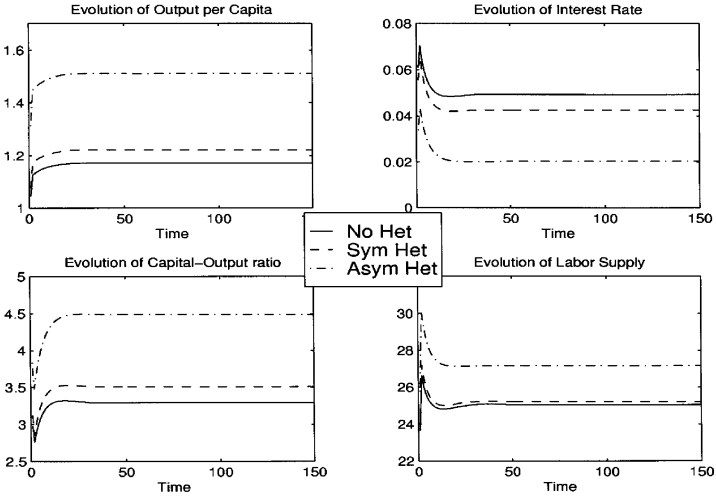


FIG. 3. Evolution of macroeconomic aggregates: reform A.

Political Economy Results for Social Security Reform

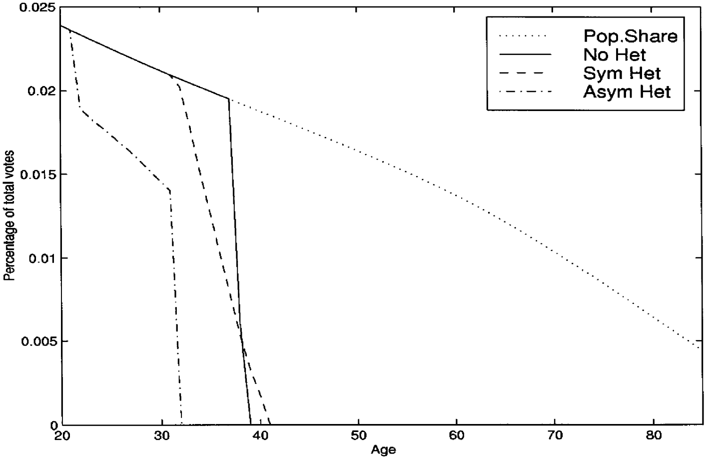


FIG. 4. Votes in favor of reform A.

Political Economy Results for Social Security Reform

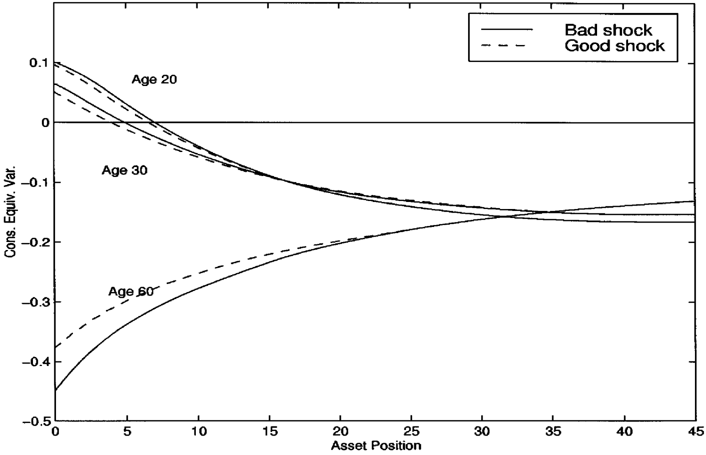


FIG. 6. Welfare effects of reform A: symm. heterogeneity.

Steady State Analysis: Results for Tax Reform

TABLE 3

AGGREGATE PROPERTIES OF INITIAL AND FINAL STEADY STATES: NEW $\tau^k = 0$

	Benchmark		No-Earnings-Risk		Endogenous Labor		Cons. Tax
	Initial	Final	Initial	Final	Initial	Final	Final
τ^k	0.397*	0.000*	0.397*	0.000*	0.397*	0.000*	0.000*
τ^n	0.269*	0.350	0.269*	0.358	0.269*	0.361	0.269* $\tau^c = 0.093$
G/Y	0.216	0.196	0.219	0.194	0.217	0.200	0.196
B/Y	0.670*	0.918	0.670*	0.965	0.670*	0.957	0.975
K/Y	3.36	3.98	3.00	3.74	3.31	3.89	4.01
Y	0.647	0.711	0.606	0.687	0.681	0.736	0.714
R (% post-tax)	2.84	3.05	3.63	3.63	2.94	3.25	2.99
Post-tax asset to labor income ratio	0.245	0.359	0.284	0.415	0.250	0.385	0.318

NOTE: Starred values indicate exogenous parameters. "Cons. tax" denotes the experiment when capital taxes are replaced with consumption taxes rather than labor income taxes in the benchmark (exogenous labor) economy.

Welfare Effects of Tax Reform

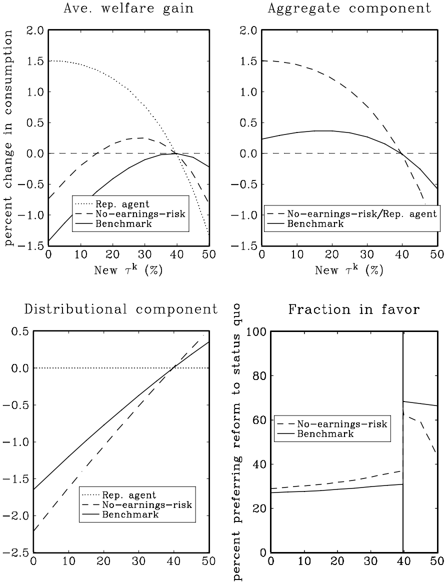


FIGURE 1

Welfare Effects of Tax Reform

TABLE 4

AGGREGATE WELFARE EFFECTS OF TAX REFORMS: NEW $\tau^k = 0$

	Benchmark	No-Earnings-Risk	Rep. Agent	Endog. Labor	Endog. Labor, No-Earnings-Risk	Cons. Tax	Cons. Tax, No-Earnings-Risk
Welfare gain	−1.42	−0.73	1.50	−3.04	−2.11	−0.41	0.55
Aggregate component	0.23	1.50	1.50	−1.18	0.86	0.21	1.50
Distributional component	−1.65	−2.23	0.00	−1.86	−2.97	−0.62	−0.95
Fractions in favor of reform:							
Low prod.	21.2			19.3		18.2	
Medium prod.	24.5			22.7		24.6	
High prod.	86.6			79.9		100.0	
Entire pop.	27.1	29.0	100.0	25.3	27.1	27.7	34.8

NOTE: For each experiment in model economies with heterogenous agents we report results with and without idiosyncratic productivity shocks. In each case the second version is labelled, "no-earnings-risk."

Welfare Effects of Tax Reform

TABLE 5

EXPECTED WELFARE GAIN FROM TAX REFORMS: NEW $\tau^k = 0$

		Wealth		
	Productivity	Zero	Median	Mean
	Low	−4.88	−4.14	1.64
Benchmark	Medium	−4.75	−4.43	0.67
	High	−2.09	−1.93	1.02
No-earnings-risk		−4.18	−3.87	1.50 (rep. agent)
	Low	−6.70	−5.48	1.16
Endog. labor	Medium	−6.55	−6.15	0.02
	High	−3.83	−3.67	−0.68
Endog. labor, no-earnings-risk		−5.62	−5.26	0.86 (rep. agent)
	Low	−2.03	−1.87	0.41
Consumption tax	Medium	−1.90	−1.80	0.34
	High	0.55	0.61	1.78
Consumption tax, no-earnings-risk		−1.00	−0.86	1.50 (rep. agent)

Transitional Dynamics of Tax Reform

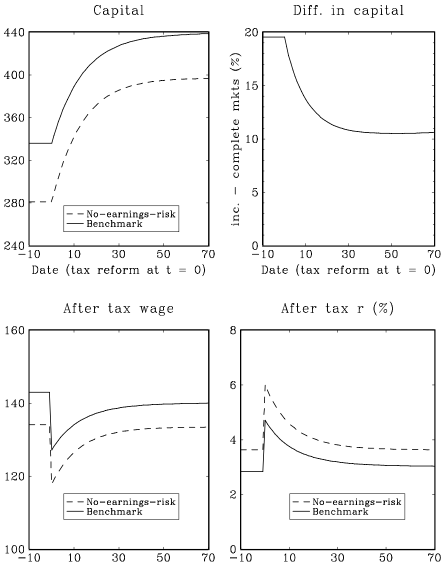


FIGURE 2

AGGREGATE CAPITAL AND AFTER-TAX PRICES: τ^k FROM 39.7% TO 0.0%

Aggregate Risk and Distributions as State Variables

- Why complicate the model? Want to talk about economic fluctuations, and its interaction with idiosyncratic uncertainty
- But: now we have to characterize and compute entire recursive equilibrium; distribution as state variable
- Note: no theoretical results about existence, uniqueness, stability, goodness of approximation

The Model

- Aggregate production function

$$Y_t = s_t F(K_t, L_t)$$

where $\{s_t\}$ is a sequence of random variables

- Let

$$s_t \in \{s_b, s_g\} = S$$

with $s_b < s_g$ and conditional probabilities $\pi(s'|s)$.

- s_b as an economic recession and s_g as an expansion
- The distribution of idiosyncratic labor productivity y_t is correlated with aggregate productivity s_t.

$$y_t \in \{y_u, y_e\} = Y$$

where $y_u < y_e$ stands for the agent being unemployed and y_e stands for the agent being employed.

- Probability of being unemployed is higher during recessions than during expansions

The Model

- Let
$$\pi(y', s'|y, s) > 0$$
be conditional probability of individual productivity y', aggregate state s' tomorrow, conditional on (y, s) today. π is a 4×4 matrix.
- Consistency requires that
$$\sum_{y' \in Y} \pi(y', s'|y, s) = \pi(s'|s) \text{ all } y \in Y, \text{ all } s, s' \in S$$
- Law of large numbers: idiosyncratic risk averages out, only aggregate risk determines number of agents in states $y \in Y$.
- *Assume* that, cross-sectionally, the fraction of the population in idiosyncratic state $y = y_u$ is only a function of the aggregate state s. Denote the cross-sectional distribution by $\Pi_s(y)$.
- This assumption imposes additional restrictions on $\pi(y', s'|y, s)$:
$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

Recursive Formulation

- Individual state variables (a, y)
- Aggregate state variables (s, Φ)
- Recursive formulation of household problem

$$v(a, y, s, \Phi) = \max_{c, a' \geq 0} \{U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', \Phi')\}$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s, \Phi) y + (1 + r(s, \Phi)) a \\ \Phi' &= H(s, \Phi, s') \end{aligned}$$

Definition

A RCE is value function $v : Z \times S \times \mathcal{M} \to R$, household policy functions $c, a' : Z \times S \times \mathcal{M} \to R$, firm policy functions $K, L : S \times \mathcal{M} \to R$, pricing functions $r, w : S \times \mathcal{M} \to R$, aggregate law of motion $H : S \times \mathcal{M} \times S \to \mathcal{M}$ s.t.

1. v, a', c are measurable wrt $\mathcal{B}(S)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$
$$\begin{aligned} r(s,\Phi) &= F_K(K(s,\Phi), L(s,\Phi)) - \delta \\ w(s,\Phi) &= F_L(K(s,\Phi), L(s,\Phi)) \end{aligned}$$
3. For all $\Phi \in \mathcal{M}$ and all $s \in S$
$$\begin{aligned} K(H(s,\Phi)) &= \int a'(a,y,s,\Phi)d\Phi \\ L(s,\Phi) &= \int y d\Phi \\ \int c(a,y,s,\Phi)d\Phi + \int a'(a,y,s,\Phi)d\Phi &= F(K(s,\Phi), L(s,\Phi)) + (1-\delta)K(s,\Phi) \end{aligned}$$
4. Aggregate law of motion H is generated by exogenous Markov chain π and policy function a'

Transition Function and Law of Motion

- Define $Q_{\Phi,s,s'} : Z \times \mathcal{B}(Z) \to [0,1]$ by

$$Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y',s'|y,s) \text{ if } a'(a,y,s,\Phi) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

- Aggregate law of motion

$$\Phi'(\mathcal{A},\mathcal{Y}) = \left(H(s,\Phi,s')\right)(\mathcal{A},\mathcal{Y}) = \int Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy)$$

Comments

- One should first define a sequential markets equilibrium. Such an equilibrium should in general exist (extension of Duffie-Shafer, unpublished?), but it is impossible to compute such an equilibrium directly. No claim of uniqueness of a sequential markets equilibrium can be made.
- Recursive equilibrium defined above generates sequential equilibrium: from the initial condition (s_0, Φ_0) generate distributions

$$\begin{aligned} \Phi_1(s^1) &= H(s_0, \Phi_0, s_1) \\ \Phi_{t+1}(s^{t+1}) &= H(s_t, \Phi_t(s^t)) \end{aligned}$$

prices

$$\begin{aligned} r_t(s^t) &= r(s_t, \Phi_t(s^t)) \\ w_t(s^t) &= w(s_t, \Phi_t(s^t)) \end{aligned}$$

Comments

- Starting from initial conditions (a_0, y_0), find households' allocations

$$\begin{aligned} c_0(a_0, y_0, s_0) &= c(a_0, y_0, s_0, \Phi_0) \\ a_1(a_0, y_0, s_0) &= a'(a_0, y_0, s_0, \Phi_0) \end{aligned}$$

  and recursively

$$\begin{aligned} c_t(a_0, y^t, s^t) &= c(a_t(a_0, y^{t-1}, s^{t-1}), y_t, s_t, \Phi_t) \\ a_{t+1}(a_0, y^t, s^t) &= a'(a_t(a_0, y^{t-1}, s^{t-1}), y_t, s_t, \Phi_t) \end{aligned}$$

  Can verify that this constitutes a sequential equilibrium.
- Existence of a recursive equilibrium in which the aggregate state only contains the current shock and the current wealth distribution? Not known. Neither is uniqueness.
- Will compute approximate equilibrium with boundedly rational agents (where approximation is not just due to numerical error). No sense as to whether this equilibrium is close to a true recursive equilibrium.

Computation of the Recursive Equilibrium

- Key computational problem: size of the state space, since aggregate wealth distribution Φ is an infinite-dimensional object
- Why do agents need to keep track of the aggregate wealth distribution? In order to forecast future capital stock and thus future prices. But for K' need entire Φ since

$$K' = \int a'(a, y, s, \Phi) d\Phi$$

- Note; if a' were linear in a, with same slope for all $y \in Y$, exact aggregation obtained and average capital stock today *is* sufficient statistic for the average capital stock tomorrow
- Trick: Approximate distribution Φ with a finite set of moments.
- Let n-dimensional vector m denote first n moments of asset distribution
- Agents use an approximate law of motion

$$m' = H_n(s, m)$$

Computation of the Recursive Equilibrium

- Note that agents are boundedly rational in the sense that moments of higher order than n of the current wealth distribution may help to more accurately forecast the first n moments tomorrow
- Now choose the number of moments and the functional form of the function H_n. Krusell and Smith pick $n = 1$ and pose

$$\log(K') = a_s + b_s \log(K)$$

for $s \in \{s_b, s_g\}$. Here (a_s, b_s) are parameters that need to be determined. Household problem

$$v(a, y, s, K) = \max_{c, a' \geq 0} \{U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', K')\}$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s, K)y + (1 + r(s, K))a \\ \log(K') &= a_s + b_s \log(K) \end{aligned}$$

- Reduction of the state space to a four dimensional space $(a, y, s, K) \in \mathbf{R} \times Y \times S \times \mathbf{R}$.

Algorithm

1. Guess (a_s, b_s)
2. Solve households problem to obtain $a'(a, y, s, K)$
3. Simulate for large number of T periods, large number N of households
   - Initial conditions for economy (s_0, K_0), for each household (a_0^i, y_0^i).
   - Draw random sequences $\{s_t\}_{t=1}^{T}$ and $\{y_t^i\}_{t=1,i=1}^{T,N}$, use decision rule $a'(a, y, s, K)$, perceived law of motion for K to generate $\{a_t^i\}_{t=1,i=1}^{T,N}$.
   - Aggregate

$$K_t = \frac{1}{N} \sum_{i=1}^{N} a_t^i$$

4. Run the regressions

$$\log(K') = \alpha_s + \beta_s \log(K)$$

to estimate (α_s, β_s) for $s \in S$. If the R^2 for this regression is high and $(\alpha_s, \beta_s) \approx (a_s, b_s)$ stop. An approximate equilibrium is found. Otherwise update guess for (a_s, b_s). If guesses for (a_s, b_s) converge, but R^2 remains low, add higher moments to the aggregate law of motion and/or use different functional form.

Calibration

- Model period 1 quarter (business cycle model)
- $CRRA$ utility with $\sigma = 1$ (i.e. log-utility)
- The time discount factor $\beta = 0.99^4 = 0.96$, i.e. $\rho = 4.1\%$
- Capital share $\alpha = 0.36$
- Annual depreciation rate $\delta = (1 - 0.025)^4 - 1 = 9.6\%$

Income Process

- Aggregate component first. Two state process; interpreted as expansion and recession

$$S = \{0.99, 1.01\}$$

Stdev. of technology shock is $\sigma_s = 0.01$

- Transition matrix symmetric

$$\pi(s_g|s_g) = \pi(s_b|s_b)$$

- Expected time in good state and bad state 8 quarters, hence

$$\begin{aligned} 8 &= [1 - \pi(s_g|s_g)]\left[1 + 2\pi(s_g|s_g) + 3\pi(s_g|s_g)^2 + \ldots\right] \\ \pi(s_g|s_g) &= \frac{7}{8} \end{aligned}$$

- Thus

$$\pi(s'|s) = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

Income Process

- Idiosyncratic component: Two state process; interpreted as employment and unemployment
- $$Y = \{0.25, 1\}$$

  Unemployed person makes 25% of the labor income of an employed person
- Transition probabilities: note that

  $$\pi(y'|s', y, s) = \frac{\pi(y', s'|y, s)}{\pi(s'|s)}$$

  or

  $$\pi(y', s'|y, s) = \pi(y'|s', y, s) * \pi(s'|s)$$
- Specify the four 2×2 matrices $\pi(y'|s', y, s)$ indicating, conditional on an aggregate transition from s to s', what the individual's probabilities of transition from employment to unemployment are.

Income Process

- Expansion: average time of unemployment equal to 1.5 quarters.

$$\begin{aligned} 1.5 &= \left[1 - \pi(y' = y_u | s' = s_g, y = y_u, s = s_g)\right] * \\ \sum_{i=1}^{\infty} i * \pi(y' &= y_u | s' = s_g, y = y_u, s = s_g)^{i-1} \\ \pi(y' &= y_u | s' = s_g, y = y_u, s = s_g) = \frac{1}{3} \end{aligned}$$

Hence $\pi(y' = y_e | s' = s_g, y = y_u, s = s_g) = \frac{2}{3}$

- Recession: average time of unemployment equal to 2.5 quarters

$$\begin{aligned} \pi(y' &= y_u | s' = s_b, y = y_u, s = s_b) = 0.6 \\ \pi(y' &= y_e | s' = s_b, y = y_u, s = s_b) = 0.4 \end{aligned}$$

Income Process

- Probability of remaining unemployed after switch from expansion to recession is 1.25 times the same probability when the economy was already in a recession

$$\pi(y' = y_u | s' = s_b, y = y_u, s = s_g) = 0.75$$
$$\pi(y' = y_e | s' = s_b, y = y_u, s = s_g) = 0.25$$

- Probability of remaining unemployed after switch from recession to expansion is 0.75 times the same probability when times were already good.

$$\pi(y' = y_u | s' = s_g, y = y_u, s = s_b) = 0.25$$
$$\pi(y' = y_e | s' = s_g, y = y_u, s = s_b) = 0.75$$

Income Process

- Recessions: unempl. rate $\Pi_{s_b}(y_u) = 10\%$. Expansions: $\Pi_{s_g}(y_u) = 4\%$.
- Consistency with aggregate transition probabilities requires

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

- This gives

$$\begin{aligned}
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_g) = 0.028 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_g) = 0.972 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_b) = 0.04 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_b) = 0.96 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_g) = 0.079 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_g) = 0.921 \\
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_b) = 0.02 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_b) = 0.98
\end{aligned}$$

- Best times for finding job when economy moves from recession to boom, worst chances when economy moves from boom into recession.

Transition Matrix for Productivity

$$\pi = \begin{pmatrix} 0.525 & 0.035 & 0.09375 & 0.0099 \\ 0.35 & 0.84 & 0.03125 & 0.1151 \\ 0.03125 & 0.0025 & 0.292 & 0.0245 \\ 0.09375 & 0.1225 & 0.583 & 0.8505 \end{pmatrix}$$

Numerical Results: Model Delivers

1. Aggregate law of motion

$$m' = H_n(s, m)$$

2. Individual decision rules

$$a'(a, y, s, m)$$

3. Time-varying cross-sectional wealth distributions

$$\Phi(a, y)$$

Aggregate Law of Motion

- Agents are boundedly rational: aggregate law of motion perceived by agents may not coincide with actual law of motion
- Only thing to forecast is K'. hence try $n = 1$
- Converged Law of Motion

$$\begin{aligned} \log(K') &= 0.095 + 0.962 \log(K) \text{ for } s = s_g \\ \log(K') &= 0.085 + 0.965 \log(K) \text{ for } s = s_b \end{aligned}$$

- How irrational are agents? Use simulated time series for aggregate capital stock with sequence of aggregate shocks $\{(s_t, K_t\}_{t=0}^{T}$. Divide sample into periods with $s_t = s_b$ and $s_t = s_g$ and run

$$\log(K_{t+1}) = \alpha_j + \beta_j \log(K_t) + \varepsilon_{t+1}^j$$

- Define

$$\hat{\varepsilon}_{t+1}^j = \log(K_{t+1}) - \hat{\alpha}_j - \hat{\beta}_j \log(K_t) \text{ for } j = g, b$$

Aggregate Law of Motion

- Define

$$\sigma_j = \left(\frac{1}{T_j} \sum_{t \in \tau_j} \left(\hat{\varepsilon}_t^j \right)^2 \right)^{0.5}$$

$$R_j^2 = 1 - \frac{\sum_{t \in \tau_j} \left(\hat{\varepsilon}_t^j \right)^2}{\sum_{t \in \tau_j} \left(\log K_{t+1} - \log \bar{K} \right)^2}$$

- If $\sigma_j = 0$ for $j = g, b$ (if $R_j^2 = 1$ for $j = g, b$) then agents do not make forecasting errors
- Results

$$R_j^2 = 0.999998 \text{ for } j = b, g$$
$$\sigma_g = 0.0028$$
$$\sigma_b = 0.0036$$

- Maximal forecasting errors for interest rates 25 years into the future is 0.1% Corresponding utility losses?

Approximate Equilibria

- In all of computational economics, since computer precision is limited, the best one can achieve is to compute an approximate equilibrium, i.e. allocations and prices in which markets almost clear (excess demand is ε away from zero).
- This approximate equilibrium may be arbitrarily far away from the true equilibrium (i.e. prices and allocations for which markets clear exactly)
- Even if first problem were absent: here only approximation to a rational expectations equilibrium
- True rational expectations equilibrium may be arbitrarily far away from the computed equilibrium

Why Quasi-Aggregation?

- Suppose all agents have linear savings functions with same marginal propensity to save

$$a'(a, y, s, K) = a_s + b_s a + c_s y$$

- Then

$$\begin{aligned} K' &= \int a'(a, y, s, K) d\Phi = a_s + b_s \int a d\Phi + c_s \bar{L} \\ &= \tilde{a}_s + b_s K \end{aligned}$$

- Exact aggregation obtains: first moment of the current wealth distribution is a sufficient statistic for Φ for forecasting aggregate capital stock tomorrow.
- In this economy: savings functions almost linear with same slope for $y = y_u$ and $y = y_e$
- Only exceptions: unlucky ($y = y_u$) liquidity constrained agents. But these agents hold negligible fraction of aggregate wealth and don't matter for aggregate capital dynamics.
- Hence quasi-aggregation!!!

Policy Functions

INCOME AND WEALTH HETEROGENEITY 879

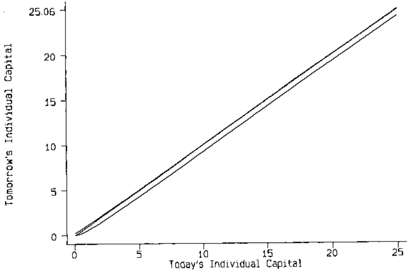


FIG. 2.—An individual agent's decision rules (benchmark model, aggregate capital = 11.7, good aggregate state).

Why is Marginal Propensity to Save Close to 1

- PILCH model with certainty equivalence and $\rho = r$

$$c_t = \frac{r}{1+r}\left(E_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} + a_t\right)$$

- Agents save out of current assets for tomorrow

$$\frac{a_{t+1}}{1+r} = \left(1 - \frac{r}{1+r}\right) a_t + G(y)$$

- Thus under certainty equivalence

$$a_{t+1} = a_t + H(y)$$

- Here agents are prudent, face liquidity constraints, but almost act as if they are certainty equivalence consumers. Why?

1. With $\sigma = 1$ agents are prudent, but not all that much
2. Unconditional standard deviation of individual income is roughly 0.2, at the lower end of the estimates used by Aiyagari
3. Negative income shocks infrequent, not very persistent.

Wealth Distribution

- Endogenous wealth distribution
- Income distribution is input into the model
- Does realistic income process lead to realistic wealth distributions?
- No: it fails to generate the high concentration of wealth at the upper end of the distribution

	1%	5%	10%	20%	30%	Gini
Data	30	51	64	79	88	0.79
Model	3	11	19	35	46	0.25

Trick: Stochastic Discount Factors

- Assume that β is stochastic and follows a three state Markov chain with $\beta \in B$ and $\gamma(\beta'|\beta)$
- Bellman equation

$$v(a, y, \beta, s, K) = \max_{c,a' \geq 0} \left\{ U(c) + \beta \sum_{y',s',\beta'} \pi(y', s'|y, s)\gamma(\beta'|\beta)v(a', y', \beta', \right.$$

$$\begin{array}{rcl} & & s.t. \\ c + a' & = & w(s, K)y + (1 + r(s, K))a \\ \log(K') & = & a_s + b_s \log(K) \end{array}$$

Trick: Stochastic Discount Factors

- Annual discount rates differ between 2.8% for the most patient agents and 5.6% for the most impatient agents:

$$B = \{0.9858, 0.9894, 0.993\}$$

- Transition matrix

$$\gamma = \begin{pmatrix} 0.995 & 0.005 & 0 \\ 0.000625 & 0.99875 & 0.000625 \\ 0 & 0.005 & 0.995 \end{pmatrix}$$

- Note: 80% of population has discount factor in the middle and 10% are on either of the two extremes
- Patient and impatient agents remain so for 50 years in expectation
- De facto three deterministic types of agents
- Patient agents dominate the wealth distribution

	1%	5%	10%	20%	30%	Gini
Data	30	51	64	79	88	0.79
Model	3	11	19	35	46	0.25
Model with $\tilde{\beta}$	24	55	73	88	92	0.82

Standard Incomplete Markets Models with Consumer Default

- Basic idea: incorporate household default on uncollateralized debt into standard SIM model.
- Institutional motivation: chapter 7 of U.S. bankruptcy code.
- Discussion based on simplified version of Chatterjee et. al., (EC, 2007).
- Focus is on explaining main mechanism of the model, not to present the most general version of Chatterjee et al. (2007).

Model: Overview

- Starting point: standard SIM GE model as in Aiyagari (1994).
- Production firms completely standard.
- Continuum (of measure 1) of infinitely lived households solve standard income fluctuation problem, can borrow at a loan-size dependent interest rate schedule, can default.
- Competitive financial intermediaries extend loans, break even after accounting for losses from default.
- Loan interest schedule determined endogenously in equilibrium.

Default Option and Consequences for Households

- Bankruptcy flag $h \in \{0, 1\}$. Summarizes past default behavior. h is a state variable.
- Current bankruptcy decision $d \in \{0, 1\}$.
- Consequence of $d = 1$: Can't save (and of course can't borrow) in current period.
- Consequence of $h = 1$:
  - Can't borrow.
  - Fraction $\gamma \in (0, 1)$ of income lost.
- State Transitions
  - If $h = 0$, then household can choose $d \in \{0, 1\}$. If $d = 0$, then $h' = 0$. If $d = 1$, then $h' = 1$.
  - If $h = 1$, then $d \in \{0\}$. With probability λ, $h' = 1$. With probability $1 - \lambda$, $h' = 0$.

Household Problem in Recursive Formulation

- Individual State variables (a, y, h).
- Assets $a \in A = \mathbb{R}$.
- Exogenous idiosyncratic income $y \in Y = [y_{\min}, y_{\max}]$, *iid* over time, identically distributed across households, with measure $\pi(.)$.
- Assume $y_{\min} > 0$ and $y_{\max} < \infty$ and a law of large numbers.
- Bankruptcy flag $h \in H = \{0, 1\}$.
- Population distribution over states (a, y, h) described by measure Φ over measurable space $(Z, \mathcal{B}(Z))$ where $Z = A \times Y \times H$ and $\mathcal{B}(Z) = \mathcal{B}(A) \times \mathcal{B}(Y) \times \mathcal{P}(H)$.

Household Budget Sets

- Household take wage w, loan price function $q(a, y, h; a')$ as given.
- Choose consumption c, assets (loans if negative) a' as well as d.
- Budget set of admissible (c, a') depends on (a, y, h) and on $d \in \{0, 1\}$.
- If $h = 0$ and $d = 0$:

$$\begin{aligned} B(a, y, h &= 0; d = 0) = \{c \geq 0, a' \in A : \\ c + q(a, y, h; a')a' &\leq wy + a\}. \end{aligned}$$

- Note: for some $(a, y) \in A \times Y$ set is empty (household has to default).
- If $h = 0$ and $d = 1$:

$$B(a, y, h = 0; d = 1) = \{c \geq 0, a' = 0 : \ c \leq wy\}.$$

- If $h = 1$:

$$B(a, y, h = 1) = \{c \geq 0, a' \geq 0 : c + q(a, y, h; a')a' \leq (1 - \gamma)wy + a\}$$

Dynamic Programming Problem of Households

- Value function $v(a, y, h)$.
- For $h = 0$:
  - If $B(a, y, h = 0; d = 0) = \varnothing$, then "involuntary default and

$$v(a, y, h = 0) = \left\{ u(wy) + \beta \int v(0, y', h' = 1) d\pi(y') \right\}$$

  - If $B(a, y, h = 0; d = 0) \neq \varnothing$, then decision between repayment and "voluntary" (strategic) default:

$$v(a, y, h = 0) = \max_{d \in \{0,1\}} \max_{(c, a') \in B(., h, d=0)} u(c) + \beta \int v(a', y', h' = 0) d\pi(y'), u(wy) + \beta \int v(0, y', h' = 1) d\pi(y')$$

Dynamic Programming Problem of Households

- For $h = 1$:

$$v(a, y, h = 1) = \max_{(c,a') \in B(a,y,h=1;d=0)}$$

$$\left\{ u(c) + \beta \left[\lambda \int v(a', y', h' = 1) d\pi(y') + (1 - \lambda) \int v(a', y', h' = 0) d\pi(y') \right] \right\}$$

- Solution is a value function $v(a, y, h)$ and optimal policy functions $c(a, y, h), a'(a, y, h)$ and $d(a, y, h)$.

Production Firms

- Measure 1 of identical, competitive firms.
- Constant returns to scale technology $F(K, L)$
- Profit maximization implies

$$\begin{aligned} w &= F_L(K, L) \\ r &= F_K(K, L) - \delta \end{aligned}$$

Financial Intermediaries

- Representative financial intermediary owns the capital stock K and buys new capital K'. Earns net return r on capital.
- Issue loans. Perfect competition loan by loan type $(a, y, h; a')$
- Consider a loan of size $a' < 0$ to a type (a, y, h) household (if $a' \geq 0$ call it a deposit).
  - Price $q(a, y, h; a')$. If $a' < 0$, intermediary gives household $q(a, y, h; a')a'$ today for promise to repay a' tomorrow.
  - (Equilibrium) default probability $p(a, y, h; a')$
  - Number $n(a, y, h; a')$ of loans of type $(a, y, h; a')$.
- Maximization problem

$$\max_{n(a,y,h;a') \geq 0} \int_{(a,y,h;a')} n(a, y, h; a') * \left[q(a, y, h; a')a' - \frac{[1 - p(a, y, h; a')]\, a'}{1 + r} \right]$$

Definition

A Stationary RCE is a value function $v(a, y, h)$ and policy functions $c(a, y, h), a'(a, y, h), d(a, y, h)$ for the households, aggregate capital and labor (K, L), quantities of loan/deposit contracts $n(a, y, h; a')$, default frequencies $p(a, y, h; a')$, loan prices $q(a, y, h; a')$, factor prices (w, r) and a probability measure Φ such that:

1. Given $(w, q), v$ solves household problem, c, a', d are optimal policies.
2. Given (w, r), (K, L) satisfy the production firms' first order condition.
3. Given $(q, p, r), n$ solves the financial intermediary's problem.
4. Consistency of default probabilities: For all $(a, y, h; a')$:

$$p(a, y, h; a') = \int d(a', y', h' = 0)\pi(y'|y)dy'$$

5. Loan markets clearing: For all $(a, y, h; a')$:

$$n(a, y, h; a') = \mathbf{1}_{\{a'(a,y,h)=a'\}}\Phi(a, y, h)$$

6. Labor/capital/goods market clearing:

$$L = \int yd\Phi = \int yd\pi$$

$$K = \int \int q(a, y, h; a')a'n(a, y, h; a')da'd\Phi$$

$$\int c(a, y, h)d\Phi + \delta K = F(K, L) - \gamma w \int y\Phi(da, dy, h = 1)$$

Characterization of Household Default Decision

- Assume that a household indifferent between defaulting or nor decides to default (CCNR call this the maximum default set). Recall: default policy $d(a, y, h)$.
- Along h-dimension: for $h = 1$, default is not an option. For $h = 0$, there is a choice.
- Along y-dimension: for a given loan level a, define default set

$$D(a) = \{y \in Y : d(a, y, h = 0) = 1\}$$

Proposition

$D(a)$ *is either empty or a closed interval.*

Default Set

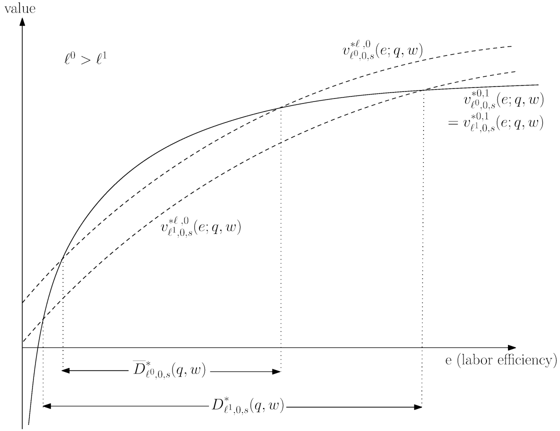


FIGURE 1.—Typical default sets conditional on household type.

Characterization of Household Default Decision

- Along the a-dimension:
  - If $a \geq 0$, then $D(a)$ is empty [it is never optimal to default on positive assets].
  - Let $\tilde{a} < a < 0$, then $D(a) \subseteq D(\tilde{a})$ [default set is expanding in indebtedness].
  - Implied default probability on a loan a' for a type $(a, y, h = 0)$:

$$p(a, y, h; a') = \pi(D(a')) = p(a')$$

Proposition

Since y' is iid, the default probability is just a function of loan size a'. Furthermore $p(a') = 0$ for all $a' \geq 0$.

Characterization of Equilibrium Loan Interest Rate Function

- Intermediaries' problem

$$\max_{n(a,y,h;a')\geq 0} \sum_{(a,y,h;a')} n(a,y,h;a')\left[q(a,y,h;a')a' - \frac{[1-p(a,y,h;a')]\,a'}{1+r}\right]$$

$$= \max_{n(a,y,h;a')\geq 0} \sum_{(a,y,h;a')} n(a,y,h;a')a'\left[q(a,y,h;a') - \frac{[1-p(a,y,h;a')]}{1+r}\right]$$

- Thus if $n(a,y,h;a') > 0$ then

$$q(a,y,h;a') = \frac{1-p(a,y,h;a')}{1+r}$$

and we assume that this is also the price for contracts not traded in equilibrium (for $n(a,y,h;a') = 0$).

The previous characterization of the household default set implies that (see their theorem 6):

1. Since $p(a, y, h; a') = p(a')$ we have $q(a, y, h; a') = q(a')$.
2. Since for all $a' \geq 0$ we have $p(a') = 0$, for those a'

$$q(a') = \frac{1}{1+r}$$

3. Since for $\tilde{a}' < a' < 0$, $p(\tilde{a}') > p(a')$ we have

$$q(\tilde{a}') \leq q(a')$$

   that is, loan interest rates are increasing in loan size.
4. There exists $\bar{a}' < 0$ small enough such that

$$q(\bar{a}') = 0.$$

   The loan size $\bar{a}'$ is an effective borrowing limit.

Augmenting the Model

- Constant probability of death ρ.
- Introduce persistent type s of households. Assume that s follows Markov chain with transitions $\pi(s'|s)$.
- Make income persistent by assuming and $y \sim \pi_s(y)$. Interpret s as partially capturing socioeconomic characteristics: model blue collar vs. white collar households.
- In model households default because of bad earnings shocks. In data they also default because of
  - Large medical bills: introduce nondiscretionary health spending shocks $\zeta(s)$.
  - Divorce: introduce preference shocks $\eta(s)$.

Bringing the Model to the Data

- Challenge is to simultaneously account for high debt levels and high default probabilities. Medical and divorce shocks necessary.
- Calibration to
  - Aggregate statistics: standard since production side is neoclassical growth model.
  - Earnings and wealth distributions: choose the appropriate earnings process.
  - Debt and bankruptcy statistics: choose medical and divorce shocks appropriately.

Quantitative Predictions of the Model

- Table 1: Reasons for filing for bankruptcy in data.
- Table 2: Extended version of the model is consistent with incidence of bankruptcy filings, average debt of those in debt.
- Figure 3: Model matches U.S. wealth distribution well.
- Figure 5, table 5: who defaults?
- Figure 6: implied loan prices (interest rates).

Reason Filing for Bankruptcy

REASONS (%) FOR FILING FOR BANKRUPTCY (PSID, 1984–1995)

1	Job loss	12.2
2	Credit misuse	41.3
3	Marital disruption	14.3
4	Health-care bills	16.4
5	Lawsuit/harassment	15.9

Source: Chakravarty and Rhee (1999)

Debt and Bankruptcy

TABLE II

DEBT AND BANKRUPTCY TARGETS FOR EACH MODEL ECONOMY

Economy	U.S.	Baseline	Extended
Reasons covered[a]	1–5	1, 2	1–5
Percent of all bankruptcies	100	53.5	100
Percent of households filing	0.54	0.29	0.54
Debt-to-income ratio (%)	0.67	0.36	0.67
Percent of households in debt	6.7	3.6	6.7

Wealth Distribution

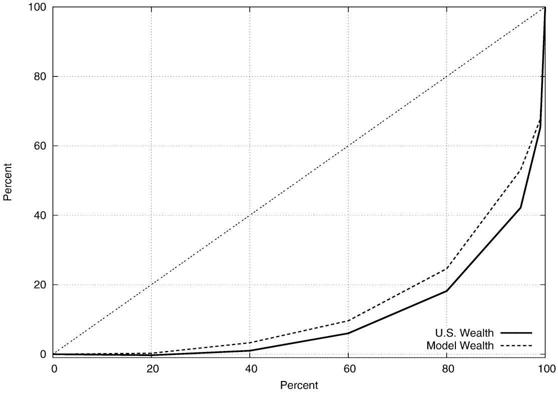


FIGURE 3.—Wealth distributions for the U.S. and extended model.

Default Probabilities in Model

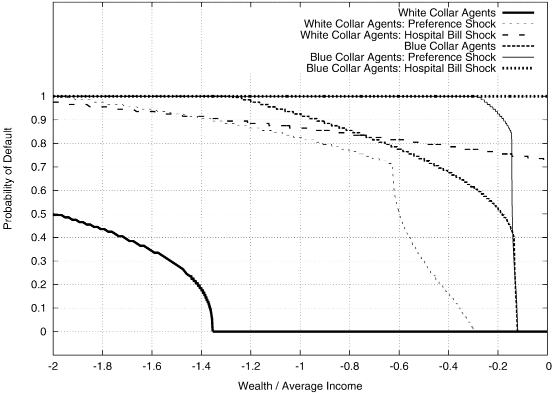


FIGURE 5.—Default probabilities by household types in the extended economy.

Earnings and Bankruptcy

TABLE V

EARNINGS AND BANKRUPTCIES: FRACTIONS (%) OF AGENTS THAT DEFAULT

	Over Total Population		Over Population in Debt		
Economy	Baseline	Extended	Baseline	Extended w/o Hosp.	Extended with Hosp.
1st quintile	0.48	0.75	9.2	10.7	13.7
2nd quintile	0.48	0.75	9.2	10.7	13.7
3rd quintile	0.48	0.75	9.2	10.7	13.6
4th quintile	0.03	0.36	4.2	7.0	10.0
5th quintile	0.00	0.09	0.0	0.4	3.2
Total	0.29	0.54	6.4	7.9	10.8

Loan Prices

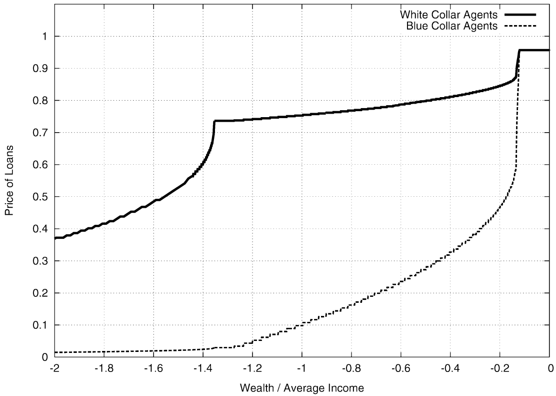


FIGURE 6.—Loan prices for blue and white collar households in the extended economy.

Complete Market Models with Frictions

- Standard Incomplete Markets Model permits only self insurance. Explicit insurance arrangements ruled out in ad hoc fashion.

- Standard Complete Markets Model: Full insurance. This prediction is rejected in the data.

- Potential Middle Ground: Complete Markets Models with micro-founded frictions:

  1. Imperfect enforceability of contracts

  2. Imperfect observability of incomes or actions

Empirical Tests of the Complete Consumption Insurance Hypothesis

- Mace (1991)
- Cochrane (1991)
- Townsend (1994)
- Attanasio and Davis (1996)
- Hayashi, Altonji and Kotlikoff (1996)
- Schulhofer-Wohl (2013)
- Mazzocco and Saini (2014)

Limited Enforceability of Contracts

- Main idea: at each point of time every agent can default on financial obligations, with *some* punishment. Punishment usually modeled as exclusion from future credit market participation.
- Initial Literature
  - International Macro: Eaton and Gersowitz (1981), Bulow and Rogoff (1989)
  - Labor: Harris and Holmstrom (1982), Thomas and Worrall (1988).
- Key Contributions in Macroeconomics: Kehoe and Levine (1993, 2001), Kocherlakota (1996) and Alvarez and Jermann (2000).

Plan of Action

1. Model with 2 (finitely many) types of agents
   1. Constrained-efficient allocations
   2. Decentralization: Kehoe and Levine (1993): Arrow Debreu equilibrium, Kocherlakota (1996): Subgame perfect equilibrium, Alvarez and Jermann (2000): Sequential markets equilibrium with state-dependent borrowing constraints that are not "too tight"
   3. Applications (consumption inequality, progressive income taxation): Krueger and Perri (2006, 2011).
2. Model with a continuum of agents.
3. Endogenizing the outside option: Krueger and Uhlig (2006, 2023), Cole, Krueger, Mailath and Park (2023).

The Model with 2 Agents

- 2 infinitely lived households, $i = 1, 2$
- Single perishable consumption good
- Symmetric endowments $e_t^i = e_t^i(s_t) > 0$, governed by random variable $s_t \in S$, a finite set
- History of shocks $s^t = (s_t, \dots s_1) \in S^t$
- Stochastic process is Markov with transition probabilities $\pi(s_{t+1}|s_t)$ and invariant distribution Π
- Fixed initial shock s_0 (or $s_0 \sim \Pi$) and

$$\pi(s^t) = \pi(s_t|s_{t-1}) * \dots \pi(s_1|s_0)$$

- Endowment processes are symmetric: if $e_t^1(s_t) = e_t^1(s^t) = e$, then there exists $\hat{s}^t$ with $\pi(s^t) = \pi(\hat{s}^t)$ and $e_t^2(\hat{s}_t) = e_t^2(\hat{s}^t) = e$
- Consumption allocation $(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}$

The Model with 2 Agents

- Preferences over consumption streams

$$U(c) = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

where $\beta \in (0,1)$ and u is strictly increasing, strictly concave and C^2 and satisfies INADA conditions.

- Example: $S = \{1, 2\}$ with income dispersion $\varepsilon \in (0,1)$

$$\begin{aligned} e_t^1(s_t &= 1) = 1+\varepsilon \text{ and } e_t^2(s_t = 1) = 1-\varepsilon \\ e_t^1(s_t &= 2) = 1-\varepsilon \text{ and } e_t^2(s_t = 2) = 1+\varepsilon \end{aligned}$$

and persistence $\delta \in [0,1]$

$$\pi = \begin{pmatrix} \delta & 1-\delta \\ 1-\delta & \delta \end{pmatrix} \text{ and } \Pi = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

and $u(c) = \log(c)$

Pareto Efficient Allocations

Definition

An allocation (c^1, c^2) is Pareto efficient if it is resource feasible

$$c^1 + c^2 = e^1 + e^2$$

for all t, all s^t, and there is no other feasible allocation $(\hat{c}^1, \hat{c}^2)$ such that $U(\hat{c}^i) \geq U(c^i)$, with at least one inequality strict.

- Note that an allocation is Pareto efficient if and only if it is resource feasible and satisfies

$$\frac{u'(c_t^1(s^t))}{u'(c_t^2(s^t))} = \gamma \text{ for all } t, \text{ all } s^t,$$

  for some $\gamma > 0$
- Strong incentives to share the endowment risk both agents face.
- Question: what is the extent of risk sharing possible if agents cannot commit to long term contracts that are not individually rational?

Constrained-Efficient Allocations

- Continuation lifetime expected utility from allocation (c^1, c^2) for agent i in node s^t

$$
\begin{aligned}
U(c^i, s^t) &= (1-\beta)u(c_t^i(s^t)) + \\
& (1-\beta)\sum_{\tau=t+1}^{\infty}\sum_{s^\tau|s^t}\beta^{\tau-t}\pi(s^\tau|s^t)u(c_\tau^i(s^\tau))
\end{aligned}
$$

- Individual rationality constraints

$$
U(c^i, s^t) \geq U(e^i, s^t) \equiv U^{i,Aut}(s_t)
$$

- Interpretation: at no point of time, no contingency any agent would prefer to walk away from the allocation (c^1, c^2), with consequence of living in financial autarky and consuming her own endowment from that node onward.

Constrained-Efficient Allocations

- Γ : set of all allocations that satisfy resource constraints and the individual rationality constraints

Definition

An allocation $(c^1, c^2) \in \Gamma$ is constrained efficient if there is no other feasible allocation $(\hat{c}^1, \hat{c}^2) \in \Gamma$ such that $U(\hat{c}^i) \geq U(c^i)$, with at least one inequality strict

- Note: individual rationality constraints have bite. For the example

$$U(c^1 = 1, s^t) < U^{1,Aut}(s_0 = 1).$$

  if $\delta = 1$. In fact, for this case only feasible and individually rational (and thus constrained-efficient) allocation is autarky.

- Note: these constraints are nasty: constraint at s^t involves consumption $c^i_{t+\tau}(s^{t+\tau})$ at all future dates and nodes. But Γ is a convex set (good for 2nd welfare theorem).

Constrained-Efficient Allocations: Questions

- How to characterize and compute constrained-efficient allocations? Thomas and Worrall (1988), Kocherlakota (1996)

- Why care about constrained efficient allocations? Can be decentralized as competitive equilibrium: Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000).

Characterization and Computation of Constrained-Efficient Allocations

- Bounds on lifetime utility:
  - Upper bound: $\bar{U}^i(s_0) = \max_{(c^1,c^2)\in\Gamma} U(c^i, s_0)$
  - Lower bound: $\underline{U}^i(s_0) = U^{i,Aut}(s_0)$
- Constrained Pareto Frontier
  $W_{s_0} : [\underline{U}^1(s_0), \bar{U}^1(s_0)] \rightarrow [\underline{U}^2(s_0), \bar{U}^2(s_0)]$

$$
\begin{aligned}
W_{s_0}(U) &= \max_{(c^1,c^2)\in\Gamma} U(c^2, s_0) \\
s.t.\ U(c^1, s_0) &\geq U
\end{aligned}
$$

Proposition

For fixed $s_0 \in S$, an allocation (c^1, c^2) is constrained efficient if and only if it solves the Pareto problem, for some $U \in [\underline{U}^1(s_0), \bar{U}^1(s_0)]$

Recursive Formulation of the Problem

- Basic idea: since the individual rationality constraints $U(c^i, s^t) \geq U^{i,Aut}(s_t)$ constrain continuation utilities of allocations, make continuation lifetime utility w (of agent 1) a state variable in the recursive problem.
- State variables $(w, s) \in \{[\underline{U}^1(s_0), \bar{U}^1(s_0)]\} \times S$
- Bellman equation

$$V(w, s) = \max_{c_1, c_2, \{w'(s')\}_{s' \in S}} \left\{ (1-\beta) u(c_2) + \beta \sum_{s' \in S} \pi(s'|s) V(w'(s'), s') \right\}$$

$$\begin{aligned} c_1 + c_2 &= e^1(s) + e^2(s) \\ V(w'(s'), s') &\geq U^{2,Aut}(s') \\ w'(s') &\in [U^{1,Aut}(s'), \bar{U}^1(s')] \\ w &= (1-\beta) u(c_1) + \beta \sum_{s' \in S} \pi(s'|s) w'(s') \end{aligned}$$

Value from Autarky

- N =card(S) equations in N unknowns

$$U^{i,Aut}(s) = (1-\beta)u(e^i(s)) + \beta \sum_{s'} \pi(s'|s) U^{i,Aut}(s')$$

- In matrix form

$$\begin{aligned} U^{i,Aut} &= (1-\beta)u(e^i) + \beta\pi U^{i,Aut} \\ U^{i,Aut} &= (I-\beta\pi)^{-1}(1-\beta)u(e^i) \end{aligned}$$

Characterization of the Recursive Problem

- Suppose V is differentiable in first argument. Would expect $V'(w,s) < 0$ and $V''(w,s) < 0$.
- Lagrange multiplier μ on

$$c_1 + c_2 = e^1(s) + e^2(s)$$

- Lagrange multiplier ν on

$$w = (1-\beta)u(c_1) + \beta \sum_{s' \in S} \pi(s'|s) w'(s')$$

- Lagrange multipliers $\beta\pi(s'|s)\lambda_1(s'), \beta\pi(s'|s)\lambda_2(s')$ on

$$\begin{aligned} w'(s') &\geq U^{1,Aut}(s') \\ V(w'(s'), s') &\geq U^{2,Aut}(s') \end{aligned}$$

Characterization of the Recursive Problem

- FOC wrt to c^i

$$\begin{aligned}(1-\beta)u'(c_2) &= \mu \\ \nu(1-\beta)u'(c_2) &= \mu\end{aligned}$$

- Thus

$$\frac{u'(c_2)}{u'(c_1)} = \nu$$

- Note: looks like in standard complete markets model, but here $\nu = \nu(w, s)$ evolves over time (and if not, there is in fact perfect risk sharing).
- If $w \uparrow$, then $\nu \uparrow$ and thus $\frac{u'(c_2)}{u'(c_1)} \uparrow$. Thus c_1 goes up relative to c_2.
- Now: characterize dynamics of w.

Characterization of the Recursive Problem

- FOC wrt to $w'(s')$ yields (after simplifying)

$$V'(w'(s'), s')(1 + \lambda_2(s')) + (\nu + \lambda_1(s')) = 0$$

and thus

$$-V'(w'(s'), s') = \frac{\nu + \lambda_1(s')}{1 + \lambda_2(s')}$$

- Since $V'' < 0$ we have that $w'(s')$ is increasing in $\nu, \lambda_1(s')$ and decreasing in $\lambda_2(s')$.
- Envelope condition

$$-V'(w, s) = \nu = \frac{u'(c_2)}{u'(c_1)}$$

- Combining yields

$$-V'(w'(s'), s') = \frac{-V'(w, s)}{1 + \lambda_2(s')} + \frac{\lambda_1(s')}{1 + \lambda_2(s')}$$

Characterization of the Recursive Problem

$$-V'(w'(s'),s') = \frac{-V'(w,s)}{1+\lambda_2(s')} + \frac{\lambda_1(s')}{1+\lambda_2(s')}$$

- Case 1: For all (w,s) and all $s' \in S$ such that $\lambda_1(s') = \lambda_2(s') = 0$

$$\begin{aligned} V'(w'(s'),s') &= V'(w,s) \\ \frac{u'(c_2')}{u'(c_1')} &= \frac{u'(c_2)}{u'(c_1)} \end{aligned}$$

- Case 2: For all (w,s) and all $s' \in S$ such that $\lambda_1(s') > 0$ and $\lambda_2(s') = 0$, then $w'(s') = U^{1,Aut}(s')$ and

$$\begin{aligned} -V'(w'(s'),s') &> -V'(w,s) \\ \frac{u'(c_2')}{u'(c_1')} &> \frac{u'(c_2)}{u'(c_1)} \end{aligned}$$

Consumption c_1 rises relative to c_2 between today and tomorrow.

Characterization of the Recursive Problem

$$-V'(w'(s'),s') = \frac{-V'(w,s)}{1+\lambda_2(s')} + \frac{\lambda_1(s')}{1+\lambda_2(s')}$$

- Case 3: For all (w,s) and all $s' \in S$ such that $\lambda_1(s') = 0$ and $\lambda_2(s') > 0$, then $V'(w'(s'),s') = U^{2,Aut}(s')$ and

$$\begin{aligned} -V'(w'(s'),s') &< -V'(w,s) \\ \frac{u'(c_2')}{u'(c_1')} &< \frac{u'(c_2)}{u'(c_1)} \end{aligned}$$

  Consumption c_2 rises relative to c_1 between today and tomorrow.

- Summary: social planner finds it optimal to smooth ratio of marginal utility whenever possible. If one of the incentive constraints bind, raise utility promise and consumption for that agent tomorrow.

Policy Functions and Sequential Allocations

- Policy functions $c^i(w,s)$ and $w'(w,s;s')$
- Initial conditions $(U,s_0) \in [\underline{U}^1(s_0), \bar{U}^1(s_0)] \times S$
- Sequential allocations

$$
\begin{aligned}
c_0^i(s_0) &= c^i(U,s_0) \\
w_1^1(s^1) &= w'(U,s_0;s_1) \\
w_1^2(s^1) &= V(w_1^1(s^1),s_1)
\end{aligned}
$$

and recursively

$$
\begin{aligned}
w_t^1(s^t) &= w'(w_{t-1}^1(s^{t-1}),s_{t-1};s_t) \\
w_t^2(s^t) &= V(w_t^1(s^t),s_t) \\
c_t^i(s^t) &= c^i(w_t^i(s^t),s_t)
\end{aligned}
$$

- Have to prove principle of optimality

Example (akin to Kehoe and Levine (2001), Krueger and Perri (2006))

- Example: $S = \{1, 2\}$ with income dispersion $\varepsilon \in (0, 1)$

$$\begin{aligned} e_t^1(s_t &= 1) = 1 + \varepsilon \text{ and } e_t^2(s_t = 1) = 1 - \varepsilon \\ e_t^1(s_t &= 2) = 1 - \varepsilon \text{ and } e_t^2(s_t = 2) = 1 + \varepsilon \end{aligned}$$

and persistence $\delta \in [0, 1]$

$$\pi = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \text{ and } \Pi = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

- Thus

$$\begin{aligned} U^{1,Aut}(s_1; \beta) &= \left(1 - \frac{\beta}{2}\right) \log(1 + \varepsilon) + \frac{\beta}{2} \log(1 - \varepsilon) \\ U^{1,Aut}(s_2; \beta) &= \left(1 - \frac{\beta}{2}\right) \log(1 - \varepsilon) + \frac{\beta}{2} \log(1 + \varepsilon) \\ U^{FB} &= \log(1) = 0 \end{aligned}$$

Characterization of Symmetric Constrained Efficient Allocation

- Goal of social planner: provide maximal consumption insurance, subject to satisfying incentive constraints.
- Conjecture that constrained efficient allocation is characterized by a single number $c_h \in [1, 1+\varepsilon]$ such that

$$\begin{aligned} c_t^1(s_t) &= c_h \text{ and } c_t^2(s_t) = 2 - c_h \text{ if } s_t = 1 \\ c_t^1(s_t) &= 2 - c_h \text{ and } c_t^2(s_t) = c_h \text{ if } s_t = 1 \end{aligned}$$

- Denote

$$\begin{aligned} U_h(c_h; \beta) &= \left(1 - \frac{\beta}{2}\right) \log(c_h) + \frac{\beta}{2} \log(2 - c_h) \\ U_l(c_h; \beta) &= \left(1 - \frac{\beta}{2}\right) \log(2 - c_h) + \frac{\beta}{2} \log(c_h) \end{aligned}$$

Characterization of Symmetric Constrained Efficient Allocation

There exist two thresholds $\beta^{Aut} < \beta^{CM} \in [0,1)$ s.t.

- If $\beta \in [0, \beta^{Aut}]$ then the constrained efficient allocation is autarkic, $c_h = 1 + \varepsilon$ and β^{Aut} satisfies $U_h'(c_h = 1 + \varepsilon; \beta^{Aut}) = 0$. Working this out gives $\beta^{Aut} = 1 - \varepsilon$.
- If $\beta \in [\beta^{CM}, 1]$ then the constrained efficient provides full insurance, $c_h = 1$ and β^{CM} satisfies

$$\begin{aligned} U^{FB} &= U^{1,Aut}(s_1; \beta^{CM}) \\ \beta^{CM} &= 2\left[\frac{\log(1+\varepsilon)}{\log(1+\varepsilon) - \log(1-\varepsilon)}\right] \end{aligned}$$

- If $\beta \in (\beta^{Aut}, \beta^{CM})$ then the constrained efficient allocation features partial insurance, with $c_h = c_h(\beta) \in (1, 1+\varepsilon)$ solving

$$\left(1 - \frac{\beta}{2}\right)\log(1+\varepsilon) + \frac{\beta}{2}\log(1-\varepsilon) = \left(1 - \frac{\beta}{2}\right)\log(c_h) + \frac{\beta}{2}\log(2 - c_h)$$

Implementation/Decentralization

- Subgame perfect equilibrium of transfer game (Kocherlakota 1996)
- Arrow-Debreu competitive equilibrium with enforcement constraints (Kehoe and Levine 1993)
- Sequential markets equilibrium with state-contingent borrowing constraints (Alvarez and Jermann 2000)

Arrow Debreu Equilibrium

- Arrow-Debreu prices $\{p_t(s^t)\}$
- Household problem

$$\max_{\{c_t^i(s^t)\}} (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi(s^t)u(c_t^i(s^t)) \text{ s.t.}$$

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{s^t\in S^t}p_t(s^t)c_t^i(s^t) &\leq \sum_{t=0}^{\infty}\sum_{s^t\in S^t}p_t(s^t)e_t^i(s^t)\\ U(c^i,s^t) &\geq U^{i,Aut}(s_t)\end{aligned}$$

- Market clearing

$$\sum_i c_t^i(s^t) = \sum_i e_t^i(s^t)$$

- Welfare theorems apply (note: consumption sets are convex)

Review: Standard Complete and Incomplete Markets Model

- Complete Markets Model

$$c_t(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \leq y_t(s^t) + a_t(s^t)$$

- Euler Equation

$$q_t(s^t, s_{t+1}) = \frac{1}{R_t(s^{t+1})} = \beta \pi_{t+1}(s^{t+1}|s^t) \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}$$

- Standard Incomplete Markets Model

$$c_t(s^t) + q_t(s^t) a_{t+1}(s^t) = y_t(s^t) + a_t(s^{t-1})$$

- Euler equation

$$q_t(s^t) = \frac{1}{R_t(s^t)} = \beta \sum_{s^{t+1}|s^t} \pi_{t+1}(s^{t+1}|s^t) \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))}$$

Sequential Market Equilibrium

- Impose borrowing constraints that are tighter than those just preventing Ponzi schemes, but "not too tight"
- Let $q_t(s^t, s_{t+1})$ denote price of the Arrow security that pays out one unit of consumption good if tomorrow's shock is s_{t+1}
- Initial asset holdings a_0^i with $\sum_i a_0^i = 0$.
- Note: there exists a one-to-one mapping between the (a_0^1, a_0^2) and the position on the Pareto frontier U
- Let $\tilde{V}(a_t^i(s^t), s_t)$ denote continuation utility of agent i at node s^t with assets $a_t^i(s^t)$

$$\begin{aligned} \tilde{V}(a_t^i(s^t), s_t) &= \max_{c^i, a^i} U(c^i, s^t) s.t. \\ c_t^i(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) &\leq e_t^i(s_t) + a_t^i(s^t) \\ a_{t+1}^i(s^{t+1}) &\geq \bar{A}^i(s^{t+1}) \end{aligned}$$

- Borrowing constraints that are not too tight

$$V(\bar{A}_t^i(s^t), s_t) = U^{i,Aut}(s_t)$$

Definition

Given $\{a_0^i\}_{i \in I}$, a SM with solvency constraints that are not too tight is allocations $\{c_t^i(s^t), a_{t+1}^i(s^{t+1})\}$, prices $\{q_t(s^t, s_{t+1})\}$ and solvency constrains $\{\bar{A}_t^i(s^t)\}$ such that

1. Given $\{q_t(s^t, s_{t+1})\}$ and $\{\bar{A}_t^i(s^t)\}$ the allocations solve the household problems
2. Markets clear: for all s^t

$$\begin{aligned} \sum_i c_t^i(s^t) &= \sum_i e_t^i(s^t) \\ \sum_i a_{t+1}^i(s^{t+1}) &= 0 \text{ for all } s^{t+1} \end{aligned}$$

3. Solvency constraints are not too tight

$$V(\bar{A}_t^i(s^t), s_t) = U^{i,Aut}(s_t) \text{ for all } i, s^t$$

Equilibrium Characterization

- $\mu_{t+1}^i(s^{t+1})$ is LM on solvency constraint for $a_{t+1}^i(s^{t+1})$, and $\lambda_t^i(s^t)$ is LM on budget constraint
- FOC's

$$\begin{aligned}
\beta^t \pi(s^t) u'(c_t^i(s^t)) &= \lambda_t^i(s^t) \\
\beta^{t+1} \pi(s^{t+1}) u'(c_{t+1}^i(s^{t+1})) &= \lambda_{t+1}^i(s^{t+1}) \\
\lambda_t^i(s^t) q_t(s^t, s_{t+1}) - \mu_{t+1}^i(s^{t+1}) &= \lambda_{t+1}^i(s^{t+1})
\end{aligned}$$

- Combining yields

$$\begin{aligned}
q_t(s^t, s_{t+1}) &= \frac{\lambda_{t+1}^i(s^{t+1})}{\lambda_t^i(s^t)} + \frac{\mu_{t+1}^i(s^{t+1})}{\lambda_t^i(s^t)} \\
q_t(s^t, s_{t+1}) &= \beta \pi(s^{t+1} | s^t) \frac{u'(c_{t+1}^i(s^t, s_{t+1}))}{u'(c_t^i(s^t))} \\
&\quad + \frac{\mu_{t+1}^i(s^{t+1})}{\lambda_t^i(s^t)}
\end{aligned}$$

Equilibrium Characterization

- Thus
$$q_t(s^t, s_{t+1}) = MRS^i(s^{t+1}) + \frac{\mu^i_{t+1}(s^{t+1})}{\lambda^i_t(s^t)}$$
- For all i with $a^i_{t+1}(s^{t+1}) > \bar{A}^i(s^{t+1})$ we have $\mu^i_{t+1}(s^{t+1}) = 0$ and thus
$$q_t(s^t, s_{t+1}) = MRS^i(s^{t+1})$$
- For all i with $a^i_{t+1}(s^{t+1}) = \bar{A}^i(s^{t+1})$ we have $\mu^i_{t+1}(s^{t+1}) \geq 0$ and thus
$$q_t(s^t, s_{t+1}) \geq MRS^i(s^{t+1})$$

Equilibrium Prices

- Arrow securities prices:

$$q_t(s^t, s_{t+1}) = \max_i \left\{ \beta\pi(s^{t+1}|s^t) \frac{u'(c^i_{t+1}(s^t, s_{t+1}))}{u'(c^i_t(s^t))} \right\}$$

- Implied Arrow-Debreu prices

$$Q(s^t|s_0) = q_0(s_0, s_1) * q_1(s^1, s_2) * \ldots * q_{t-1}(s^{t-1}, s_t)$$

- For an *arbitrary* allocation $\{c^i_t(s^t)\}$ implied interest rate are said to be high (HIR) if

$$\sum_{t\geq 0} \sum_{s^t} Q(s^t|s_0) * \sum_i c^i_t(s^t) < \infty$$

Theoretical Results (Alvarez and Jermann, EC 2000)

- Key question: under what conditions is a constrained efficient allocation a SM equilibrium allocation (i.e. what do we need for the second welfare theorem to hold)?

Proposition

Any constrained-efficient consumption allocation with HIR can be decentralized (with appropriate initial conditions $\{a_0^i\}$) as a SM equilibrium with solvency constraints that are not too tight.

- Immediate questions:
  - Under what conditions have constrained efficient allocations HIR?
  - Are there SM equilibria with allocations that are not constrained-efficient or that do not have HIR? Yes!

Autarky as Equilibrium

Proposition

The autarkic allocation can always be decentralized (with $a_0^i = 0$) as a SM equilibrium with solvency constraints $\bar{A}_t^i(s^t) = 0$ that are not too tight. Prices are given by

$$q_t(s^t, s_{t+1}) = \max_i \left\{ \beta\pi(s^{t+1}|s^t) \frac{u'(e_{t+1}^i(s^t, s_{t+1}))}{u'(e_t^i(s^t))} \right\}$$

This result is independent of whether autarky is constrained-efficient and whether autarky has HIR.

Proposition

If the autarkic allocation has HIR, it is a constrained efficient allocation (and thus the only feasible allocation).

Nonautarkic Equilibrium

Proposition

Suppose that a constrained efficient allocation $\{c_t^i(s^t)\}$ *has some risk sharing, that is, for all* i *there exists an* s^t *such that*

$$U(c^i, s^t) > U(e^i, s^t)$$

Then the implied interest rates are high.

Corollary

Any nonautarkic constrained efficient consumption allocation can be decentralized as a sequential markets equilibrium with borrowing constraints that are not too tight.

Back to the Example

- Autarky

$$q_t(s^t, s_{t+1}) = \frac{\beta}{2} * \begin{cases} 1 & \text{if } s_t = s_{t+1} \\ \frac{1+\varepsilon}{1-\varepsilon} & \text{if } s_t \neq s_{t+1} \end{cases}$$

If $\beta < \beta^{Aut}$, then autarky has HIR and thus is the only constrained-efficient allocation

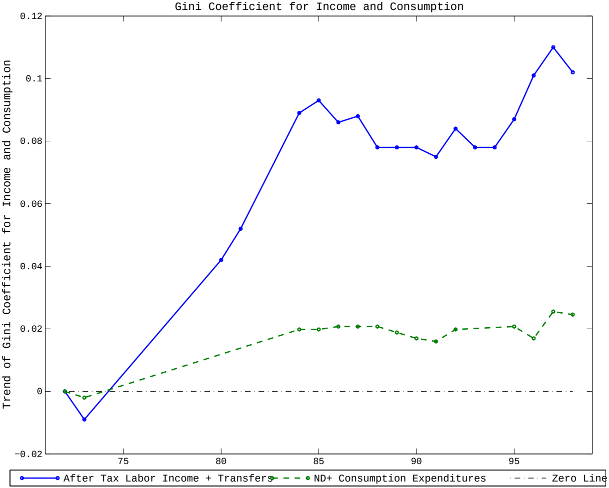
- Partial insurance

$$q_t(s^t, s_{t+1}) = \frac{\beta}{2} * \begin{cases} 1 & \text{if } s_t = s_{t+1} \\ \frac{u'(2-c_h(\beta))}{u'(c_h(\beta))} = \frac{c_h(\beta)}{2-c_h(\beta)} & \text{if } s_t \neq s_{t+1} \end{cases}$$

Since there is partial insurance, $U_h'(c_h(\beta); \beta) > 0$ and $\beta > \beta^{Aut}$, and thus allocation characterized by c_h has HIR:

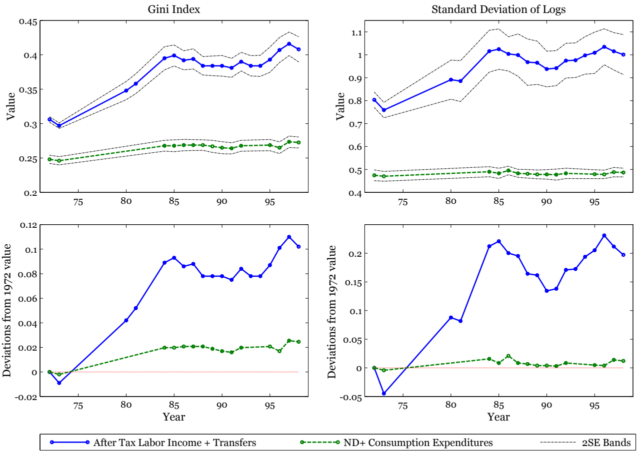
$$\begin{aligned} \left(1 - \frac{\beta}{2}\right) u'(c_h) &> \frac{\beta}{2} u'(2 - c_h) \\ \frac{u'(2 - c_h)}{u'(c_h)} &< (2/\beta - 1) \end{aligned}$$

Application: Income and Consumption Inequality: Krueger and Perri (2006)



Application: Income and Consumption Inequality: Krueger and Perri (2006)

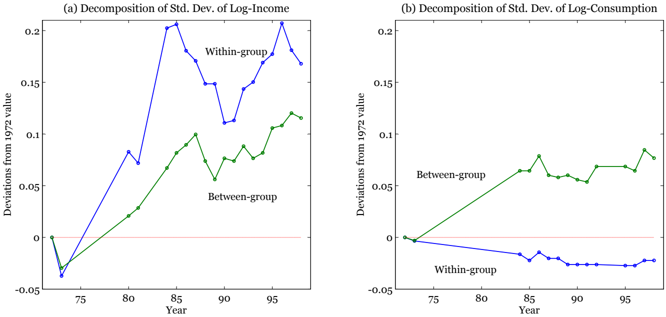
Figure 1. The Evolution of Income and Consumption Inequality in the US



Note: Circles represent actual data points. All indexes are computed on cross sections of households in the CE survey. Income and Consumption are per adult equivalent. Standard deviations are based on the residuals from regressing, in each cross section, household consumption or income on controls for age and race of the reference person

Application: Income and Consumption Inequality: Krueger and Perri (2006)

Figure 2. Decomposition of Income and Consumption Inequality: Data



Note: Circles represent actual data points

Applications: Back to the Example (with Persistence)

- Aggregate state of the world $s_t \in S = \{1, 2\}$
- Endowments

$$e^1(s_t) = \begin{cases} 1+\varepsilon & \text{if } s_t = 1 \\ 1-\varepsilon & \text{if } s_t = 2 \end{cases}$$
$$e^2(s_t) = \begin{cases} 1-\varepsilon & \text{if } s_t = 1 \\ 1+\varepsilon & \text{if } s_t = 2 \end{cases}$$

- Transition probabilities

$$\pi = \begin{pmatrix} \delta & 1-\delta \\ 1-\delta & \delta \end{pmatrix}$$

- Parameter $\delta \in (0, 1)$ measures persistence, $\varepsilon \in [0, 1)$ measures variability of income process
- Stationary distribution given by $\Pi(s) = \frac{1}{2}$. Assume that $\Pi(s_0) = \frac{1}{2}$

Continuation Utility from Autarky

- Continuation utilities from autarky satisfy

$$
\begin{aligned}
& U(1+\varepsilon) \\
= \; & (1-\beta)u(1+\varepsilon) + \delta U(1+\varepsilon) + (1-\delta)U(1-\varepsilon) \\
& U(1-\varepsilon) \\
= \; & (1-\beta)u(1-\varepsilon) + \delta U(1-\varepsilon) + (1-\delta)U(1+\varepsilon)
\end{aligned}
$$

- Solving this yields

$$
U(1+\varepsilon) =
$$

$$
\frac{1}{D}\left\{(1-\beta)\,u(1+\varepsilon) + \beta(1-\delta)\left[u(1+\varepsilon) + u(1-\varepsilon)\right]\right\}
$$

$$
U(1-\varepsilon) =
$$

$$
\frac{1}{D}\left\{(1-\beta)\,u(1-\varepsilon) + \beta(1-\delta)\left[u(1-\varepsilon) + u(1+\varepsilon)\right]\right\}
$$

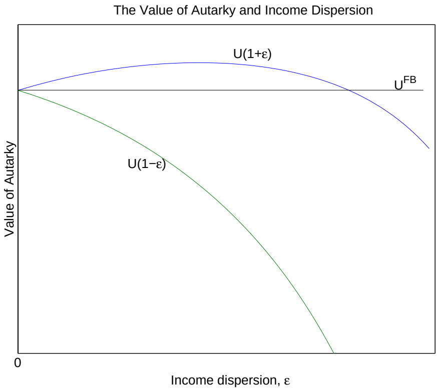
$$
D = \frac{(1-\beta\delta)^2 - (\beta-\beta\delta)^2}{1-\beta} > 0
$$

- Weighted sum of utility from consuming endowment today and expected future utility

Properties of Utility from Autarky

1. $U(1+\varepsilon)|_{\varepsilon=0} = U(1-\varepsilon)|_{\varepsilon=0} = u(1) \equiv U^{FB}$
2. $U(1+\varepsilon)$ and $U(1-\varepsilon)$ are strictly concave and differentiable for all $\varepsilon \in [0,1)$
3. $\frac{dU(1-\varepsilon)}{d\varepsilon} < 0$ for all $\varepsilon \in [0,1)$
4. $\left.\frac{dU(1+\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} > 0$ and $\lim_{\varepsilon \to 1} \frac{dU(1+\varepsilon)}{d\varepsilon} < 0$
5. There exists a unique $\varepsilon_1 = \arg\max_{\varepsilon \in [0,1)} U(1+\varepsilon)$. For all $\varepsilon \in [0, \varepsilon_1)$ we have $\frac{dU(1+\varepsilon)}{d\varepsilon} > 0$ and for all $\varepsilon \in (\varepsilon_1, 1)$ we have $\frac{dU(1+\varepsilon)}{d\varepsilon} < 0$. The threshold ε_1 satisfies $\frac{u'(1-\varepsilon_1)}{u'(1+\varepsilon_1)} = \frac{2}{\beta} - 1 \in (1, \infty)$
6. There exists at most one $\varepsilon_2 \in (0,1)$ such that $U(1+\varepsilon_2) = u(1)$. If ε_2 exists, it satisfies $\varepsilon_2 > \varepsilon_1$

Application: Income and Consumption Inequality: Krueger and Perri (2006)



Comparative Statics: Changes in ε and δ

- Constrained-efficient consumption distribution features maximal insurance, subject to satisfying the IR constraint of currently rich agent.

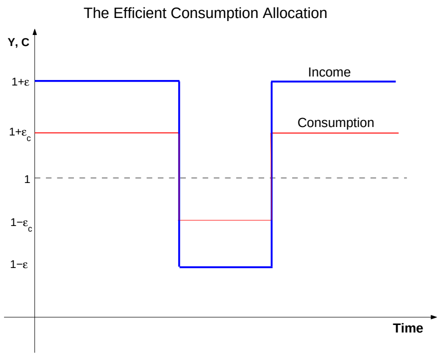
Proposition

Constrained-efficient consumption distribution is characterized by a number $\varepsilon_c(\varepsilon) \geq 0$. The agent with income $1+\varepsilon$ consumes $1+\varepsilon_c(\varepsilon)$ and the agent with income $1-\varepsilon$ consumes $1-\varepsilon_c(\varepsilon)$ regardless of her past history. The number $\varepsilon_c(\varepsilon)$ is the smallest non-negative solution of the following equation

$$\max(U^{FB}, U(1+\varepsilon)) = U(1+\varepsilon_c(\varepsilon))$$

- Three cases
  - $\varepsilon \geq \varepsilon_2$: $U(1+\varepsilon_c(\varepsilon)) = U^{FB}$ and $\varepsilon_c(\varepsilon) = 0$.
  - $\varepsilon \leq \varepsilon_1$: $U(1+\varepsilon_c(\varepsilon)) = U(1+\varepsilon)$ and $\varepsilon_c(\varepsilon) = \varepsilon$.
  - $\varepsilon \in (\varepsilon_1, \varepsilon_2)$: $0 < \varepsilon_c(\varepsilon) < \varepsilon$, $\frac{\partial \varepsilon_c(\varepsilon)}{\partial \varepsilon} < 0$.

Application: Income and Consumption Inequality: Krueger and Perri (2006)



Comparative Statics: Changes in ε and δ

Proposition

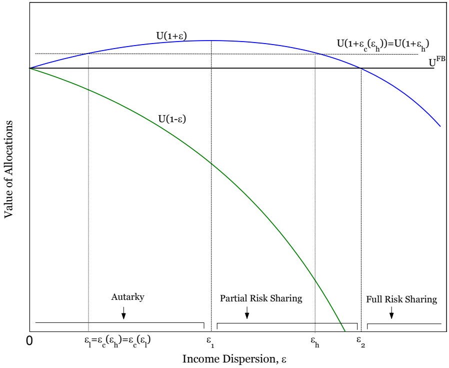
For given δ, starting from a given income dispersion ε a marginal increase in ε leads to a decrease in consumption inequality if and only if $\varepsilon_c(\varepsilon) < \varepsilon$ (in the initial equilibrium there is positive risk sharing). The decrease is strict if and only if $0 < \varepsilon_c(\varepsilon) < \varepsilon_0$ (in the initial equilibrium there is positive, but not complete risk sharing)

Proposition

For a given income dispersion ε a marginal increase in persistence δ leads to an increase in consumption inequality. The increase is strict if and only if $0 < \varepsilon_c(\varepsilon, \delta) < \varepsilon$ (in the initial equilibrium there is positive, but not complete risk sharing)

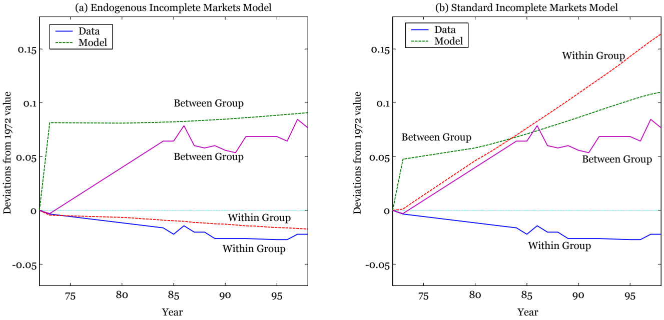
Application: Income and Consumption Inequality: Krueger and Perri (2006)

Figure 3. Characterizing the Efficient Allocation



Application: Income and Consumption Inequality: Krueger and Perri (2006)

Figure 6. Decomposition of Standard Deviation of Log Consumption: Data and Models



A Continuum Economy

- Continuum of infinitely lived agents
- Idiosyncratic income process $\{y_t\}$, assumed to be a finite state Markov chain with transition probabilities π and invariant distribution Π
- Endowment shock history y^t, with probability $\pi(y^t|y_0)$
- law of large numbers
- initial income distribution Π
- $\Phi(a_0, y_0)$: initial distribution over assets and income
- Same commitment problem as before

Plan

- Compute and characterize constrained-efficient allocations
- Decentralize them as equilibria with borrowing constraints that are not too tight, in the spirit of Alvarez and Jermann (2000)
- Note: agents' initial wealth level determines how much expected discounted lifetime utility w_0 this agent can obtain
- Will later establish mapping between (a_0, y_0) and (w_0, y_0)
- Initial distribution of utility promises and income $\Psi(w_0, y_0)$

Constrained-Efficient Allocations

- So far: solved for constrained-efficient consumption allocation by maximizing one agent's expected utility, subject to promise-keeping, individual rationality and resource constraints
- With continuum of agents: dual approach pioneered by Atkeson and Lucas (1992, 1995): minimize cost of delivering a given level of promised utility to a particular agent; distribution of utility promises is adjusted to preserve resource feasibility
- Atkeson and Lucas:
  1. Define constrained efficiency
  2. Formulate sequential social planners problem that solves for constrained-efficient allocations
  3. Make this problem recursive
  4. Prove that policy functions from recursive problem induce constrained-efficient sequential consumption allocations $\{c_t(w_0, y^t)\}$
- Here: discuss recursive problem directly

Recursive Problem Determining Stationary Constrained Efficient Allocations

- Relative "price" of resources today versus tomorrow $R \in (1, \frac{1}{\beta})$.

- Social planner allocates current consumption and lifetime utility from tomorrow on. Promised lifetime utility w as a state variable.

- Deals with one household of type (w, y) at a time.

Dynamic Programming Problem

- Individual state variables (w, y)
- Choices: current consumption c, expected utility from tomorrow, conditional on tomorrow's income shock y', $w'(y')$. Bellman equation

$$\begin{aligned} V(w,y) &= \min_{c,w'(y')} \left\{ c + \frac{1}{R} \sum_{y' \in Y} \pi(y'|y) V(w'(y'), y') \right\} \\ w &= u(c) + \beta \sum_{y' \in Y} \pi(y'|y) w'(y') \\ w'(y') &\geq U^{Aut}(y') \end{aligned}$$

- $V(w, y)$ is resource cost planner has to spend to fulfill utility promises w, without violating individual rationality constraint of agent
- Note: if income process is *iid*, then state variable y disappears
- Operator induced by FE is contraction; unique fixed point V is differentiable, strictly convex; resulting policies $c(w, y)$ and $w'(w, y; y')$ continuous functions; c is strictly increasing in w and w' is either constant at $U^{Aut}(y')$ or strictly increasing in w.

First Order Conditions

$$
\begin{aligned}
1 &= \lambda u'(c) \\
\frac{1}{R}\pi(y'|y)V'(w'(y'),y') &= \lambda\beta\pi(y'|y) + \mu(y')\beta\pi(y'|y) \\
V'(w,y) &= \lambda = 1/u'(c)
\end{aligned}
$$

- Combining yields

$$
\begin{aligned}
V'(w,y) &= \frac{V'(w'(y'),y')}{R\beta} - \mu(y') \\
u'(c) &= \frac{1}{V'(w,y)}
\end{aligned}
$$

- Note: if V is strictly convex in w, then consumption is strictly increasing in w. If one can characterize dynamics of w, can characterize dynamics of c.

Characterization of Optimal Policies with iid Shocks

- If income is *iid* then

$$V'(w) = \frac{V'(w'(y'))}{R\beta} - \mu(y')$$

and one can indeed prove that V is strictly convex and differentiable (Krueger and Perri, 2011)

- States $y', \hat{y}'$ for which IR constraint is not binding: $\mu(y') = \mu(\hat{y}') = 0$

$$V'(w) = \frac{V'(w'(y'))}{R\beta} = \frac{V'(w'(\hat{y}'))}{R\beta}$$

and thus $w'(y') = w'(\hat{y}')$

- States $\tilde{y}'$ for which IR is binding, $\mu(\tilde{y}') > 0$

$$\begin{aligned} V'(w'(\tilde{y}')) &= R\beta V'(w) + R\beta\mu(\tilde{y}') > \\ R\beta V'(w) &= V'(w'(y')) \\ U^{Aut}(\tilde{y}') &= w'(\tilde{y}') > w'(y'). \end{aligned}$$

Characterization of Optimal Policies with iid Shocks

- For each w there exists a $w'^*(w)$ such that

$$w'(y') = \max\{w'^*(w), U^{Aut}(y')\}$$

  that is, tomorrow's promises are held constant across y', unless the constraint is binding and $w'(y') = U^{Aut}(y')$.
- If $R\beta = 1$, then $w'^*(w) = w$

$$w'(y') = \max\{w, U^{Aut}(y')\}$$

- If $\beta R < 1$, then $w'^*(w) < w$

$$w'(y') = \max\{w'^*(w), U^{Aut}(y')\}$$

- Thus value of βR crucial for dynamics of the model.

State Space with iid Shocks

- Define $\underline{w} = \min_{y \in Y} U^{Aut}(y)$ and $\bar{w} = \max_{y \in Y} U^{Aut}(y)$
- For all w we have $w'(w; y') \geq \underline{w}$
- For $y_{\max} = \arg\max_{y \in Y} U^{Aut}(y)$ we have

$$w'(U^{Aut}(y_{\max}), y_{\max}) = U^{Aut}(y_{\max})$$

  and for all $w > U^{Aut}(y_{\max})$ we have

$$w'(w, y') \leq w'(w, y_{\max}) \leq U^{Aut}(y').$$

- For *iid* income shocks the state space $W = [\underline{w}, \bar{w}]$.

Feasibility and Aggregation

- For fixed interest rate R, dynamic programming problem delivers value function $V(w, y)$ and policy functions $c(w, y)$ and $w'(w, y; y')$
- Want associated stationary distribution over utility promises and income shocks Ψ_R
- First: Markov transition function Q_R induced by π and $w'(w, y; y')$
- State space $Z = W \times Y$ and $\mathcal{B}(Z) = \mathcal{B}(W) \times \mathcal{P}(Y)$.
- Transition function $Q_R : Z \times \mathcal{B}(Z) \to [0, 1]$

$$Q_R((w, y), (A, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) & \text{if } w'(w, y; y') \in A \\ 0 & \text{else} \end{cases}$$

- Invariant measure Ψ_R over (w, y)

$$\Psi_R(A, \mathcal{Y}) = \int Q_R((w, y), (A, \mathcal{Y})) d\Psi_R$$

- Has to prove that invariant measure exists and is unique (e.g. by applying Hopenhayn and Prescott's 1991 EC theorem).
- Consumption distribution mimics Ψ_R since $c(w, y)$ strictly increasing in w.

Determination of Intertemporal "Price"

- Want to find R^* that induces allocation satisfying resource balance
- Total resources available

$$\int y d\Psi_R = \sum_y y\Pi(y)$$

- Total resources allocated by the planner

$$\int c(w,y) d\Psi_R$$

- Excess resource requirements

$$d(R) = \int c(w,y) - y d\Psi_R$$

- Find an R^* such that $d(R^*) = 0$ by searching over $R \in (1, \frac{1}{\beta})$ for such R^*

Theory

- For the *iid* case can show that $d(R)$ is well-defined (for each R there exists a unique invariant measure Ψ_R), continuous and increasing in R on $(1, \frac{1}{\beta})$
- With additional assumptions on u and the income process one can show that $\lim_{R\to\frac{1}{\beta}} d(R) > 0$ and that $\lim_{R\to 1} d(R) \leq 0,$ proving existence of a stationary constrained-efficient consumption allocation.
- Since it is hard to prove that $d(R)$ is strictly increasing, uniqueness is hard to establish

Decentralization

- Can decentralize a constrained efficient stationary consumption allocation as a sequential markets equilibrium with borrowing constraints that are not too tight (as in Alvarez and Jermann) or as standard Arrow Debreu equilibrium.
- Agents trade consumption allocations and assets whose payoff is conditional on *individual* income realizations.
- Initial condition $\Phi(a_0, y_0)$
- Standard Arrow Debreu equilibrium: IR constraints directly loaded into household budget sets

$$
\begin{aligned}
\sum_t \sum_{y^t} p_t(y^t)\left(c_t(a_0, y^t) - y_t\right) &\leq a_0 \\
U(c; y^t) &\geq U^{Aut}(y_t)
\end{aligned}
$$

Decentralization

- Sequential Market equilibrium: household problem

$$V(a_t(a_0,y^t),y^t;\bar{A}) = \max_{\{c_s(a_0,y^s),a_{s+1}(a_0,y^{s+1})\}} u(c_t(a_0,y^t)) + \sum_{s+1}\sum_{y^s|y^t}\beta^{s-t}\pi(y^s|y^t)u(c_s(a_0,y^s))$$

$$c_s(a_0,y^s) + \sum_{y_{s+1}} q_s(y^s,y_{s+1})a_{s+1}(a_0,y^{s+1}) = y_s + a_s(a_0,y^s)$$

$$a_{s+1}(a_0,y^{s+1}) \geq \bar{A}(y^s,y_{s+1})$$

- Solvency constraints are not too tight if

$$V(\bar{A}(y^t,y_{t+1}),y^{s+1};\bar{A}) = U^{Aut}(y_{t+1})$$

- Market clearing

$$\int \sum_{y^t|y_0 t}\left[c_t(a_0,y^t) - y_t\right]\pi(y^t|y_0)d\Phi(a_0,y_0) = 0$$

$$\int \sum_{y^{t+1}|y_0} a_{t+1}(a_0,y^{t+1})\pi(y^{t+1}|y_0)d\Phi(a_0,y_0) = 0$$

Decentralization

- Stationary equilibrium: Cross sectional distribution of asset is constant over time.
- Note: easy to augment this definition to a production economy:
    - Factor prices

$$\begin{aligned} w_t &= F_L(K_t, L_t) \\ r_t &= F_K(K_t, L_t) - \delta \end{aligned}$$

    - Household budget constraint: replace y_t with $w_t y_t$
    - Market clearing

$$\begin{aligned} L_t &= \int \sum_{y^t|y_0 t} y_t \pi(y^t|y_0) d\Phi \\ C_t &= \int \sum_{y^t|y_0 t} c_t(a_0, y^t) \pi(y^t|y_0) d\Phi \\ K_{t+1} &= \frac{\int \sum_{y^{t+1}|y_0} a_{t+1}(a_0, y^{t+1}) \pi(y^{t+1}|y_0)}{1 + r_{t+1}} \\ K_{t+1} &= F(K_t, L_t) + (1-\delta)K_t - C_t \end{aligned}$$

Decentralization of Constrained Efficient Allocations

- Prices of Arrow securities

$$q_t(y^t, y_{s+1}) = q(y_{t+1}|y_t) = \frac{\pi(y_{t+1}|y_t)}{R}$$

where R is intertemporal "price" from social planner problem

- From q's can derive Arrow-Debreu prices as before (see below).
- Form of $q_t(y^t, y_{s+1})$ is simple by no arbitrage: For arbitrary y^t, if one wants to buy 1 consumption good tomorrow in all states. Two ways:
  - buy one unit of $a_{t+1}(y^t, y_{t+1})$ for all y_{t+1}. Cost $\sum_{y_{t+1}} q_t(y^t, y_{t+1})$
  - For arbitrary $\hat{y}_{t+1}$ buy $\frac{1}{\pi(\hat{y}_{t+1}|y_t)}$ units of $a_{t+1}(y^t, \hat{y}_{t+1})$ from measure 1 households with y_t. By LLN fraction $\pi(\hat{y}_{t+1}|y_t)$ have $y_{t+1} = \hat{y}_{t+1}$. This guarantees 1 unit of consumption. Cost $\frac{q_t(y^t, \hat{y}_{t+1})}{\pi(\hat{y}_{t+1}|y_t)}$
- Thus for all $\hat{y}_{t+1}$ (since y^t was arbitrary)

$$q_t(y^t, \hat{y}_{t+1}) = \pi(\hat{y}_{t+1}|y_t) \sum_{y_{t+1}} q_t(y^t, y_{t+1}) = \frac{\pi(\hat{y}_{t+1}|y_t)}{R_t}$$

- With restriction to stationary equilibria: $R_t = R$

Utility Promises and Initial Assets

- Want: link between (w_0, y_0), (a_0, y_0) and $\Psi(w_0, y_0)$, $\Phi(a_0, y_0)$
- Have: stationary constrained-efficient distribution $\Psi(w, y)$, corresponding R^* and associated $V(w, y)$, $h(w, y)$ and $g(w, y; y')$
- Construct sequential constrained-efficient consumption allocations $\{c_t(w_0, y_0)\}$ for agent with initial conditions (w_0, y_0)
- Define Arrow-Debreu prices as

$$\begin{aligned} Q(y_0) &= \Pi(y_0) \\ Q(y^t) &= q(y_t|y_{t-1}) * \ldots q(y_1|y_0)\Pi(y_0) \end{aligned}$$

Utility Promises and Initial Assets

- Initial assets associated with (w_0, y_0) are given as

$$a_0(w_0, y_0) = \sum_{t=0}^{\infty} \sum_{y^t|y_0} Q(y^t)\left(c_t(w_0, y^t) - y_t\right)$$

- Associated equilibrium consumption allocations

$$c_t(a_0, y^t) = c_t(a_0^{-1}(a_0, y_0), y^t)$$

  where $w_0 = a_0^{-1}(a_0, y_0)$
- Distribution $\Phi(a_0, y_0)$ is determined as

$$\Phi(a_0, y_0) = \Psi(a_0^{-1}(a_0, y_0), y_0)$$

- Can show: for initial distribution $\Phi(a_0, y_0)$, the allocation $c_t(a_0, y^t)$ and prices $q(y'|y)$ are a competitive equilibrium with borrowing constraints that are not too tight
- Alternatively, form a competitive equilibrium in the spirit of Kehoe and Levine (1993), with equilibrium prices $\{Q(y^t)\}$.

Endogenizing Outside Option: Krueger-Uhlig ('06, '21)

- So far outside option is given by autarky. Where does this come from? Question remains if allows households to save after default.
- Now: endogenize outside options through competition among financial intermediaries.
- Suppose households have outside options $U^{Out} = \{U^{Out}(y)\}_{y \in Y}$ to be determined in equilibrium.
- Financial intermediaries solve the cost minimization problem of the constrained planner, but with $\{U^{Out}(y)\}_{y \in Y}$ instead of $\{U^{Aut}(y)\}_{y \in Y}$.
- Equilibrium outside option determined by zero profits: NPV of consumption contract equals NPV of the endowment stream. No punishment from default. Just go to next intermediary.
- KU (2006): Exogenous interest rate R. Show decentralization as SM equilibrium with borrowing constraints $\bar{A}^i(s^t) \equiv 0$.
- KU (2021): Endogenize R in stationary equilibrium with production, continuum of agents (Aiyagari (1994) with limited commitment).
- Alternative endogenous outside option: allow endogenous group formation and group deviations (Cole, Krueger, Mailath and Park, 2021).

Private Information

- Partial Equilibrium: Green (1987) and Thomas and Worrall (1990)
- General equilibrium: Phelan and Townsend (1991) and Atkeson and Lucas 1992, 1995

Partial Equilibrium

- One risk-neutral financial intermediary, lives forever, discounts future at $\beta \in (0, 1)$.
- Risk averse household, also discounts future at β.
- Agent has stochastic income process $\{y_t\}$, assumed to be *iid* with finite support $Y = \{y_1, y_2, \ldots y_N\}$ and probabilities $(\pi_1, \pi_2, \ldots, \pi_N)$;
- Long-term insurance contract. Both parties can commit to long-term contracts
- In absence of private information optimal consumption allocation for agent, subject to financial intermediary breaking even:

$$c_t(y^t) = E(y_t) = E(y) = 1$$

- Agent hands over realized income in every period and receives constant consumption from financial intermediary.

Partial Equilibrium

- Lifetime utility agent is

$$U(c) = (1-\beta)\sum_{t=0}^{\infty}\sum_{y^t}\pi_t(y^t)\beta^t u(c_t(y^t)) = u(1)$$

- Expected utility of the financial intermediary is

$$W(c) = (1-\beta)\sum_{t=0}^{\infty}\sum_{y^t}\pi_t(y^t)\beta^t \left(y_t - c_t(y^t)\right) = 0$$

- Problem: private information. If agent is promised constant consumption independent of report, will always report $y_t = y_1$, keep difference between true income and report, receive 1 from intermediary, do strictly better. Principal would lose money:

$$W = (1-\beta)\sum_{t=0}^{\infty}\sum_{y^t}\pi_t(y^t)\beta^t (y_1 - 1) = y_1 - 1 < 0.$$

- Basic trade-off: provision of insurance and the provision of incentives for truth telling.

Partial Equilibrium

- Want to construct efficient long-term insurance contract between two parties with informational frictions.
- Recursive problem: minimize cost, subject to satisfying incentives, promise keeping

$$
\begin{aligned}
V(w) &= \min_{\{t_s,w_s\}_{s=1}^N} \sum \pi_s \left[(1-\beta)t_s + \beta V(w_s)\right] \\
& \quad \text{s.t.} \\
w &= \sum \pi_s \left[(1-\beta)u(t_s+y_s) + \beta w_s\right] \\
& \quad (1-\beta)u(t_s+y_s) + \beta w_s \\
&\geq (1-\beta)u(t_k+y_s) + \beta w_k \;\forall s,k \\
t_s &\in [-y_s, \infty) \\
w_s &\in [\underline{w}, \bar{w}]
\end{aligned}
$$

Partial Equilibrium

- Truth-telling constraints: let state s with y_s be realized. If agent reports truth, receives transfers t_s, continuation utility w_s, for lifetime utility

$$u(t_s + y_s) + \beta w_s.$$

- If falsely reports y_k, receives transfers t_k, continuation utility w_k, for lifetime utility

$$u(t_k + y_s) + \beta w_k.$$

- Constraints state it is in agents' interest to truthfully reveal income.
- **Assumption:** $u : [0, \infty) \to \mathbf{R}$ *is strictly increasing, strictly concave, at least twice differentiable and satisfies the Inada conditions.*
- Start of contract. How is w determined? Higher initial w means more lifetime utility for agent, but less lifetime utility for principal. Expected lifetime utility for principal, is given by

$$W(w) = 1 - V(w)$$

- Vary w to trace constrained Pareto frontier between principal, agent.
- Key w is the w that solves $V(w) = 1$. Lifetime utility of the agent that yields 0 profits for the principal.

Properties of the Recursive Problem

- Contraction mapping argument: functional equation has unique solution V
- Can bound V above and below.
- Cheapest way to provide w is by constant consumption (but not incentive compatible)
- Define $c^{fb}(w)$ as $u(c) = w$ or $c^{fb}(w) = u^{-1}(w)$.
- Cost for constant consumption stream $c^{fb}(w)$

$$\underline{V}(w) = c^{fb}(w) - 1.$$

- Obviously

$$\underline{V}(w) \leq V(w)$$

  with strict inequality if any of the incentive constraints is binding.
- Upper bound: principal can give constant transfers $t_s = t$. This induces truth-telling.

Properties of the Recursive Problem

- To satisfy promise-keeping need

$$\sum_s \pi_s u(y_s + t) = w$$

- Denote transfer by $\bar{t}(w)$. Cost of policy is given by

$$\bar{V}(w) = \bar{t}(w)$$

- Obviously

$$V(w) \leq \bar{V}(w)$$

- Since $c^{fb}(w)$ is strictly increasing and strictly convex, so is $\underline{V}(w)$. Furthermore $\underline{V}'(\underline{w}) = 0$ and $\underline{V}'(\bar{w}) = \infty$. Finally $\underline{V}(\underline{w}) = -1$ and $\underline{V}(\bar{w}) = \infty$.
- Function $\bar{t}(w)$ and thus $\bar{V}(w)$ are strictly increasing and strictly convex, with $\bar{V}(\bar{w}) = \infty$.
- $V(w)$ is strictly increasing in w. Will also assert that it is strictly convex and differentiable.

Proposition

In an efficient contract, lower income reports are rewarded with higher current transfers, but lower continuation utilities: $t_{s-1} \geq t_s$ *and* $w_s \geq w_{s-1}$ *for all* $s \in S$

Proof.

Consider the constraints

$$\begin{aligned}(1-\beta)u(t_s + y_s) + \beta w_s &\geq (1-\beta)u(t_{s-1} + y_s) + \beta w_{s-1} \\ (1-\beta)u(t_{s-1} + y_{s-1}) + \beta w_{s-1} &\geq (1-\beta)u(t_s + y_{s-1}) + \beta w_s\end{aligned}$$

Adding them results in

$$\begin{aligned}u(t_s + y_s) + u(t_{s-1} + y_{s-1}) &\geq u(t_{s-1} + y_s) + u(t_s + y_{s-1}) \\ u(t_s + y_s) - u(t_{s-1} + y_s) &\geq u(t_s + y_{s-1}) - u(t_{s-1} + y_{s-1})\end{aligned}$$

Since u is strictly concave and $y_s > y_{s-1}$ we have $t_{s-1} \geq t_s$. From first constraint it follows that $w_s \geq w_{s-1}$. □

Proposition

Suppose that the true income state is s and that all local constraints of the form

$$
\begin{aligned}
(1-\beta)u(t_s+y_s)+\beta w_s &\geq (1-\beta)u(t_{s+1}+y_s)+\beta w_{s+1} \\
(1-\beta)u(t_s+y_s)+\beta w_s &\geq (1-\beta)u(t_{s-1}+y_s)+\beta w_{s-1}
\end{aligned}
$$

hold. Then all other incentive constraints are satisfied.

Proof.

Show: if these constraints satisfied, household does not want to report $s-1$ when true state is $s+1$. From previous result $t_s \leq t_{s-1}$, thus

$$u(t_s + y_{s+1}) - u(t_s + y_s) \geq u(t_{s-1} + y_{s+1}) - u(t_{s-1} + y_s)$$

Multiply (411) by $(1-\beta)$, add to second yields

$$(1-\beta)u(t_s + y_{s+1}) + \beta w_s \geq (1-\beta)u(t_{s-1} + y_{s+1}) + \beta w_{s-1}$$

$$(1-\beta)u(t_{s+1} + y_{s+1}) + \beta w_{s+1} \geq (1-\beta)u(t_s + y_{s+1}) + \beta w_s$$

and thus

$$\begin{aligned} & (1-\beta)u(t_{s+1} + y_{s+1}) + \beta w_{s+1} \\ \geq \quad & (1-\beta)u(t_s + y_{s+1}) + \beta w_s \\ \geq \quad & (1-\beta)u(t_{s-1} + y_{s+1}) + \beta w_{s-1} \end{aligned}$$

□

Proposition

The local downward constraints

$$(1-\beta)u(t_s + y_s) + \beta w_s \geq (1-\beta)u(t_{s-1} + y_s) + \beta w_{s-1}$$

are always binding, the local upward constraints

$$(1-\beta)u(t_s + y_s) + \beta w_s \geq (1-\beta)u(t_{s+1} + y_s) + \beta w_{s+1}$$

are never binding.

Proof.

Omitted □

A Simple Example

- $N = 2$
- Dynamic program becomes

$$
\begin{aligned}
V(w) &= \min_{\{t_1,t_2,w_1,w_2\}} \pi\left[(1-\beta)t_1 + \beta V(w_1)\right] \\
&\quad +(1-\pi)\left[(1-\beta)t_2 + \beta V(w_2)\right] \\
&\quad \text{s.t.} \\
w &= \pi\left[(1-\beta)u(t_1+y_1) + \beta w_1\right] \\
&\quad +(1-\pi)\left[(1-\beta)u(t_2+y_2) + \beta w_2\right] \\
&\quad (1-\beta)u(t_2+y_2) + \beta w_2 \\
&\geq (1-\beta)u(t_1+y_2) + \beta w_1
\end{aligned}
$$

A Simple Example

- First order and envelope conditions

$$\begin{aligned} & \pi(1-\beta) - \lambda\pi(1-\beta)u'(t_1+y_1) + \\ & \mu(1-\beta)u'(t_1+y_2) \\ = & \; 0 \end{aligned}$$

$$\begin{aligned} & (1-\pi)(1-\beta) - \lambda(1-\pi)(1-\beta)u'(t_2+y_2) \\ & -\mu(1-\beta)u'(t_2+y_2) \\ = & \; 0 \end{aligned}$$

$$\begin{aligned} \pi\beta V'(w_1) - \lambda\pi\beta + \mu\beta &= 0 \\ (1-\pi)\beta V'(w_2) - \lambda(1-\pi)\beta - \mu\beta &= 0 \\ V'(w) &= \lambda \end{aligned}$$

- Rewriting yields

$$\lambda = V'(w) = V'(w_1) + \frac{\mu}{\pi} = V'(w_2) - \frac{\mu}{1-\pi}$$

A Simple Example

- Thus $w_1 < w < w_2$
- Furthermore

$$\begin{aligned}
1 &= \lambda u'(t_1 + y_1) - \frac{\mu}{\pi} u'(t_1 + y_2) \\
&= \left(\lambda - \frac{\mu}{\pi}\right) u'(t_1 + y_1) \\
&\quad + \frac{\mu}{\pi} \left[u'(t_1 + y_1) - u'(t_1 + y_2)\right] \\
&> \left(\lambda - \frac{\mu}{\pi}\right) u'(t_1 + y_1) \\
1 &= \left(\lambda + \frac{\mu}{1 - \pi}\right) u'(t_2 + y_2)
\end{aligned}$$

- Thus

$$\begin{aligned}
1 &= V'(w_2) u'(c_2) \\
&= V'(w_1) u'(c_1) + \frac{\mu}{\pi} \left[u'(c_1) - u'(t_1 + y_2)\right] \\
&> V'(w_1) u'(c_1)
\end{aligned}$$