

Credit Risk, Credit Derivatives and Firm Value Based Models

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Abstract

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1 Introduction

Nowadays there have been developed many instruments to transfer credit risk. These instruments are called credit derivatives. There have also been developed many model and methods to evaluate credit risk. They range from practical market methods to theory guided methods relying on firm value. In this note, first some well known instruments for transferring credit risk are discussed and then, second, firm value based models on evaluating credit risk are studied. Of course, there are other evaluation methods of credit risk, for example, intensity based models or credit rating models but here we want to focus on firm value based models. Those ones have a sound theoretical foundation and, they are based on the theoretical development of the 1970, put forward by Black and Scholes (1973) and Merton (1974). Further theoretical foundations of this approach can be found in Schönbucher (2003), Grüne and Semmler (2005) and Grüne, Semmler and Bernard (2006).

2 The Relevance of Credit Derivatives

The market for credit derivatives was created in the early 1990s in London and New York and it is fastest growing derivative market at the moment. Considering only the period between June 2001 and June 2004, the notional amounts outstanding in billions of US dollars were 695 and 4.477 respectively according to a recent survey of the Bank for International Settlements, Switzerland (see table 3 in the appendix). That is a growth of more than 500 per cent in only three years.

Participants in the market for credit derivatives can be divided into five major groups. Banks form the largest group with a fraction of about 47 per cent. The second largest group consists of insurances and re-insurances which cover about 23 per cent of the market's notional outstanding. Other groups are hedge funds (8 per cent) and investment funds (5 per cent) as well as industrials (4 per cent) of different branches.

Table 1 Market share by instrument type (rounded numbers)

Instrument	Share (%)
Credit default swaps (including FtDs)	67
Synthetic balance sheet CDOs	12
Tranched portfolio default swaps	9
Credit-linked notes, asset repackaging, asset swaps	7
Credit spread options	2
Managed synthetic CDOs	2
Total return swaps	1
Hybrid credit derivatives	0,2

Source: Risk (Patel, 2002).

When one takes a look at the derivative market with respect to instrument types, one can see that credit default swaps (CDS) represent about 67 per cent of all transactions made in that field (see table 1). A reason for this may be the standards for "plain vanilla" CDSs developed by the International Swaps and Derivatives Association (ISDA), leading to lower transaction costs and simplifying the whole business. Further types are discussed later in this paper.

Purposes for using credit derivatives are, as the types of instruments themselves, manifold. One can think of using credit derivatives as investments, for the credit risk management of bond portfolios, for hedging counterparty or country risk in isolated cases, as a funding opportunity for banks through the securitisation of loan portfolios or for portfolio optimization for bond and loan portfolio managers. Referring to former times, a bank could only manage its credit risk at origination. During the whole lifetime of a loan the risk remained on the books until the loan was paid off or the obligor defaulted. With the possibilities of these instruments, however, a bank and all the other previously mentioned institutions are able to conduct active risk management. Due to these features and the fact that credit is now a trading asset, the market of credit derivatives is growing and should keep growing in the future.

After giving a short introduction about the important role credit derivatives play in the financial world today, the terminology of the general credit derivative is described. Next we provide an overview of different types of credit derivatives, and give an example to show how they are used to conduct active risk management. In the second part of the paper, we talk about the idea of firm's value models and their connection to credit derivatives. In particular the Black/Scholes-Merton model and Moody's KMV are discussed.

3 Terminology

A useful definition of credit derivatives is formulated by Phillip Schönbucher (2003):

”A credit derivative is a derivative security that has a payoff which is conditioned on the occurrence of a credit event. The credit event is defined with respect to a reference credit (or several reference credits), and the reference credit asset(s) issued by the reference credit. If the credit event has occurred, the default payment has to be made by one of the counterparties. Besides the default payment a credit derivative can have further payoffs that are not default contingent. This definition can be extended to include derivative securities whose payoffs are materially affected by credit events and derivatives on defaultable underlying securities.”

For most derivatives, one can use the following definitions:

- **A** is the counterparty which receives a payment in the event of a default
- **B** is the counterparty which has to make the payment in the event of a default
- **C** is the reference credit
- **Reference entity/reference credit** is the issuer of the reference obligation/reference credit asset whose default triggers the credit event
- **Reference obligations/reference credit asset** is a set of assets issued by the reference entity

- **Credit event/default event** occurs e.g. for the following reasons:
 - Bankruptcy
 - Failure to pay with certain requirements
 - Obligation default
 - Ratings downgrade below given thresholds (only for ratings-triggered credit derivatives)
- **Default payment** is the payment which has to be made by **B** if a credit event occurs

4 Some Types of Credit Derivatives

4.1 Total Return Swaps (TRS)

In a total return swap (or total rate of return swap), **A** wants to change its entire payoff from a defaultable investment (e.g. a bond, denoted by **C** with the entire payoff **B** receives from its default-free Libor investment).

There are several effects appearing from this contract. First, **B** is long the **C**-bond without having paid for this investment. Therefore **B** normally has to put collateral (this can be the **C**-bond, which legally still belongs to **A**), depending on its creditworthiness. Second, **A** has hedged its exposure to the **C**-bond and bears a certain counterparty risk now, but which should be minimized because of the collateral.

Concerning the purpose of credit derivatives, **A** transmits the credit AND market risk of the **reference credit C** to **B** and ensures a risk-free Libor interest rate plus a certain spread, reflecting the creditworthiness of **B**.

4.2 Credit Default Swaps (CDS)

The most important difference between a TRS and a CDS is the matter of isolating credit risk. While a TRS transfers both credit AND market risk (whereas a certain risk remains

for counterparty **A** because only the risk of one of the **reference credit** is transferred, not the whole default risk), the default risk of this type of credit derivative is completely isolated.

In a credit default swap (or credit swap), **B** takes the default risk of **A**'s defaultable asset and has to make a **default payment** if a **credit event** occurs. In exchange for this service, **A** pays a fee for the default protection.

With respect to the **default payment**, there are several possibilities. A physical delivery requires the delivery of the reference assets against a repayment at par. When a cash settlement is arranged, **B** has to pay the difference between the post-default market value and the face value of the asset. A default digital swap, in contrast, demands a fixed amount of money, agreed to at the time of the contract.

Since **A** and **B** can declare any asset of **C** they want, they are able to widen the range of assets so that the default risk of **C** is completely transferred.

According to the International Swaps and Derivatives Association (ISDA), the following information should be part of a CDS contract:

- The reference obligor and his reference assets
- The definition of a credit event that is to be insured
- The notional of the CDS
- The start of the CDS
- The maturity date
- The credit default swap spread
- The frequency and day count convention for the spread payments
- The payment of the credit event and its settlement

4.3 Collateralized Debt Obligations (CDO)

Collateralized debt obligations belong to the group of exotic credit derivatives as their construction is very special. The aim of a CDO is to securitize a complete portfolio of defaultable assets like a basket of bonds or loans in order to sell these securities and the credit risk of the assets with them.

The way a CDO is born looks like this: first, a portfolio of defaultable assets is set up and then sold to a company, exclusively created for this aim and denoted by special purpose vehicle (SPV). The second step is to divide the portfolio into several tranches in a way that every single tranche can be securitized and sold to investors with different risk aversions and different demands for the yield, respectively. The obligations sold by the SPV are collateralized by the underlying debt portfolio.

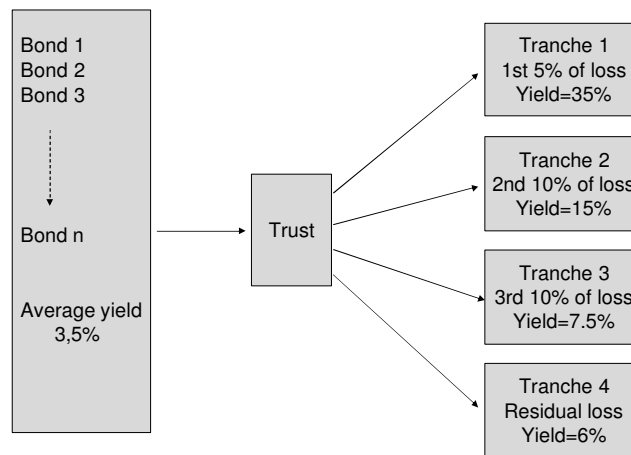


Figure 1: Collateralized debt obligation

According to the tranche an investor owns, he or she is confronted with more or less risk. Assuming the investor has obligations of the first tranche, in the example given in figure 1 he or she suffers already from the first 5 per cent of losses the portfolio gains. Since the risk of losing money is very high in this case, the yield one gets is correspondingly very high, too. Normally, it is a multiple of the average yield of the assets of the portfolio.

An investor of the fourth tranche, in contrast, is only burdened with a loss when already more than 25 per cent of the assets of the portfolio defaulted. Of course, people investing in this tranche have a lower expected yield than the average expected portfolio yield.

4.4 Example of a CDS with Real Quotes

The following example should give an idea how a plain vanilla credit default swap looks in practice. Given the bid/offer quotes of a market maker in table 2, one can think through several cases.

Table 2 Credit default swap quotes (basis points)

Company	Rating	Maturity			
		3 years	5 years	7 years	10 years
Toyota Motor Corp	AaI/AAA	16/24	20/30	26/37	32/53
Merrill Lynch	Aa3/AA-	21/41	40/55	41/83	56/96
Ford Motor Company	A+/A	59/80	85/100	95/136	118/159
Enron	BaaI/BBB+	105/125	115/135	117/158	182/233
Nissan Motor Co.Ltd.	BaI/BB+	115/145	125/155	200/230	244/274

Looking at Toyota, the market maker is prepared to buy three-year default protection for 16 basis points per year and sell three-year default protection for 24 basis points per year and so on (Hull 2002).

Supposing that a bank had several hundred million dollars of loans outstanding to Enron and was concerned about its exposure. It could buy a \$100 million five-year CDS on Enron from the market maker for 135 basis points or \$1.35 million per year. This would shift part of the bank's Enron credit exposure to the market maker (Hull 2002).

Another possibility could be an exchange in the bank's credit risk. If the bank is interested in shifting part of its credit risk to another industry, it could, for example, sell a five-year \$100 million CDS on Nissan for \$1.25 million per year while buying a similar CDS on Enron at the same time. The net cost of this strategy would be 10 basis points

or \$100,000 per year. So the bank had changed part of its credit risk from Enron for a certain credit risk of Nissan. Due to the differences in these industries, one can say that the bank has diversified its credit exposure (Hull 2002).

5 Firm Value Based Models and Black and Scholes

So far we have talked about the characteristics of credit derivatives in general and how to use them as tools for active risk management. Now we will focus on the pricing of credit derivatives using a specific modeling approach: the approach of firm's value models.

To be able to price credit derivatives, we have to know something about the default risk (credit risk) of the underlying asset. Modeling the default risk is the aim of credit derivatives pricing models such as intensity and spread-based models. Compared to those, firm's value models use a much more fundamental approach to valuing defaultable debt and in addition try to provide a link between the values of equity and debt of the firm.

Firm's value models assume a fundamental process V , denoting the total value of the assets of the firm that has issued the bonds in question. V is described as a stochastic process, influenced by the prices of all securities issued by the firm. A very important point of this type of model is that all claims on the firm's value are modelled as derivative securities with the firm's value as underlying.

Black and Scholes (1973) and Merton (1974) were the first people modeling credit risk with what we know today as a firm's value model. Modeling credit risk means modeling default probability. In their consideration a default could only occur at maturity of the debt, i.e. if the difference *firm value V minus outstanding debt* at maturity is negative, a default happens, otherwise the firm continues to exist. Merton (1974) explicitly treated the corporate liability from the perspective of derivative pricing. We will come to another and more realistic view later. For further theoretical development see, Schönbucher (2003), Grüne and Semmler (2005) and Grüne, Semmler and Bernard (2006).

As already mentioned above, the value V of the firm's assets is described as a stochastic process. Fischer Black, Myron Scholes and Robert C. Merton set up for V the following

geometric Brownian motion:

$$dV = \mu V dt + \sigma V dW \quad (1)$$

or

$$\frac{dV}{V} = \mu dt + \sigma dW \quad (2)$$

where the variable σ is the volatility of firm value, the variable μ is the expected rate of return and dW as a Wiener process (for the derivation of this equation see Hull 2002, 11.3).

From now on in this model, the prices of both debt $\bar{B}(V, t)$ and shares $S(V, t)$ are functions of the firm's value V and the time t . What Black and Scholes (1973) and Merton (1974) did was a breakthrough. They showed that both equity and debt of the firm can be seen as derivative securities on the value V of the firm's assets. The payoff structure of these derivative securities looks like this (\bar{D} is the exercise price):

$$\bar{B}(V, t) = \min(\bar{D}, V) \quad (3)$$

$$S(V, t) = \max(V - \bar{D}, 0) \quad (4)$$

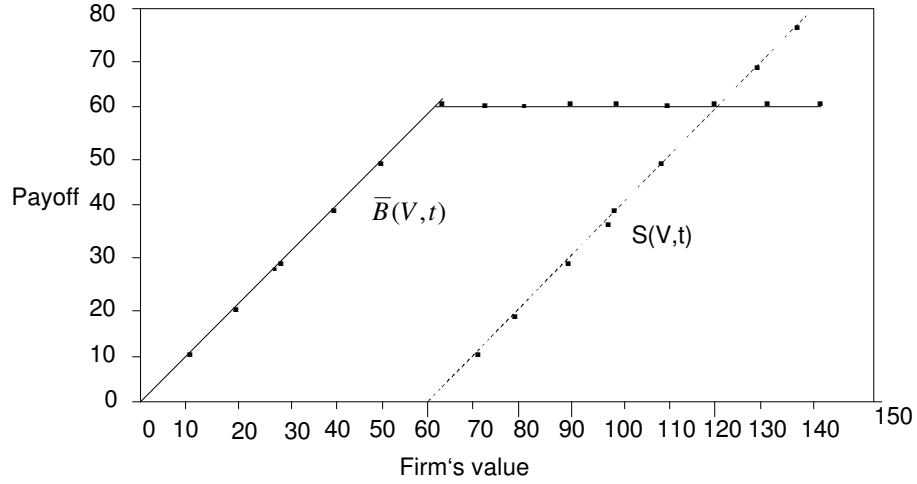


Figure 2: Payoffs of shares and bonds at $t = T$ for $\bar{D} = 60$

As we are interested in pricing equity and debts of the firm and credit derivatives, respectively, we set up a risk-neutral portfolio by hedging one bond with Δ -shares. The value of the portfolio is:

$$\Pi = \bar{B}(V,t) + \Delta S(V,t) \quad (5)$$

The change in value can be derived from Ito's lemma (see appendix 11A in Hull 2002) and is:

$$\begin{aligned} d\Pi &= d\bar{B}\Delta dS \quad (6) \\ &= \left(\frac{\partial \bar{B}}{\partial t} + \frac{1}{2} \frac{\partial^2 \bar{B}}{\partial V^2} + \Delta \frac{\partial S'}{\partial t} + \frac{1}{2} \Delta \frac{\partial^2 S'}{\partial V^2} \right) dt \\ &\quad + \left(\frac{\partial \bar{B}}{\partial V} + \Delta \frac{\partial S}{\partial V} \right) dV \end{aligned}$$

To be fully hedged and to have a predictable return, the number of shares must be:

$$\Delta = -\frac{\partial \bar{B}/\partial V}{\partial S/\partial V} \quad (7)$$

This leads to the well known Black-Scholes partial differential equation:

$$\frac{\partial}{\partial t}S + \frac{1}{2}\sigma^2V^2\frac{\partial^2}{\partial V^2}S + rV\frac{\partial}{\partial V}S - \Gamma S = 0 \quad (8)$$

Now we can compute the value of a share with the Black-Scholes formula C^{BS} for a European call option on V . The expiry date is denoted by T , the exercise price by \bar{D} , the underlying volatility by σ and the interest rate by r_f :

$$S(V, t) = C^{BS}(V, t; \bar{D}, \sigma, r_f) \quad (9)$$

$$= VN(d_1) - e^{-r_f(T-t)}\bar{D}N(d_2) \quad (10)$$

where

$$d_1 = \frac{\ln(V/\bar{D}) + (r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (11)$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (12)$$

Note that in the risk neutral case the V in equ. (10) refers to the current value of the firm, but of course it is determined by the discounted future income stream of the firm. Yet in the risk free case we can have

$$\begin{aligned} S(V, t) &= C^{BS}(V, t; \bar{D}, \sigma, r_f) \\ &= e^{-r_f(T-t)}(VN(d_1)e^{r_f(T-t)} - \bar{D}N(d_2)) \end{aligned}$$

A Gauss computer program for the above evaluation of corporate debt from the perspective of derivative pricing is available.¹ Schönbucher (2003, ch.) extends the model

¹available upon request

by also taking into account a safety covenant acting as a default barrier. He also introduces bankruptcy cost, and a time varying interest rate, following a Brownian motion, for example, co-varying with stock market shocks. The firm value approach tries to model the whole obligor at once through linking the debt and equity with a hedge. A large and important disadvantage of the model is that one does not observe the process V with its driving factors.

6 Computing Firm Value and Creditworthiness

In Grüne and Semmler (2005) the firm value is derived from an intertemporal behavior of firms. There, however, only for the deterministic case. Yet, in Grüne, Semmler and Bernard (2006) the stochastic case is also considered.

We give a formal presentation of the deterministic model. We can say in the bilateral contract between a creditor and debtor there are two problems involved. The first pertains to the computation of debt and the second to the computation of the debt ceiling. The first problem is usually answered by employing an equation of the form

$$\dot{B}(t) = \theta B(t) - f(t), \quad B(0) = B_0$$

where $B(t)$ is the level of debt² at time t , θ the interest rate determining the credit cost and $f(t)$ the net income of the agent. The second problem can be settled by defining a debt ceiling such as

$$B(t) \leq C, \quad (t > 0)$$

or less restrictively by

$$\sup_{t \geq 0} B(t) < \infty$$

²Note that all subsequent state variables are written in terms of efficiency labor along the line of Blanchard (1983).

or even less restrictively by the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\theta t} B(t) = 0. \quad (13)$$

The ability of an obligator to service the debt, i.e. the feasibility of a contract, will depend on the obligator's source of income. Along the lines of intertemporal models of borrowing and lending³ we model this source of income as arising from a stock of capital $k(t)$, at time t , which changes with the investment rate $j(t)$ at time t through

$$\dot{k}(t) = j(t) - \sigma(k(t)), \quad k(0) = k_0. \quad (14)$$

In our general model both the capital stock and the investment are allowed to be multivariate. As debt service we take the net income from the investment rate $j(t)$ at capital stock level $k(t)$ minus some minimal rate of consumption.⁴ Hence

$$\dot{B}(t) = \theta B(t) - f(k(t), j(t)), \quad B(0) = B_0 \quad (15)$$

where $\theta B(t)$ is the credit cost. Note that the credit cost is not necessarily a constant factor (a constant interest rate). We call $B^*(k)$ the creditworthiness of the capital stock k . The problem to be solved is how to compute B^* .

If there is a constant credit cost factor (interest rate), $\theta = \frac{H(B,k)}{B}$, then, it is easy to see, $B^*(k)$ is the present value of k or the asset price of k :

$$B^*(k) = \underset{j}{Max} \int_0^{\infty} e^{-\theta t} f(k(t), j(t)) dt - B(0) \quad (16)$$

³Prototype models used as basis for our further presentation can be found in Blanchard (1983), Blanchard and Fischer (1989) or Turnovsky (1995).

⁴In the subsequent analysis of creditworthiness we can set consumption equal to zero. Any positive consumption will move down the creditworthiness curve. Note also that public debt for which the Ricardian equivalence theorem holds, i.e. where debt is serviced by a non-distortionary tax, would cause the creditworthiness curve to shift down. In computing the "present value" of the future net surpluses we do not have to assume a particular interest rate. Yet, in the following study we neither elaborate on the problem of the price level nor on the exchange rate and its effect on net debt and creditworthiness.

s.t.

$$\dot{k}(t) = j(t) - \sigma(k(t)), \quad k(0) = k_s \quad (17)$$

$$\dot{B}(t) = \theta B(t) - f(k(t), j(t)), \quad B(0) = B_0. \quad (18)$$

The more general case is, however, that θ is not a constant. As in the theory of credit market imperfections we generically may let θ depend on k and B , see below.⁵ Employing a dynamic model of the firm⁶ we can use the following net income function that takes account of adjustment investment and adjustment cost of capital.

$$f(k, j) = k^\alpha - j - j^\beta k^{-\gamma} \quad (19)$$

where $\sigma > 0, \alpha > 0, \gamma > 0$ are constants.⁷ In the above model $\sigma > 0$ captures both a constant growth rate of productivity as well as a capital depreciation rate. Blanchard (1983) used $\beta = 2, \gamma = 1$ to analyze the optimal indebtedness of a firm (see also Blanchard and Fischer 1989, Chap. 2).

The maximization problem (16)-(18) can be solved by using the necessary conditions of the Hamiltonian for (16)-(17). Thus we maximize

$$Max_j \int_0^\infty e^{-\theta t} f(k(t), j(t)) dt$$

s.t. (17).

The Hamiltonian for this problem is

⁵The more general theory of creditworthiness with state dependent credit cost is provided in Grüne, Semmler and Sieveking (2004). Note that instead of relating the credit cost inversely to net worth, as in Bernanke, Gertler and Gilchrist (1998), one could use the two arguments, k and B , explicitly.

⁶The subsequent model can be viewed as a standard RBC model where the stochastic process for technology shocks is shut down and technical change is exogenously occurring at a constant rate.

⁷Note that the production function k^α may have to be multiplied by a scaling factor. For the analytics we leave it aside here.

$$\begin{aligned}
H(k, x, j, \lambda) &= \max_j H(k, x, j, \lambda) \\
H(k, x, j, \lambda) &= \lambda f(k, j) + x(j - \sigma k) \\
\dot{x} = \frac{-\partial H}{\partial k} + \theta x &= (\sigma + \theta)x - \lambda f_k(k, j).
\end{aligned}$$

We denote x as the co-state variable in the Hamiltonian equations and λ is equal to 1.⁸ The function $f(k, j)$ is strictly concave by assumption. Therefore, there is a function $j(k, x)$ which satisfies the first order condition of the Hamiltonian

$$f_j(k, j) + x = 0 \quad (20)$$

$$j = j(k, x) = \left(\frac{x - 1}{k^{-\gamma} \cdot \beta}\right)^{\frac{1}{\beta-1}} \quad (21)$$

and j is uniquely determined thereby. It follows that (k, x) satisfy

$$\dot{k} = j(k, x) - \sigma k \quad (22)$$

$$\dot{x} = (\sigma + \theta)x - f_k(k, j(k, x)) \quad (23)$$

The isoclines can be obtained by the points in the (k, x) space for $\beta = 2$ where $\dot{k} = 0$ satisfies

$$x = 1 + 2\sigma k^{1-\gamma} \quad (24)$$

and where $\dot{x} = 0$ satisfies

$$x_{\pm} = 1 + \vartheta k^{1-\gamma} \pm \sqrt{\vartheta^2 k^{2-2\gamma} + 2\vartheta k^{1-\gamma} - 4\alpha\gamma^{-1}k^{\alpha-\gamma}} \quad (25)$$

where $\vartheta = 2\gamma^{-1}(\sigma + \theta)$. Note that the latter isocline has two branches.

If the parameters are given, the steady state – or steady states, if there are multiple ones – can be computed and then the local and global dynamics studied. We scale the production function by α .⁹

⁸For details of the computation of the equilibria in the case when one can apply the Hamiltonian, see Semmler and Sieveking (1998), appendix.

⁹We have multiplied the production function by $a = 0.30$ in order to obtain sufficiently separated equilibria, and take $c = 0$. We employ the following parameters: $\alpha = 1.1$, $\gamma = 0.3$, $\sigma = 0.15$, $\theta = 0.1$.

There is another solution technique which allows one to solve for firm value by using a dynamic programming approach. The alternative solution method uses the Hamilton-Jacobi-Bellman (HJB) equation.

In this appendix we present the solution technique of how to find the solution of the HJB-equation. We describe an algorithm which enable us to compute the asset price of the firm for the HJB equation of a type such as (1) which will give us the present value borrowing constraint. We show of how one can explicitly compute firm value using modern dynamic decision theory.

The HJB-equation for our problem reads

$$\theta V = \max_j [k^\alpha - j - j^2 k^{-\gamma} + V'(k)(j - \sigma k)] \quad (26)$$

Using the HJB equation we also can compute the steady state equilibria.

For the steady state, for which $0 = j - \sigma k$ holds, we obtain:

$$V(k) = \frac{f(k, j)}{\theta} \quad (27)$$

$$V'(k) = \frac{f'(k, j)}{\theta} = \frac{\frac{\partial}{\partial k}(k^\alpha - \sigma k - \sigma^2 k^{2-\gamma})}{\theta} \quad (28)$$

Using the information of (27)-(28) in (26) gives, ater taking the derivatives of (26) with respect to j , the steady states for the stationary HJB equation:

$$-1 - 2jk^{-\gamma} + \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2 - \gamma)k^{1-\gamma}}{\theta} = 0 \quad (29)$$

Note that hereby $j = \sigma k^{10}$. Given our parameters the equation may admit multiple steady states.

We specify the company's technology parameters to be $\sigma = 0.15, A = 0.29, \alpha_i = 0.7, \beta_i = 2, \gamma = 0.3$ and $\theta = 0.1$. The remaining parameters are specified below.

¹⁰Note that this gives us the same equilibria as using the Hamiltonian approach.

As for the numerical procedure an example was computed for different k 's in the compact interval $[0, 2]$, using dynamic programming with control range $j \in [0, 0.25]$. The dynamic programming algorithm (DP) used here is built on the HJB equation and is explained in Grüne and Semmler (2004). From this algorithm we obtain the figure below which approximates the present value curve $V(k)$ representing firm value.

We have considered our deterministic formulation above. In this case, debt is issued, but with no default premium. Thus, the credit cost is given by $H(k, B) = \theta B$. We have used the above mentioned DP algorithm in order to solve the discounted infinite horizon problem (16)-(18). Figure 3 shows the corresponding value function representing the present value curve, $V(k)$. The present value curve represents the asset value of the company for initial conditions $k(0)$ and thus its creditworthiness.

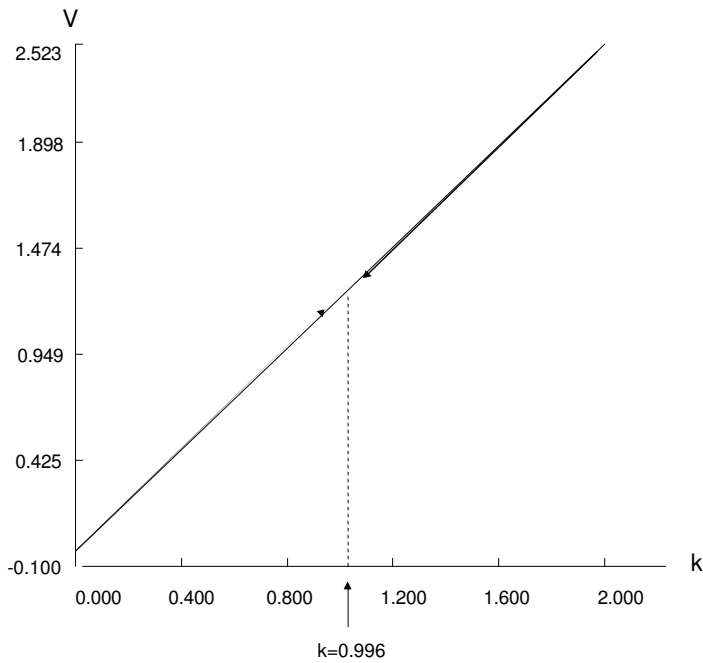


Figure 3: Present Value of Company's Capital Assets

The debt control problem is solved whenever debt is below the firm's asset value, so that we have $V - B \geq 0$. The optimal investment strategy is not constrained and thus the asset value which represents the maximum debt capacity V , is obtained by a solution for

an unconstrained optimal investment strategy, represented by the present value curve in Figure 3. For initial values of the capital assets above or below k^* , the optimal trajectories tend to the domain of attraction $k^* = 0.996$. For all initial conditions, the debt dynamics remain bounded as long $V - B \geq 0$, thus allowing the company's equity holders to exercise the option of retiring the debt. Any initial debt above the present value curve will be explosive and the company will lose its creditworthiness, since it will not be able to pay its obligations.

For the more general case where a default premium is to be paid we can use the following function to represent risk premia:

$$H(k(t), B(t)) = \frac{\alpha_1}{\left(\alpha_2 + \frac{N(t)}{k(t)}\right)^\mu} \theta B(t)$$

For the model (16)-(18) with a risk premium included in the company's borrowing cost, it is not possible to transform the model into a standard infinite horizon optimal control problem. This results because debt is now an additional constraint on the optimization problem. Hence, we need to use another method firm value and one can undertake experiments for different shapes of the credit cost function representing different alternative functions for the risk premium. An important class of functions for risk premia is defined by the steepness of the slope defined by the parameter α_2 , for details, see Grüne, Semmler and Bernard (2006). There are also results reported as to what extent the value of this company is affected by a default premium.

7 Moody's KMV

Due to the difficulties in computing the present value for firm value models¹¹ a practical implementation has been developed which comes with solutions to this problem. The KMV model, named after the founders Kealhover, McQuown and Vasicek (2001), models credit risk and the default probability of a firm as follows.

¹¹A more practical method of computing firm value is proposed in Benninga (1998, chs. 2-3).

7.1 The Distance-to-Default

The model states that there are three main elements determining the default probability of a firm:

- **Value of assets** is the market value of the firm's assets.
- **Asset risk** is the uncertainty or risk of the asset value. This is a measure of the firm's business and industry risk.
- **Leverage** is the extent of the firm's contractual liabilities. It is the book value of liabilities relative to the market value of assets.

As in equs (3) and (10) the default risk of the firm increases when the value of the assets approaches the book value of the liabilities. The firm defaults when the market value of the assets is smaller than the book value of the liabilities.

According to Peter Crosbie and Jeff Bohn (2003) who wrote the paper *Modelling Default Risk* for Moody's, their studies do not confirm this thesis in general. Not all the firms which reach the point where the asset value goes below the book value of their liabilities default. There are many which continue and serve their debt. The reason for this can be found in the long-term liabilities which enable the firms to continue their business until the debt becomes due. The firms may also have credit lines at their disposal.

Crosbie and Bohn draw the conclusion that the default point, the asset value at which the firm will default, generally lies somewhere between total liabilities and short-term liabilities. The relevant net worth of the firm is therefore defined as:

$$[\text{Market Net Worth}] = [\text{Market Value of Assets}] - [\text{Default Point}] \quad (30)$$

If the market net worth of a firm is zero, the firm is assumed to default. To measure the default risk, one can combine all three elements determining the default probability in a single measure of default risk, the distance-to-default:

$$\begin{aligned}
[\text{Distance-to-Default}] &= \frac{[\text{Market Net Worth}]}{[\text{Size of One Standard Deviation of the Asset Value}]} & (31) \\
&= \frac{[\text{Market Value of Assets}]-[\text{Default Point}]}{[\text{Asset Volatility}]} & (32)
\end{aligned}$$

The distance-to-default is the number of standard deviations the asset value is away from default. The default probability can then be computed directly from the distance-to-default if the probability distribution of the asset value is known.

7.2 The Probability of Default

Crosbie and Bohn (2003) give 6 variables that determine the default probability of a firm over some horizon, from now until time H (see figure 4):

1. The current asset value
2. The distribution of the asset value at time H
3. The volatility of the future assets value at time H
4. The level of the default point, the book value of the liabilities
5. The expected rate of growth in the asset value over the horizon
6. The length of the horizon, H

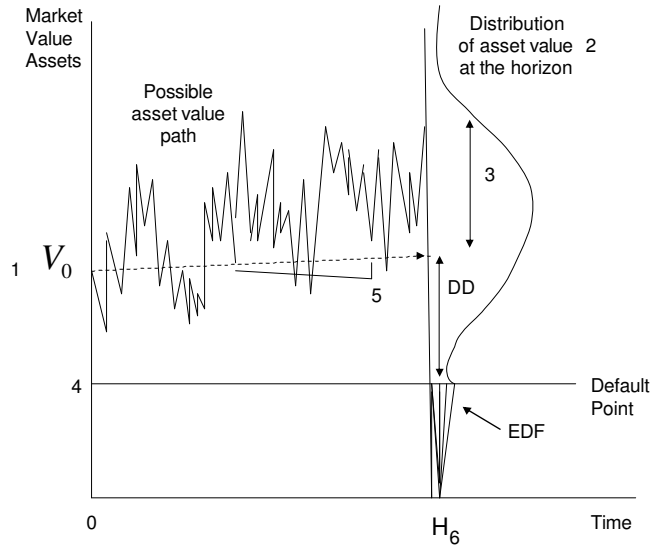


Figure 4:

The probability of default (expected default frequency or EDFvalue) can be computed with the aid of the measure we calculated above and data on historical default and bankruptcy frequencies. The database that Moody's uses consists of more than 400,000 company-years of data and more than 4,900 incidents of default or bankruptcy (see figure 4). From this data, a frequency table can be generated which relates the likelihood of default to the distance-to-default measure.

For example, a firm that is 7 standard deviations away from default has an expected default frequency (EDF value) of 5 per cent which leads to a rating of AA. In this case, Moody's analysis the default history of the fraction of firms which were 7 standard deviations away from the default point and defaulted over the next year. According to Crosbie and Bohn (2003), Moody's tested the relationship between distance-to-default and default frequency for industry, size, time and other effects and has found that the relationship is constant across all of these variables.

Those relationships can be developed in mathematical terms. According to the Black-Scholes model and as above in equ. (1) presumed, the market value of the firm's underlying assets is described by the following stochastic process:

$$dV_A = \mu V_A dt + \sigma_A V_A dz \quad (33)$$

where

V_A, dV_A are the firm's asset value and change in asset value

μ, σ_A are the firm's asset value drift rate and volatility

dz is a Wiener process

The probability of default that the market value of the firm's assets will be less than the book value of the firm's liabilities by the time the debt matures:

$$\begin{aligned} p_t &= Pr[V_A^t \leq X_t \mid V_A^0 = V_A] \\ &= Pr[\ln V_A^t \leq \ln X_t \mid V_A^0 = V_A] \end{aligned} \quad (34)$$

where

p_t is the probability of default by time t

V_A^t is the market value of the firm's assets at time t

X_t is the book value of the firm's liabilities due at time t

The change in the value of the firm's assets is described by (16), so the value at time t, V_A^t , given that the value at time 0 is V_A , is:

$$\ln V_A^t = \ln V_A + \left(\mu - \frac{\sigma_A^2}{2} \right) t + \sigma_A \sqrt{t} \varepsilon \quad (35)$$

where

μ is the expected return on the firm's asset

ε is the random component of the firm's return

Equation (18) describes the asset value path shown in figure 3. Combining (17) and (18), one can write the probability of default as:

$$p_t = Pr \left[\ln V_A + \left(\mu - \frac{\sigma_A^2}{2} \right) t + \sigma_A \sqrt{t} \varepsilon \leq \ln X_t \right] \quad (36)$$

or

$$p_t = Pr \left[-\frac{\ln \frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \geq \varepsilon \right] \quad (37)$$

Since the Black-Scholes model assumes that ε is normally distributed, one can write the default probability as:

$$p_t = N \left[-\frac{\ln \frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \right] \quad (38)$$

Since the distance-to-default measure is nothing else than the number of standard deviations that the firm is away from default, one can write this measure with the Black-Scholes notation as:

$$[\text{Distance-to-Default}] = \frac{\ln \frac{V_A}{X_t} + \left(\mu - \frac{\sigma_A^2}{2} \right) t}{\sigma_A \sqrt{t}} \quad (39)$$

Given an example that we compute a distance-to-default from equation (22) that equals 3.0, the probability of default using equation (21) will then be 13 basis points or 13 per cent. In practice, this distance-to-default measure is adjusted to include several other factors which play a role in measuring the default probability.

8 Empirical Evidence for Firm Value Based Models

There are several advantages and disadvantages that firm value based models have in practice. The predictions of firms value based models on the dynamics of share and debt prices of firms, are discussed briefly in this section. After a few empirical papers are discussed the general importance of these models will be evaluated.

While the majority of firm value based models predict a hilly shape for the term structure of credit spreads, Litterman and Iben (1991) showed that this is only true for rating classes of firms with bad rating. For other classes, like investment-grade rating classes, they observed increasing credit spreads rather than hilly ones.

The aim of another empirical work the one by Lardic and Rouzeau (1999), was to reproduce the risk ranking of obligors using firm value models. The test was designed not to study the real market value of the firms but to derive the risk level of firms in such a way that allowed to differentiate between riskier and less risky assets. The results however showed that the models were not able to reproduce the risk ranking of obligors. Instead, they were only able to recognize changes in the credit quality of the same obligor.

Longstaff and Schwartz (1995) investigated credit spread movements. With their tests using Moody's corporate bond yield averages, they found that there is a negative correlation between spreads and rates, meaning that firm value based models cannot be used for hedging purposes.

Concerning the pricing accuracy of firm value based models, Eom et al. (2000) run a test where they priced corporate bonds using the current share prices and balance sheet data of firms that issued the bonds. According to this test where the dynamics of the spreads were not included, it was found that there are pricing errors in all models.

Approximating data on fundamental is an essential strength of firm value based models, but defining the actual firm value can be really an complex issue. The problems can quickly become too complex to be handled by empirical tests. Despite all the complications one has to deal with when using firm value models, a more practical approach like Moody's KMV shows that one can obtain acceptable results and a better pricing performance with some pragmatic approach (see section 7).

Appendix 1

Table A1

OTC derivatives market¹

Triennial Central Bank Survey of Foreign Exchange and Derivatives Market Activity

Amounts outstanding in billions of US dollars

	Notional amounts		Gross market values	
	End-June	End-June	End-June	End-June
	2001	2004	2001	2004
GRAND TOTAL	99,659	220,058	3,045	6,394
A. Foreign exchange contracts	20,434	31,510	967	1,118
Outright forwards and forex swaps	13,275	16,764	548	483
Currency swaps	4,302	7,939	339	506
Options	2,824	6,806	80	150
Other	33	7	0	0
B. Interest rate contracts ²	75,813	177,432	1,748	4,581
FRAs	7,678	14,399	32	211
Swaps	57,220	137,277	1,531	3,978
Options	10,913	25,756	185	393
Other	2	0	0	0
C. Equity-linked contracts	2,039	5,094	220	321
Forwards and swaps	373	774	55	72
Options	1,666	4,320	164	249
D. Commodity contracts ³	674	1,354	88	177
Gold	278	360	25	47
Other	396	995	63	130
Forwards and swaps	235	541	–	–
Options	162	453	–	–

¹ All figures are adjusted for double counting. Notional amounts outstanding have been adjusted by halving positions vis-a-vis other reporting dealers. Gross market values have been calculated as the sum of the total gross positive market value of contracts and the absolute value of the gross negative market value of contracts with non-reporting counterparties.

² Single currency contracts only.

³ Adjustments for double-counting partly estimated.

⁴ Gross market values after taking into account legally enforceable bilateral netting agreements.

⁵ Sources: FLOW TRADEdata, Future industry Association; various futures and options exchanges.

Appendix 2: The Numerical Solution of the Model

We here briefly describe the dynamic programming algorithm as applied in Grüne and Semmler (2004) that enables us to numerically solve the dynamic model as proposed in section 3. The feature of the dynamic programming algorithm is an adaptive discretization of the state space which leads to high numerical accuracy with moderate use of memory.

Such algorithm is applied to discounted infinite horizon optimal control problems of the type introduced in section 3. In our model variants we have to numerically compute $V(x)$ for

$$V(x) = \max_u \int_0^\infty e^{-r} f(x, u) dt$$

s.t. $\dot{x} = g(x, u)$

where u represents the control variable and x a vector of state variables.

In the first step, the continuous time optimal control problem has to be replaced by a first order discrete time approximation given by

$$V_h(x) = \max_j J_h(x, u), \quad J_h(x, u) = h \sum_{i=0}^{\infty} (1 - \theta h) U f(x_h(i), u_i) \quad (\text{A1})$$

where x_u is defined by the discrete dynamics

$$x_h(0) = x, \quad x_h(i+1) = x_h(i) + hg(x_i, u_i) \quad (\text{A2})$$

and $h > 0$ is the discretization time step. Note that $j = (j_i)_{i \in \mathbb{N}_0}$ here denotes a discrete control sequence.

The optimal value function is the unique solution of a discrete Hamilton-Jacobi-Bellman equation such as

$$V_h(x) = \max_j \{hf(x, u_o) + (1 + \theta h)V_h(x_h(1))\} \quad (\text{A3})$$

where $x_h(1)$ denotes the discrete solution corresponding to the control and initial value x after one time step h . Abbreviating

$$T_h(V_h)(x) = \max_j \{hf(x, u_o) + (1 - \theta h)V_h(x_h(1))\} \quad (\text{A4})$$

the second step of the algorithm now approximates the solution on grid Γ covering a compact subset of the state space, i.e. a compact interval $[0, K]$ in our setup. Denoting the nodes of Γ by $x^i, i = 1, \dots, P$, we are now looking for an approximation V_h^Γ satisfying

$$V_h^\Gamma(X^i) = T_h(V_h^\Gamma)(X^i) \quad (\text{A5})$$

for each node x^i of the grid, where the value of V_h^Γ for points x which are not grid points (these are needed for the evaluation of T_h) is determined by linear interpolation. We refer to the paper cited above for the description of iterative methods for the solution of (A5). Note that an approximately optimal control law (in feedback form for the discrete dynamics) can be obtained from this approximation by taking the value $j^*(x) = j$ for j realizing the maximum in (A3), where V_h is replaced by V_h^Γ . This procedure in particular allows the numerical computation of approximately optimal trajectories.

In order to distribute the nodes of the grid efficiently, we make use of a posteriori error estimation. For each cell C_l of the grid Γ we compute

$$\eta_l := \max_{k \in C_l} | T_h(V_h^\Gamma)(k) - V_h^\Gamma(k) |$$

More precisely we approximate this value by evaluating the right hand side in a number of test points. It can be shown that the error estimators η_l give upper and lower bounds for the real error (i.e., the difference between V_j and V_h^Γ) and hence serve as an indicator for a possible local refinement of the grid Γ . It should be noted that this adaptive refinement of the grid is very effective for computing steep value functions and models with multiple equilibria, see Grüne and Semmler (2004).

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