Real – financial market interaction: A KMG portfolio approach

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Abstract

An integrated monetary growth model of Keynes–Metzler–Goodwin type with a portfolio approach to its three asset markets (money, bonds, equities) is introduced to study the interaction between the real and the financial part of market economies. Beneath expectations and governmental behavior, profits and their implied dividend payments influence the behavior of asset markets, which determine interest rate and Tobin's q by means of a general equilibrium approach. Tobin's q drives firms' investment in business fixed capital and performs the link from financial markets to good markets. We study the model in intensive form, the comparative statics of its asset market module, its unique interior steady state solution and its stability, the latter by way of an 8D dynamical system and its various subsystems, proving local stability assertions and the existence of so called Hopf–bifurcation that give rise to limit cycles in the 8D state space.

JEL classification **E12, E32, E41, E44**

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1 Introduction

The goal of this paper is to present a Keynesian macrodynamic model of a growing monetary economy, that builds on the analysis of the working KMG model[1] of Chiarella and Flaschel (2000a) and Chiarella, Flaschel, Groh, and Semmler (2000) and that explains the real–financial interaction in Keynesian dynamics in a more satisfactory way than in the working KMG model from which it has been derived. In this latter model type, asset markets influence the real dynamics only in a very traditional way, by means of an LM curve that gave rise to a stable relationship between the nominal rate of interest, the output capital ratio and real balances per unit of capital. Furthermore neither bond dynamics nor the evolution of the stock of equities could there influence the real part of the economy due to the lack of wealth and interest income effects on aggregate demand. The present paper will now introduce a portfolio theory of asset market behavior in the place of a single LM curve and will thereby improve the representation of asset market dynamics considerably, though wealth and interest income effects will still be ignored. Nevertheless bond and equity stock dynamics now feeds back into the real part of the economy, yet still by a single route namely through Tobin's average q as one important argument in the investment behavior of firms.

Our KMG approach to macrodynamics investigates the interaction of all important markets of the macroeconomy (for labor, goods, money, bonds and equities) still in a non stochastic environment without explicit utility maximization of households and profit maximization of firms.[2] Households behavioral equations are in the tradition of the Kaldorian approach (Kaldor 1940) with differentiated saving habits and are not derived by optimizing a hypothetical utility function of workers or capitalists. On the one hand, this method reflects our skepticism about the relevance of utility maximization for aggregate behavioral relationships (in an economy with labor and goods market disequilibrium) and on the other hand it allows us to leave the model sufficiently simple in order to concentrate on the description and analysis of asset market dynamics.[3] Combining a full disequilibrium approach in the real part of the economy with a general equilibrium approach in the financial part gives rise to various interesting considerations of the dynamics which then drives the economy. The model therefore presents an integrated approach to macrodynamics that accounts for all budget constraints of all types of agents in the economy, exhibits a uniquely determined steady state solution surrounded by a variety of interesting propagation or the feedback mechanisms existing in the economy. It is therefore a consistently formulated

[1] Keynes-Metzler-Goodwin model. This model type makes use of labor and goods market disequilibrium adjustment processes in the tradition of Goodwin and Metzler, respectively. It therefore considers explicitly the interaction of income distribution with economic growth, the interaction of disappointed sales expectations of firms and resulting unintended inventory changes and a simple LM theory of the money market which allows the investigation of Keynes– as well as Mundell–effects of wage and price inflation or deflation.

[2] See however Chiarella, Flaschel, Groh, Köper, and Semmler (1999) for improvements of this approach into this latter direction.

[3] See again Chiarella, Flaschel, Groh, Köper, and Semmler (1999) for improvements of this approach with respect to workers consumption and savings behavior.

integrated dynamical model on the aggregate level which exhibits a rich dynamic structure with a type of high order dynamics that has not been investigated from the theoretical perspective in the macroeconomic literature.

As already stated, the core of the model is given by a Keynes–Metzler–Goodwin (KMG) approach to integrated macrodynamics as developed in Chiarella and Flaschel (2000a) and further analyzed in Chiarella, Flaschel, Groh, and Semmler (2000).[4] Foundations to such integrated macrodynamical model building were already laid in Flaschel, Franke, and Semmler (1997). Further work is in preparation that extends the KMG approach to macrodynamics towards the treatment of small or interacting open economies, see Chiarella, Flaschel, Franke, and Lux (2000) and towards a theoretical as well as numerical analysis of modern macroeconometric model building in Chiarella, Flaschel, and Semmler (2000). The level of macrodynamic modeling that is reached in the present paper already goes beyond the work just cited and represents part of the basis of future work with the topic 'Firms, Finance, and Economic Policy' that will continue the above cited approaches towards a deeper treatment of firms behavior, the influence of financial markets on the real economy and the treatment of fiscal and monetary policy rules that may help to improve the dynamics that typically characterizes models of KMG growth.

The main properties of this approach should be briefly presented. The economy consists of various private agents: workers, asset holders and firms. The public sector consists of the government and the central bank. Concerning the good market, there exists a production good exclusively produced by firms, that can be, on the one hand consumed by the workers, asset holders or the government, and on the other hand invested as business fixed capital or used for inventory investment by firms. Firms do not have perfect foresight with respect to the demand for goods and do not adjust their output instantaneously towards the level of aggregate demand. Hence, in order to be able to satisfy actual and future demands, they have to hold stocks of inventories of produced goods. The adjustment policy for reaching a desired stock of inventories is modeled in a Metzlerian way (Metzler 1941).

The labor market is assumed to take place under a Keynesian regime in the sense, that any demand can be satisfied by an always positive excess supply of labor at the actual wage rate. Goodwin's contribution to the model is the dynamic interaction of employment and the real wage rate (Goodwin 1969).

We want to model a monetary economy with various financial assets in order to investigate their interaction with the real parts, namely good market and labor market. There are various assets: money and short term bonds issued by the government, and equities issued by firms in order to finance investments. All this financial assets are exclusively held by the asset holders.

In section 2 we develop the extensive form of the model and give a detailed explanation of its structure. In section 3 the intensive form of the dynamics is derived in order to allow for steady state considerations on the basis of eight

[4]See also Chiarella and Flaschel (1996), Chiarella and Flaschel (1998) and Chiarella and Flaschel (1999), Chiarella and Flaschel (2000b), the latter two for the treatment of open economies in the KMG framework.

autonomous laws of motion that, as will be shown, indeed exhibit a unique point of rest or steady state. The stability of the full 8D dynamical system is analyzed in section 4 by way of a sequence of subsystems of increasing dimension. In two further sections we extend the model towards a treatment of cash balances held (by firms) for transaction purposes and towards a dividend policy of firms that allow for retained earnings of firms. Finally conclusions are drawn in section **??**. Two appendices provide detailed mathematical proofs for the propositions of previous sections and the notation that is employed in the paper.

2 A portfolio approach to KMG growth theory

In this section we provide the extensive or structural form of our growth model of KMG type, now exhibiting a portfolio equilibrium block in the place of the LM theory of the short–run rate of interest and the dynamic adjustment equations for the prices of the other assets of Chiarella, Flaschel, Groh, and Semmler (2000). We split the model into appropriate modules which primarily concern the sectors of the economy, namely households, firms, and the government (fiscal and monetary authority), but also represent the wage–price–interaction and the asset markets.

2.1 Households

As discussed in the introduction we disaggregate the sector of households in worker households and asset holder households. We begin with the description of workers' behavior:

Workers' households

$$\begin{aligned} \omega &= w/p, && (1)\\ C_w &= (1-\tau_w)\omega L^d, && (2)\\ S_w &= 0, && (3)\\ \hat{L} &= n = \text{const.} && (4) \end{aligned}$$

Equation (1) gives the definition of the real wage ω before taxation, where w denotes the nominal wage and p the actual price level. We operate in a Keynesian framework with sluggish wage and price adjustment processes, hence we take the real wage to be given exogenously at each moment in time. Further we follow the Keynesian framework by assuming that the labor demand of firms can always be satisfied out of the given labor supply, i.e., we do not allow for regime switches as they are discussed in Chiarella, Flaschel, Groh, and Semmler (2000, ch. 5). Then, according to (2), real income of workers equals the product of real wages times labor demand, which net of taxes $\tau_w \omega L^d$, equals workers' consumption, since we do not allow for savings of the workers as postulated in (3).[5] No savings implies, that wealth of workers is zero at every point in time. This in particular means

[5] See Chiarella, Flaschel, Groh, and Semmler (2000) for the inclusion of workers' savings into a KMG framework.

that the workers do not hold any money and that they consume instantaneously their disposable income.[6] As is standard in theories of economic growth, we finally assume in equation (4) a constant growth rate n of the labor force L based on the assumption that labor is supplied inelastically at each moment in time. The parameter n can be easily reinterpreted to be the growth rate of the working population plus the growth rate of labor augmenting technical progress.

The modeling of the asset holders income, consumption and wealth is described by the following set of equations:

Asset holders' households

$$\begin{aligned}
\rho^e &= (Y^e - \delta K - \omega L^d)/K && (5)\\
C_c &= (1-s_c)[\rho^e K + rB/p - T_c], \quad 0 < s_c < 1, && (6)\\
S_p &= s_c[\rho^e K + rB/p - T_c] && (7)\\
&= (\dot{M} + \dot{B} + p_e \dot{E})/p, && (8)\\
W_c &= (M + B + p_e E)/p, \quad W_c^n = pW_c. && (9)
\end{aligned}$$

The first equation (5) of this module of the model defines the expected rate of return on real capital ρ^e to be the ratio of the currently expected real cash flow and the real stock of business fixed capital K. The expected cash flow is given by expected real revenues from sales Y^e diminished by real depreciation of capital δK and the real wage sum ωL^d. We assume that firms pay out all expected cash flow in form of dividends to the asset holders. These dividend payments are one source of income for asset holders. The second source is given by real interest payments on short term bonds (rB/p) where r is the nominal interest rate and B the stock of such bonds. Summing up these types of interest incomes and taking account of lump sum taxes T_c in the case of asset holders (for reasons of simplicity) we get the disposable income of asset holders within the square brackets of equation (6), which together with a postulated fixed propensity to consume $(1 - s_c)$ out of this income gives us the real consumption of asset holders.

Real savings of pure asset owners is real disposable income minus their consumption as exposed in equation (7). They can allocate it in form of money $\dot{M}$, or buy other financial assets, namely short-term bonds $\dot{B}$ or equities $\dot{E}$ at the price p_e, the only financial instruments that we allow for in the present reformulation of KMG growth. Hence, savings of asset holders must be distributed to these assets as stated in equation (8). Real wealth of pure asset holders is defined on this basis in equation (9) as the sum of the real cash balance, real short term bond holdings and real equity holdings of asset holders. Note that the short term bonds are assumed to be fixed price bonds with a price of one, $p_b = 1$, and a flexible interest rate r.

We now describe the demand equations of asset owning households for financial assets following Tobin's general equilibrium approach (Tobin 1969):

$$M^d = f_m(r, r_e^e) W_c^n \qquad (10)$$

[6] We explain in an appendix that money holdings for transaction purposes is here only considered with respect to firms which is just the opposite assumption of what is usually considered in the macroeconomic literature.

$$B^d = f_b(r, r_e^e)W_c^n \quad (11)$$
$$p_e E^d = f_e(r, r_e^e)W_c^n \quad (12)$$
$$W_c^n = M^d + B^d + p_e E^d \quad (13)$$

The demand for money balances of asset holders M^d is determined by a function $f_m(r, r_e^e)$ which depends on the interest rate on short run bonds r and the expected rate of return on equities r_e^e. The value of this function times the nominal wealth W^n gives the nominal demand for money M^d, i.e., f_m describes the portion of nominal wealth that is allocated to pure money holdings. Note that this formulation of money demand is not based on a transaction motive, since the holding of transaction balances is the job of firms in the present paper. We also do not assume that the financial assets of the economy are perfect substitutes, but indeed assume that financial assets are imperfect substitutes by the approach that underlies the above block of equations. But what is the motive for asset holders to hold a fraction of their wealth in form of money, when there is a riskless interest bearing asset? In our view it is reasonable to employ a speculative motive: Asset holders want to hold money in order to *be able* to buy other assets or goods with zero or very low transaction costs. This of course assumes that there are (implicitly given) transaction costs when fixed price bonds are turned into money. Köper (2000b) will modify this framework by assuming that money holdings equal $M3$ and that bonds are flexprice or long-term bonds which give rise to capital gains or losses just as the equities of the present paper.

The nominal demand for bonds is determined by $f_b(r, r_e^e)$ and the nominal demand for equities by $f_e(r, r_e^e)$, which again describe the fractions that are allocated to these forms of financial wealth. From equation (9) we know that actual nominal wealth equals the stocks of financial assets held by the asset holders. We assume, as is usual in portfolio approaches, that the asset holders do demand assets of an amount which equals in sum their nominal wealth as stated in equation (9). In other words, they just reallocate their wealth in view of new information on the rates of returns on their assets and thus take care of their wealth constraint.

What is left to model in the households sector is the expected rate of return on equities r_e^e which consists of real dividends per equity ($\rho^e pK/p_e E$), and expected capital gains, π_e, the latter being nothing else than the expected growth rate of equity prices.

$$r_e^e = \frac{\rho^e pK}{p_e E} + \pi_e \quad (14)$$

In order to complete the modeling of asset holders' behavior we thus have to describe the evolution of π_e. We here assume that there are two types of asset holders, which differ with respect to their expectation formation of equity prices. There are *chartists* who in principle employ an adaptive expectations mechanism:

$$\dot{\pi}_{ec} = \beta_{\pi_{ec}}(\hat{p}_e - \pi_{ec}), \quad (15)$$

where $\beta_{\pi_{ec}}$ is the adjustment speed towards the actual growth rate of equity prices. The other asset holders, the *fundamentalists*, employ a forward looking

expectation formation mechanism:

$$\dot{\pi}_{ef} \quad = \quad \beta_{\pi_{ef}}(\bar{\eta} - \pi_{ef}) \tag{16}$$

where $\bar{\eta}$ is the fundamentalists' expected long run inflation rate of share prices. Assuming that the aggregate expected inflation rate is a weighted average of the two expected inflation rates, where the weights are determined according to the sizes of the groups, we postulate

$$\pi_e \quad = \quad \alpha_{\pi_{ec}}\pi_{ec} + (1 - \alpha_{\pi_{ec}})\pi_{ef}. \tag{17}$$

Here $\alpha_{\pi_{ec}} \in (0, 1)$ is the ratio of chartists to all asset holders.

2.2 Firms

We consider the behavior of firms by means of two submodules. The first describes the production framework and their investment in business fixed capital and the second introduces the Metzlerian approach of inventory cycles concerning expected sales, actual sales and the output of firms.

Firms: production and investment

$$\begin{aligned}
\rho^e &= (pY^e - wL^d - p\delta K)/(pK), && (18)\\
Y^p &= \bar{y}^p K, && (19)\\
U_c &= Y/Y^p, && (20)\\
L^d &= Y/\bar{x}, && (21)\\
V &= L^d/L = Y/(xL), && (22)\\
q &= p_e E/(pK), && (23)\\
I &= i_1(q-1)K + i_2(U_c - \bar{U}_c)K + nK, && (24)\\
\hat{K} &= I/K, && (25)\\
p_e\dot{E} &= pI + p(\dot{N} - \mathcal{I}) && (26)
\end{aligned}$$

Firms are assumed to pay out dividends according to expected profits (expected sales net of depreciation and minus the wage sum), see the above module of the asset owning households. The rate of expected profits ρ^e is expected real profits per unit of capital as stated in equation (18). For producing output firms utilize a production technology that transforms demanded labor L^d combined with business fixed capital K into output. For convenience we assume that the production takes place by a fixed proportion technology.[7] According to (19) potential output Y^p is therefore given in each moment of time by the fixed coefficient $\bar{y}^p$ times the existing stock of physical capital. Accordingly, the utilization of productive capacities is given by the ratio U_c of actual production Y and the potential output Y^p. The fixed proportions in production also give rise to constant output-labor coefficient $\bar{x}$, by means of which we can deduce labor demand from goods market

[7] See Chiarella, Flaschel, Groh, and Semmler (2000) for the treatment of neoclassical smooth factor substitution.

determined output as in equation (21). The ratio L^d/L thus defines the rate of employment of the model.

The economic behavior of firms also comprises the investment decision into business fixed capital, which is determined independently from households savings decision. We here model investment decisions per unit of capital as function of the deviation of Tobin's q, see Tobin (1969), from its long run value 1, and the deviation of actual capacity utilization from a normal rate of capital utilization, and add an exogenously given trend term, here given by the natural growth rate n in order to allow this rate to determine the growth path of the economy in the usual way. We employ here Tobin's average q which is defined in equation (23). It is the ratio of the nominal value of equities and the reproduction costs for the existing stock of capital. Investment in business fixed capital is enforced when q exceeds one, and is be reduced when q is smaller then one. This influence is represented by the term $i_1(q-1)$ in equation (24). The term $i_2(U_c - \bar{U}_c)$ models the component of investment which is due to the deviation of utilization rate of physical capital from its non accelerating inflation value $\bar{U}_c$. The last component, nK, takes account for the natural growth rate n which is necessary for steady state analysis if natural growth is considered as exogenously given. Equation (26) is the budget constraint of the firms. Investment in business fixed capital and unintended changes in the inventory stock $p(\dot{N} - \mathcal{I})$ must be financed by issuing equities, since equities are the only financial instrument of firms in this paper. Capital stock growth finally is given by net investment per unit of capital I/K in this demand determined modeling of the short–run equilibrium position of the economy.

Next we model the inventory dynamics in the model following Metzler (1941) and Franke (1992). This approach is a very useful concept for describing the goods market disequilibrium dynamics with all of its implications.

Firms output adjustment:

$$
\begin{aligned}
N^d &= \beta_{n^d} Y^e, && (27)\\
\mathcal{I} &= nN^d + \beta_n (N^d - N), && (28)\\
Y &= Y^e + \mathcal{I}, && (29)\\
Y^d &= C + I + \delta K + G, && (30)\\
\dot{Y}^e &= nY^e + \beta_{y^e}(Y^d - Y^e), && (31)\\
\dot{N} &= Y - Y^d, && (32)\\
S_f &= Y - Y^e = \mathcal{I}, && (33)
\end{aligned}
$$

where $\beta_{n^d}, \beta_n, \beta_{y^e} \geq 0$.

As stated in equation (27), the desired stock of physical inventories is denoted by N^d and is assumed to be a fixed proportion of the expected sales. The planned investments $\mathcal{I}$ in inventories follow a sluggish adjustment process towards the desired stock N^d according to equation (28). Taking account of this additional demand for goods we write the production Y to be set equal to the expected sales of firms plus $\mathcal{I}$ in equation (29). For explaining the expectation formation for good demand, we need the actual total demand for goods which is given by

consumption (of private households and the government) and gross investment by firms (30). By knowing the actual demand Y^d, which is always served, the dynamics of expected sales is given in equation (31). It models these expectations to be the outcome of an error correction process, that incorporates also the natural growth rate n in order take account of the fact that this process operates in a growing economy. The adjustment of sales expectations is driven by the prediction error $(Y^d - Y^e)$, with an adjustment speed that is given by β_{y^e}. Actual changes in the stock of inventories are given by the deviation of production from goods demanded (32). The savings of the firms S_f is as usual defined by income minus consumption. Because firms are assumed to not consume anything, their income equals their savings and is given by the excess of production over expected sales, $Y - Y^e$. According to the production account in figure 1 the gross accounting profit of firms finally is $\rho^e pK + p\mathcal{I} = pC + pI + p\delta K + p\dot{N} + pG$. Plugging in the definition of ρ^e from equation (18), we compute that $pY^e + p\mathcal{I} = pY^d + p\dot{N}$ or equivalently $p(Y - Y^e) = \mathcal{I}$ as stated in equation (33).

Uses	Resources
Production Account of Firms:	
Depreciation $p\delta K$	Private consumption pC
Wages wL^d	Gross investment $pI + p\delta K$
Gross accounting profits $\Pi = \rho^e pK + p\mathcal{I}$	Inventory investment $p\dot{N}$
	Public consumption pG
Income Account of Firms:	
Dividends $\rho^e p_y K$	Gross accounting profits Π
Savings $p\mathcal{I}$	
Accumulation Account of Firms:	
Gross investment $pI + p\delta K$	Depreciation $p\delta K$
Inventory investment $p\dot{N}$	Savings $p\mathcal{I}$
	Financial deficit FD
Financial Account of Firms:	
Financial deficit FD	Equity financing $p_e\dot{E}$

Figure 1: *Accounting sheets of the firms' sector*

2.3 Governmental sector

The role of the government in this paper is to provide the economy with public (unproductive) services within the limits of its budget constraint. Public purchases (and interest payments) are financed through taxes, through newly printed money, or newly issued fixed-price bonds ($p_b = 1$). The budget constraint gives

rise to some repercussion effects between the public and the private sector.

$$
\begin{aligned}
T &= \tau_w \omega L^d + T_c, && (34)\\
T_c &= \bar{t}_c^n K + rB/p, \qquad \bar{t}_c^n = \text{const.} && (35)\\
G &= \bar{g} K, \qquad \bar{g} = \text{const.} && (36)\\
S_g &= T - rB/p - G, && (37)\\
\hat{M} &= \bar{\mu}, && (38)\\
\dot{B} &= pG + rB - pT - \dot{M}. && (39)
\end{aligned}
$$

We model the tax income consisting of taxes on wage income and lump sum taxes on capital income T_c. The latter is assumed for reasons of analytical simplicity solely (for the time being) in way, that makes aggregate demand independent of the interest payments of the government, which in particular simplifies steady state calculations significantly, adding to our simplification of not including wealth effects on consumption into our model.[8]

For the real purchases of the government for providing governmental services we assume, again as in Sargent (1987), that they are a fixed proportion $\bar{g}$ of real capital, which taken together allows to represent fiscal policy by means of simple parameters on the intensive form level of the model and in the steady state considerations to be discussed later on. The real savings of the government, which is a deficit if it has a negative sign, is defined in equation (37) by real taxes minus real interest payments minus real public services. Again for reasons of simplicity the growth rate of money is given by a constant $\bar{\mu}$. Equation (38) is the monetary policy rule of the central bank and money is assumed to enter the economy via open market operations of the central bank, which buys short-term bonds from the asset holders when issuing new money. Then the changes in the short-term bonds supplied by the government are given residually in equation (39), which is the budget constraint of the governmental sector. This representation of the behavior of the monetary and the fiscal authority clearly shows that the treatment of policy questions is not yet a central part of the paper. See Köper (2000a) for an explicit treatment of government interest payments.

2.4 Wage–Price–Interaction

We now turn to the last sector of our model which is the wage-price sector picking up the Rose approach (Rose 1990) of two short-run Phillips curves, i) the wage Phillips curve and ii) the price Phillips curve:

$$
\begin{aligned}
\hat{w} &= \beta_w (V - \bar{V}) + \kappa_w \hat{p} + (1 - \kappa_w)\pi, && (40)\\
\hat{p} &= \beta_p (U_c - \bar{U}_c) + \kappa_p \hat{w} + (1 - \kappa_p)\pi, && (41)\\
\dot{\pi} &= \beta_\pi (\alpha \hat{p} + (1 - \alpha)(\bar{\mu} - n) - \pi). && (42)
\end{aligned}
$$

where $\beta_w, \beta_p, \beta_\pi \geq 0$, $0 \leq \alpha \leq 1$, and $0 \leq \kappa_w, \kappa_p \leq 1$. This approach makes use of the assumption that relative changes in money wages are influenced by

[8] See Sargent (1987) for another application of this assumption.

demand pressure in the market for labor and price inflation (cost-pressure) terms and that price inflation in turn depends on demand pressure in the market for goods and on money wage (cost-pressure) terms. Wage inflation therefore is described in equation (40) on the one hand by means of a demand pull term $\beta_w(V - \bar{V})$, which tells us that relative changes in wages depends positively on the gap between actual employment V and its NAIRU value $\bar{V}$. On the other hand, the cost push elements in wage inflation is the weighted average of short-run (perfectly anticipated) price inflation $\hat{p}$ and medium run expected overall inflation π, where the weights are given by κ_w and $1 - \kappa_w$. The price Phillips curve is quite similar, it displays a demand pull and a cost push component, too. The demand pull term is given by the gap between capital utilization and its NAIRU value, $(U_c - \bar{U}_c)$, and the cost push element is the κ_p and $1 - \kappa_p$ weighted average of short run wage inflation $\hat{w}$ and expected medium run overall inflation π.

What is left to model is the expected medium run inflation rate π. We postulate in equation (42) that changes in expected medium run inflation are due to an adjustment process towards a weighted average of the current inflation rate and steady state inflation. Thus we introduce here a simple kind of forward looking expectations into the economy. This adjustment is driven by an adjustment velocity β_π.

It is obvious from this description of the model that it is, on the one hand, already a very general description of macroeconomic dynamics. On the other hand, it is still dependent on some very special assumptions, in particular with respect to financial markets and the government sector. This can be justified at the present stage of analysis by observing that many of its simplifying assumptions are indeed typical for macrodynamic models, which attempt to provide a complete description of a closed monetary economy with labor, goods markets and three markets for financial assets, see in particular the model of Keynesian dynamics of Sargent (1987).

2.5 Capital markets

We have not yet discussed the determination of the nominal rate of interest r and the price of equities p_e and thus have not yet formulated how capital markets are organized. Following Tobin's portfolio approach (1969), see also Franke and Semmler (1999), we here simply postulate that the following equilibrium conditions always hold and thus determine the above two prices concerning bonds and equities as statically endogenous variables of the model. Note here that all asset supplies are given magnitudes at each moment in time and that r_e^e is given by $\frac{\rho^e pK}{p_e E} + \pi_e$ and thus varies at each point in time solely due to variations in the share price p_e.

$$M = M^d \, (= f_m(r, r_e^e) W_c^n) \tag{43}$$

$$B = B^d \, (= f_b(r, r_e^e) W_c^n) \tag{44}$$

$$p_e E = p_e E^d \, (= f_e(r, r_e^e) W_c^n) \tag{45}$$

In our model we thus support the view that the secondary market is the market, where the prices or interest rates for the financial assets are determined such that these markets are cleared at all moments in time. This implies, that the newly issued assets do not have significant impact effects on these prices.[9]

The trade between the asset holders induces a process that makes asset prices fall or rise in order to equilibrate demands and supplies. In the short run (in continuous time) the structure of wealth of asset holders, W_c^n is, disregarding changes in the share price p_e, given to them and for the model. This implies that the functions $f_m()$, $f_b()$, and $f_e()$, introduced in equations 10 to 12 must satisfy the following well known conditions:

$$f_m(r, r_e^e) + f_b(r, r_e^e) + f_e(r, r_e^e) = 1 \tag{46}$$

$$\frac{\partial f_m(r, r_e^e)}{\partial i} + \frac{\partial f_b(r, r_e^e)}{\partial i} + \frac{\partial f_e(r, r_e^e)}{\partial i} = 0, \quad \forall\, i \in \{r, r_e^e\} \tag{47}$$

These conditions guarantee that the number of independent equations is equal to the number of statically endogenous variables (r, p_e) that the asset markets are assumed to determine at each moment in time.

We postulate that the financial assets display the gross substitution property, which means that the demand for all other assets increase whenever the price of another asset rises. For a formal definition see for example Mas-Colell, Whinston, and Green (1995, p. 611).

$$\frac{\partial f_b(r, r_e^e)}{\partial r} > 0, \quad \frac{\partial f_m(r, r_e^e)}{\partial r} < 0, \quad \frac{\partial f_e(r, r_e^e)}{\partial r} < 0, \tag{48}$$

$$\frac{\partial f_e(r, r_e^e)}{\partial r_e^e} > 0, \quad \frac{\partial f_m(r, r_e^e)}{\partial r_e^e} < 0, \quad \frac{\partial f_b(r, r_e^e)}{\partial r_e^e} < 0. \tag{49}$$

The above discussion concentrates on stocks and their impact on asset prices, including the so-called Walras' law of stocks. The following proposition shows in addition that also the Walras' law of flows does hold in our modeled economy, representing an important consistency check of the model.

Proposition 1 *Assume that the issue of new bonds and money of the government are absorbed by the asset holders. Then: Every new issued amount of equities of firms will be met by the demand for equities by the asset holders.*

[9]This representation of the secondary markets as markets characterized by stock equilibrium at each moment in time may be turned into flow equilibrium conditions (including then new issue, possibly on primary markets) if it is assumed that desired stocks only give rise to sluggish desired adjustments to such target values, for example in the following way, where these demand flows are and can then to be coordinated with the new issue of money, bonds and equities.

$$\begin{aligned} \dot{M} &= \dot{M}^d = \beta_{fm}(f_m(\ldots)W^n - M) \\ \dot{B} &= \dot{B}^d = \beta_{fb}(f_b(\ldots)W^n - B) \\ p_e\dot{E} &= p_e\dot{E}^d = \beta_{fe}(f_e(\ldots)W^n - p_eE) \end{aligned}$$

Proof: For proving this proposition we refer to the definitions of nominal savings of the three considered sectors:

$$\begin{aligned} S_p^n &= \dot{M}^d + \dot{B}^d + p_e\dot{E}^d && (50)\\ S_g^n &= -\dot{M} - \dot{B} && (51)\\ S_f^n &= p\mathcal{I} && (52) \end{aligned}$$

The assumption made means that $\dot{M}^d = \dot{M}$ and $\dot{B}^d = \dot{B}$ holds. By definition we know that ex post investments equal savings. Investment is given by the investment into business fixed capital plus actual inventory investment. Savings are the sum of the savings of all sectors.

$$\begin{aligned} & pI + p\dot{N} &&= S_p^n + S_g^n + S_f^n \\ \Leftrightarrow \quad & pI + p\dot{N} &&= \dot{M}^d + \dot{B}^d + p_e\dot{E}^d - \dot{M} - \dot{B} + p\mathcal{I} \\ \Leftrightarrow \quad & pI + p\dot{N} &&= p_e\dot{E}^d + p\mathcal{I} \\ \Leftrightarrow \quad & pI + p(\dot{N} - \mathcal{I}) &&= p_e\dot{E}^d \end{aligned}$$

From equation (26) we conclude that $p_e\dot{E}^d = p_e\dot{E}$, which means that the demand for new equities equals its supply. ∎

3 The model in intensive form

In this section we derive the intensive form of the model, i.e., we will express all stock and flow variables in the laws of motion to be derived, and also in the needed algebraic equations, per unit of capital. We thus divide nominal stock and flow variables by the nominal value of the capital stock pK and all real ones by K, the real capital stock. This allows the determination of a (unique) steady state solution as interior point of rest of the state space considered.

We begin with the intensive form of some necessary definitions or identities, which we need needed for representing the dynamic system in a sufficiently comprehensible form. Note here that the function q used in this block of equations will be determined and discussed later on, in subsection 3.2, where the comparative statics of the portfolio part of the model is investigated.

$$\begin{aligned} Y/K &= y &&= (1+\beta_{n^d}(n+\beta_n))y^e - \beta_n\nu \\ L^d/K &= l^d &&= y/\bar{x} \\ & V &&= l^d/l \\ & U_c &&= y/y^p \\ & \rho^e &&= y^e - \delta - \omega l^d \\ C/K &= c &&= (1-\tau_w)\omega l^d + (1-s_c)(y^e - \delta - \omega l^d - \bar{t}_c^n) \\ I/K &= i &&= i_1(q-1) + i_2(U_c - \bar{U}_c) + n \\ Y^d/K &= y^d &&= c + i + \delta + \bar{g} \\ p_eE/(pK) &= q &&= q(m, b, \rho^e, \pi_e), \qquad \text{see subsection 3.2} \\ & r_e^e &&= \rho^e/q + \pi_e \\ & \pi_e &&= \alpha_{\pi_e}\pi_{ec} + (1-\alpha_{\pi_e})\pi_{ef} \end{aligned}$$

The above equations describe output and employment per unit of capital, the rate of utilization of the existing stock of labor and capital, the expected rate of profit, consumption, investment and aggregate demand per unit of capital, Tobin's average q, and the expected rate of return on equities (including expected capital gains π_e)

Now we translate the laws of motion of the dynamically endogenous variables into capital intensive form. The law of motions for the nominal wages and price level stated in equations (40) and (41) are interacting instantaneously and thus depend on each other. Solving these two linear equations for $\hat{w}$ and $\hat{p}$ gives

$$\begin{aligned} \hat{w} &= \kappa\left(\beta_w(V-\bar{V})+\kappa_w\beta_p(U_c-\bar{U}_c)\right)+\pi, && (53)\\ \text{and} \qquad \hat{p} &= \kappa\left(\beta_p(U_c-\bar{U}_c)+\kappa_p\beta_w(V-\bar{V})\right)+\pi, && (54)\end{aligned}$$

with $\kappa = (1-\kappa_w\kappa_p)^{-1}$. For a detailed computation see Chiarella and Flaschel (2000a) and here appendix A.1. From these two inflation rates one can compute the growth law of real wages $\omega = w/p$ by means of the definitional relationship $\hat{\omega} = \hat{w} - \hat{p}$. This gives us

$$\hat{\omega} = \kappa[(1-\kappa_p)\beta_w(V-\bar{V})+(\kappa_w-1)\beta_p(U_c-\bar{U}_c)], \qquad (55)$$

where κ is $(1-\kappa_w\kappa_p)^{-1}$. The next set of equations explains the dynamic laws of the expected rate of inflation, the labor capital ratio,the expected sales, and the stock of inventories in intensive form:

$$\begin{aligned} \dot{\pi} &= \alpha\beta_\pi\kappa[\beta_p(U_c-\bar{U}_c)+\kappa_p\beta_w(V-\bar{V})]+(1-\alpha)\beta_\pi(\bar{\mu}-n-\pi), && (56)\\ \hat{l} &= n-i = -i_1(q-1)-i_2(U_c-\bar{U}_c), && (57)\\ \dot{y}^e &= \beta_{y^e}(y^d-y^e)+(n-i)y^e, && (58)\\ \dot{\nu} &= y-y^d-i\nu, && (59)\end{aligned}$$

Equation (56) is almost the same as in the extensive modeling, but here the term $\hat{p}-\pi$ is substituted according to equation (54). Equation (57), the law of motion of relative factor endowment, is given by the (negative) of the investment function as far as its dependence on asset markets and the state of the business cycle are concerned. Equation (58) is obtained by way of the time derivative of y^e as follows:

$$\dot{y}^e = \tfrac{d(Y^e/K)}{dt} = \tfrac{\dot{Y}^eK-Y^e\dot{K}}{K^2} = \tfrac{\dot{Y}^e}{K} - y^ei = \beta_{y^e}(y^d-y^e)+y^e(n-i).$$

In essentially the same way one gets equation (59). The laws of motion governing the expectations about the equity prices are not changed by the intensive form modeling and thus again read as follows:

$$\begin{aligned} \dot{\pi}_{ef} &= \beta_{\pi_{ef}}(\bar{\eta}-\pi_{ef}), && (60)\\ \dot{\pi}_{ec} &= \beta_{\pi_{ec}}(\hat{p}_e-\pi_{ec}). && (61)\end{aligned}$$

In the following only the value of aggregate capital gains expectations is needed. But for the computation of this we need the historic values of actual inflation

in equity prices $\hat{p}_e$ for which we lack a law of motion, because the general equilibrium approach tells us $\hat{p}_e$ is such that asset markets are in equilibrium. We follow Sargent (1987, pp. 117), by employing the integral representation of the expectation about equity price inflation, which leads us to the following definition of aggregate expectation of equity price inflation:

$$\begin{aligned}\pi_e(t) &= \alpha_{ec}\left[\pi_{ec}(t_0)e^{-\beta_{\pi_{ec}}(t-t_0)} + \beta_{\pi_{ec}}\int_{t_0}^{t} e^{-\beta_{\pi_{ec}}(t-s)}\hat{p}_e(s)ds\right] \\ &\quad + (1-\alpha_{ec})\left[(\pi_{ef}(t_0)-\bar{\eta})e^{-\beta_{\pi_{ef}}t} + \bar{\eta}\right], \qquad (62)\end{aligned}$$

where $\pi_{ec}(t_0)$ and $\pi_{ef}(t_0)$ are the initial values of the expectations about growth in equity prices, performed by the chartist and the fundamentalists at time t_0. The details for obtaining this equation are given in the mathematical appendix A.2.

Finally, the laws of motion for real balances and real bonds per unit of capital have to be derived. Based on the knowledge of the laws for inflation $\hat{p}$ and investment i we can derive the differential equation for bonds per unit of capital shown in equation (63) from the following expression:

$$\dot{b} = \frac{d(B/pK)}{dt} = \frac{\dot{B}}{pK} - b(\hat{p}+i)$$

where $\dot{B}$ is given by equation (39). The same idea is used for the changes in the money supply. We thus get finally the following two differential equations:

$$\begin{aligned}\dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w\omega l^d - \bar{\mu}m \\ &\quad - b\left(\kappa[\beta_p(U_c-\bar{U}_c) + \kappa_p\beta_w(V-\bar{V})] + \pi + i\right), \qquad (63) \\ \dot{m} &= m\bar{\mu} - m(\kappa[\beta_p(U_c-\bar{U}_c) + \kappa_p\beta_w(V-\bar{V})] + \pi + i). \qquad (64)\end{aligned}$$

According to the above, the dynamics in extensive form can therefore be reduced to nine (eight) differential equations, where however the law of motion for share prices has not been determined yet, or to seven differential and one integral equation which is easier to handle than the alternative representation, since there is no law of motion for the development of future share prices to be calculated then. Note with respect to these dynamics that economic policy (fiscal and monetary) is still represented in very simple terms here, since money supply is growing with a given rate and since government expenditures and taxes on capital income net of interest payments per unit of capital are given parameters. This makes the dynamics of the government budget constraint, see the law of motion for bonds per unit of capital b, a very trivial one, as in Sargent (1987, ch. 5), and thus reserves the problems associated with these dynamics in the literature a matter for future research. The advantage that fiscal policy can be discussed in very simple way here by means of three parameters solely.

Comparing the present dynamics with those of the working KMG model of Chiarella and Flaschel (2000a) and Chiarella, Flaschel, Groh, and Semmler (2000) shows that there are now two variables from the financial sector that feed back to

the real dynamics in this extended system, the bond to capital ratio b representing the evolution of government debt and Tobin's average q. The first (dynamic) variable however only influences the real dynamics since it is one of the factors that influences the statically endogenous variable q which in turn enters the investment function as a measure of the firms' performance. Government bonds do not influence the economy in other ways, since there are no wealth effects in consumption yet and since the interest income channel to consumption has been suppressed by the assumption on tax collection concerning capital income. In addition, the interest rate channel of the earlier KMG approaches, where the real rate of interest as compared to the real profit rate entered the investment function, is now absent from this function. The nominal interest rate as determined by portfolio equilibrium thus does not matter in the present formulation of the model, where Tobin's q in the place of this interest rate provides now the channel by which investment behavior is reacting to the results brought about by the financial markets.

The present dynamics has no longer laws of motion that are left implicit in its discussion (the bond and the share price dynamics of the working KMG models cited above), but is now a completely formulated dynamics, yet still one where the real financial interaction is represented in basic terms. Price inflation (via real balances and real bonds) and the expected rate of profit (via the dividend rate of return) influence the behavior of asset markets via laws of motion for them, while the reaction of asset markets feeds back into the real part of the economy instantaneously through the change in Tobin's q that is caused by them (and the dynamics of expected capital gains).

3.1 Steady state considerations

In this subsection we show the existence of a steady state in the modeled economy. We here stress that this can be done independently of the knowledge supplied in the next section on the comparative statics of the asset market equilibrium system, since Tobin's q is given by 1 in the steady state via the real part of the model and since the portfolio equations can be uniquely solved in conjunction with the government budget constraint for the three variables r, m, b which they then determine. Note that m, b are data in the short-run analysis of the behavior of asset markets of the next subsection (where q, r are determined on their basis as the variables that bring the asset markets into equilibrium), while m, b are variables in the long run that are to be derived from asset market equilibrium conditions and the government budget constraint.

In the following variables with a "$\check{}$" on top denote the steady state value of the corresponding variable.

Proposition 2 *Assume $s_c > \tau_w$ and $s_c\check{\rho}^e > n + \bar{g} - \bar{t}_c^n$. Assume furthermore that the parameter ϕ used below has a positive numerator, i. e. ,the government runs a primary deficit in the steady state, (and thus between zero and one if money supply is growing). The dynamic system given by equations (55) to (64) possesses a unique interior steady state solution $(\check{\omega}, \check{l}, \check{m} > 0)$ with equilibrium on the asset*

markets, iff the fundamentalists long run reference inflation rate of equity prices equals the steady state inflation rate of good prices

$$\bar{\eta} = \check{\hat{p}},$$

and

$$\begin{aligned} & \lim_{r\to 0}(f_m(r,\check{\rho}^e+\check{\pi}_e)+f_b(r,\check{\rho}^e+\check{\pi}_e)) \;<\; \bar{\phi} \\ \text{and} \quad & \lim_{r\to \infty}(f_m(r,\check{\rho}^e+\check{\pi}_e)+f_b(r,\check{\rho}^e+\check{\pi}_e)) \;>\; \bar{\phi} \end{aligned}$$

holds true with $\bar{\phi} = \frac{\bar{g}-\bar{t}_c^n-\tau_w\check{\omega}\check{l}^d}{\bar{g}-\bar{t}_c^n-\tau_w\check{\omega}\check{l}^d+\bar{\mu}}$.[10]

Proof: If the economy rests in a steady state, then all intensive variables stay constant and all time derivatives of the system become zero. Thus by setting the left hand side of the system of equations (55) to (64) equal to zero, we can deduce the steady state values of the variables.

From equation (57) we can derive that $\check{i} = n$ holds, from (58) we get $\check{y}^e = \check{y}^d$, and from (64): $\bar{\mu} = (\kappa[\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V})] + \pi + i)$. The last relation plugged in into equation (42) and using $\check{i} = n$ we get with $\alpha\beta_\pi \neq -(1-\alpha)\beta_\pi$ that $\bar{\mu} - n - \pi = 0$ and $\kappa[\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V})] = 0$. Thus we have the following two equations for $U_c - \bar{U}_c$ and $V - \bar{V}$:

$$\begin{aligned} U_c - \bar{U}_c &= -\kappa_p\beta_w(V-\bar{V})/\beta_p \\ U_c - \bar{U}_c &= (1-\kappa_p)\beta_w(V-\bar{V})/[(1-\kappa_w)\beta_p] \end{aligned}$$

By assumption we have $\beta_p, \beta_w > 0$ and $0 \leq \kappa_p, \kappa_w leq 1$. Then $V - \bar{V}$ must equal zero in order to fulfill the last two equations. When $V = \bar{V}$, then according to (55) we know that $U_c = \bar{U}_c$. Then equation (57) leads to $\check{q} = 1$.

With these relations one can easily compute the unique steady state values of the variables y^e, l, π, ν, ω:

$$\check{y}^e = \frac{\check{y}}{1+n\beta_{n^d}}, \quad \text{with } \check{y} = \bar{U}_c\bar{y}^p \tag{65}$$

$$\check{l} = \check{y}/(\bar{V}\bar{x}) \tag{66}$$

$$\check{\pi} = \bar{\mu} - n \tag{67}$$

$$\check{\nu} = \beta_{n^d}\check{y}^e \tag{68}$$

$$\check{\omega} = \frac{\check{y}^e - n - \delta - \bar{g} - (1-s_c)(\check{y}^e - \delta - \bar{t}_c^n)}{(s_c - \tau_w)\check{l}^d}, \tag{69}$$

$$\check{\rho}^e = \check{y}^e - \delta - \check{\omega}\check{l}^d \tag{70}$$

All these values are determined on the good and labor markets. The steady state value of the real wage has been in particular derived from the goods market

[10]Note with respect to this part of the proposition that the steady state values used in the above assumption are calculated before this assumption is applied to a determination of the steady state value of the nominal rate of interest.

equilibrium condition that must hold in the steady state and it is positive under the assumptions made in proposition 2.

We next take account of the asset markets, which determine the values of the short-term interest rate r (which is now uniquely in charge to clear the asset markets), but now in conjunction with the determination of the steady state for m, b, where $m+b$ is determined through the government budget constraint. This is the case because the steady state rate of return on equities relies, on the one hand, on solely on $\check{\rho}^e$, since q has been determined through the condition $i = n$ and shown to equal one in steady state, and, on the other hand, on the expected inflation rate of share prices, which equals the goods price inflation rate in the steady state as will be shown below;

$$\check{r}_e^e \quad = \quad \check{\rho^e} + \check{\pi}_e.$$

The steady state values of the two kinds of expectations about the inflation rate of equity prices (of chartists and fundamentalists) are

$$\check{\pi}_{ef} = \bar{\eta} \quad \wedge \quad \check{\pi}_{ec} = \bar{\eta} \tag{71}$$

from which one can derive that $\check{\pi}_e = \bar{\eta} = \check{\hat{p}} = \check{\pi} = \bar{\mu} - n$ must hold: We have seen that, in the steady state, Tobin's q equals one and its time derivative equals zero, $\dot{q} = 0$:

$$\begin{array}{rrcl} & \dot{q} & = & 0 \\ \Rightarrow & \frac{(\dot{p}_e E + p_e \dot{E})pK - p_e E(\dot{p}K + p\dot{K})}{p^2 K^2} & = & 0 \\ \Rightarrow & \frac{\dot{p}_e E + p_e \dot{E}}{pK} & = & \hat{p} + n \end{array}$$

According to equation (26) we have $p_e \dot{E} = pI + p(\dot{N} - \mathcal{I})$ we thus get in the steady state $p_e \dot{E} = pI$. Inserting this into the last implication shown we get $\hat{p}_e = \hat{p}$ and thus as an important finding that $\bar{\eta} = \bar{\mu} - n$ must hold in order to allow for a steady state.

Now we determine the steady state values of the stocks of real cash balances and the stock of bonds. These values have to be determined in conjunction with the steady state interest rate $\check{r}$ which is now solely responsible for clearing the asset markets, because Tobin's $q = 1$ has already been determined on the real markets.

The budget constraint of the government is given in intensive form by

$$\dot{b} + \dot{m} \quad = \quad \bar{g} - \bar{t}_c^n - \tau_w \omega l^d - (b+m)(\hat{p} + i). \tag{72}$$

One therefore obtains in the steady state

$$\check{b} + \check{m} \quad = \quad (\bar{g} - \bar{t}_c^n - \tau_w \omega l^d)/\bar{\mu} \tag{73}$$

Furthermore, consider the asset demand functions (10) and (11).

$$m \quad = \quad f_m(r, r_e^e)(m + b + q), \quad q = 1 \tag{74}$$

$$b \quad = \quad f_b(r, r_e^e)(m + b + q), \quad q = 1 \tag{75}$$

The left side of the last two equations are the supplied amounts and the right sides represent the demand for the assets m, b.

Using now equation (73)

$$\bar{\mu}(\check{m} + \check{b}) \quad = \quad \bar{g} - \bar{t}_c^n - \tau_w \omega l^d \tag{76}$$

From this system of three linear independent equations (74) to (76) one can deduce the three unique steady state values $\check{r}$, $\check{b}$, and $\check{m}$ which we will show below.

Beginning with the steady state interest rate we sum equations (74) and (75) and multiplying by $\bar{\mu}$:

$$\bar{\mu}(\check{m} + \check{b}) \quad = \quad (\check{f}_m + \check{f}_b)\bar{\mu}(\check{m} + \check{b} + 1)$$

where $\check{f}_m$ and $\check{f}_b$ denote the values of $f_m(\check{r}, \check{\rho}^e + \check{\pi}_e)$ and $f_b(\check{r}, \check{\rho}^e + \check{\pi}_e)$ respectively. Plugging in the budget constraint in the form of equation (76) we get

$$\check{f}_m + \check{f}_b \quad = \quad \bar{\phi},$$

with $\bar{\phi} = \frac{\bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d}{\bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d + \bar{\mu}}$. From property (47) and (49) we can conclude that

$$\frac{\partial (f_m + f_b)}{\partial r} > 0 \tag{77}$$

which implies that the cumulated demand for money and bonds is a strictly increasing function in the variable r.

If $\lim_{r \to 0}(f_m(r, \check{\rho}^e + \check{\pi}_e) + f_b(r, \check{\rho}^e + \check{\pi}_e)) < \bar{\phi}$ and $lim_{r \to \infty}(f_m(r, \check{\rho}^e + \check{\pi}_e) + f_b(r, \check{\rho}^e + \check{\pi}_e)) > \bar{\phi}$ then by monotony and continuity there must be a value of r, which equilibrates the asset markets in the above aggregated form. Then, steady state supplies of m, b can be calculated by equations (74) and (75) in a unique way, based on the steady state interest rates $r = \check{r}$ and $r_e^e = \check{\rho}^e + \pi_e$. This concludes the uniquely determined derivation of steady state values for our dynamical system (55) to (64) which in turn when inserted into this system indeed imply that the dynamics is at a point of rest in this situation. ■

We observe finally that the calculation of the steady state value of the rate of wage and the rate of profit can be simplified when it is assumed that government expenditures are given by $\bar{g} + \tau_w \omega l^d$ in the place of only $\bar{g}$.

3.2 The comparative statics of the asset markets

After we have specified the extensive and intensive form of the model and have shown the existence and uniqueness of an interior steady state solution of the intensive form we now focus on the short–run comparative statics of the financial markets module of the system. We thus now derive in particular the function $q = q(m, b, \rho^e, \pi_e)$ already made use of in the intensive form presentation of the model, which is now needed to investigate the stability properties of the model close to its steady state solution, see the next subsection.

Assuming that the asset demand functions display the property which gave us a unique interior steady state solution in the preceding subsection, see proposition

2, it is now possible to approximate these demand functions by linear functions in a neighborhood of the steady state in order to derive the stability properties of the next subsection. These linearized versions of the asset demand functions can be written as

$$\begin{aligned} f_m^l(r, r_e^e) &= \alpha_{m0} - \alpha_{m1} r - \alpha_{m2}(\rho^e/q + \pi_e) \\ f_b^l(r, r_e^e) &= \alpha_{b0} + \alpha_{b1} r - \alpha_{b2}(\rho^e/q + \pi_e) \\ f_e^l(r, r_e^e) &= \alpha_{e0} - \alpha_{e1} r + \alpha_{e2}(\rho^e/q + \pi_e) \end{aligned}$$

where the superscript l denotes the linearized form and where

$$\alpha_{ij} \geq 0 \quad \forall i \in \{b, m, e\}, j \in \{0, 1, 2\}.$$

Because of proposition 1 it is sufficient to focus on the first two asset market equilibrium conditions in all subsequent equilibrium considerations. These two equilibrium conditions now read:

$$m = (\alpha_{m0} - \alpha_{m1} r - \alpha_{m2}(\rho^e/q + \pi_e))(m + b + q), \tag{78}$$

$$b = (\alpha_{b0} + \alpha_{b1} r - \alpha_{b2}(\rho^e/q + \pi_e))(m + b + q). \tag{79}$$

Solving (78) and (79) for the interest rate r we obtain:

$$r_{\text{LM}} = \frac{\alpha_{m0} - \alpha_{m2}(\rho^e/q + \pi_e) - m/(m + b + q)}{\alpha_{m1}} \tag{80}$$

$$\text{and} \quad r_{\text{BB}} = \frac{-\alpha_{b0} + \alpha_{b2}(\rho^e/q + \pi_e) + b/(m + b + q)}{\alpha_{b1}} \tag{81}$$

The LM–subscript denotes the interest rate that equals demand for real balances and real money supply and the BB–subscript denotes the interest rate that equals real bond demand and supply. Figure 2 displays examples of these two functions. The intersection of the LM–curve and the BB–curve then provides the equilibrium values for the short-term interest rate r and Tobin's q. The figure only shows examples of such functions and as we know that the functions are not linear in q we do not know yet whether the equilibrium exists and is unique. Note however that we are only considering a neighborhood of the steady state solution for $r, q, m, b, \rho^e, \pi_e$ where the latter must of course fulfill the above equilibrium conditions for the asset markets. In order to show that r, q exists and is uniquely determined for all m, b, ρ^e, π_e sufficiently close to this steady state solution we therefore have to show that the assumptions of the implicit function theorem are valid at the steady state.

Proposition 3 *Adopting the assumptions of proposition 2. There is a unique solution (r, q) to the equations (74) and, (75), which thus clears the asset markets, for all values of m, b, ρ^e, π_e in an appropriately chosen neighborhood of the interior steady state solution of the dynamics (55) to (64).*

Proof: We have to show that the Jacobian of the system

$$\begin{aligned} f_m(r, q)(m + b + q) - m &= 0 \\ f_b(r, q)(m + b + q) - b &= 0 \end{aligned}$$

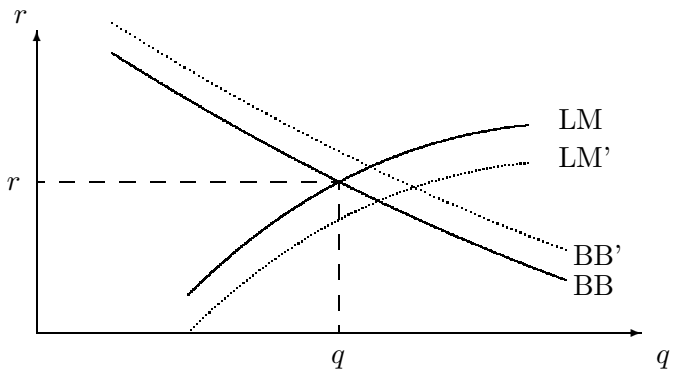

Figure 2: *LM–BB–Curves: The dashed lines show the simultaneously shifted curves of the LM– and BB–curve when one of the statically exogenous variables* ρ^e, π_e, q, m *rises or* b *falls.*

is regular with respect to the variables r, q, which means that

$$\begin{vmatrix} \frac{\partial}{\partial r}(f_m(r,q)(m+b+q)-m) & \frac{\partial}{\partial q}(f_m(r,q)(m+b+q)-m) \\ \frac{\partial}{\partial r}(f_b(r,q)(m+b+q)-b) & \frac{\partial}{\partial q}(f_b(r,q)(m+b+q)-b) \end{vmatrix} \neq 0$$

must hold true. We know for the signs of the entries in this Jacobian:

$$\begin{pmatrix} - & + \\ + & + \end{pmatrix}$$

which immediately implies the regularity of this Jacobian. ∎

We have thus shown that the financial markets can always be cleared through adjustments of the short-term interest rate and Tobin's q. But how do these two variables react in the short-run when the above given statically exogenous variables change (in time)? We consider this questions first on the level of the partial equilibrium curves shown in figure 2. We can derive as dependencies of the two-short run interest functions r_{LM}, r_{BB} from ρ^e, π_e, q, m and on this level also from q the following relationships:

$$\underset{\quad - \; - \; - \; + \; +}{r_{\text{LM}}(\,\rho^e,\,\pi_e,\,m,\,b,\,q\,)} \qquad \text{and} \qquad \underset{\quad + \; + \; - \; + \; -}{r_{\text{BB}}(\,\rho^e,\,\pi_e,\,m,\,b,\,q\,)} \tag{82}$$

These results come directly from the partial derivatives of the functions in equations (80) and (81).

Equations (80) and (81) together build up an equilibrium condition by $r_{\text{LM}} = r_{\text{BB}}$:

$$\frac{\alpha_{m0} - \alpha_{m2}(\rho^e/q + \pi_e) - m/(m+b+q)}{\alpha_{m1}} - \frac{-\alpha_{b0} + \alpha_{b2}(\rho^e/q + \pi_e) + b/(m+b+q)}{\alpha_{b1}} = 0 \tag{83}$$

Applying the implicit function theorem then gives the following qualitative dependencies of Tobin's q^*

$$\begin{array}{lll} q^*(\underset{+}{\rho^e},\underset{+}{\pi_e},\underset{+}{m},\underset{+}{b}) & \forall & q > (\frac{\alpha_{b1}}{\alpha_{m1}} - 1)m \\ q(\underset{+}{\rho^e},\underset{+}{\pi_e},\underset{+}{m},\underset{-}{b}) & \forall & q < (\frac{\alpha_{b1}}{\alpha_{m1}} - 1)m \end{array} \tag{84}$$

The corresponding calculations are given in the mathematical appendix A.3.

We thus know that the first situation must apply locally around the steady state if $(\frac{\alpha_{b1}}{\alpha_{m1}}-1)\check{m} < 1$ holds true while the other one holds in the opposite case.[11] We thus get the results that an increase in ρ^e the basis for the dividend rate of return unambiguously increases Tobin's q just as an increase in the expected capital gains π_e. Furthermore, an increase in m also pushes q upwards and thus increases investment, just as an increase in m would do it in the presence of a negative dependence of the rate of investment on the rate of interest, the Keynes effect or traditional models of the AS-AD variety. The positive influence of m on q thus mirrors the Keynes effect of traditional Keynesian short-run equilibrium analysis. The nominal rate of interest is however no longer involved in the real part of the model as it is here formulated which permits to ignore the comparative statics of this interest rate here.

Results with respect to the influence of bonds b on a change in Tobin's q are however ambiguous and depend on the steady state value of real balances m as well as on the parameters that determine the interest rate sensitivity of money and bonds demand. But we get more insights into the formation of Tobin's q by means of the following lemma:

Lemma 1 *In a neighborhood around the steady state, the partial derivative of Tobin's q with respect to cash balances exceeds the partial derivative of q with respect to bond holdings:*

$$\frac{\partial q}{\partial m} > \frac{\partial q}{\partial b}$$

Proof: According to appendix A.4 we can rewrite the inequality of the proposition by

$$-\frac{\det \frac{\partial(F_1,F_2)}{\partial(r,m)}}{\det \frac{\partial(F_1,F_2)}{\partial(r,q)}} > -\frac{\det \frac{\partial(F_1,F_2)}{\partial(r,b)}}{\det \frac{\partial(F_1,F_2)}{\partial(r,q)}}$$

we know that the denominator is negative and we get equivalently:

$$\begin{array}{rrcl} & \det \frac{\partial(F_1,F_2)}{\partial(r,m)} & > & \det \frac{\partial(F_1,F_2)}{\partial(r,b)} \\ \Leftrightarrow & -\alpha_{m1}b + \alpha_{b1}(b+q) & > & \alpha_{m1}(m+q) - \alpha_{b1}m \\ \Leftrightarrow & \alpha_{b1}(m+b+q) & > & \alpha_{m1}(m+b+q) \\ \Leftrightarrow & \alpha_{b1} & > & \alpha_{m1} \end{array}$$

[11] We do not pay attention here to the border case where $(\frac{\alpha_{b1}}{\alpha_{m1}} - 1)\check{m} = 1$ holds true. Note here also that the α_{ij} sum to one for $j = 0$ and to zero for $j = 1, 2$ which implies that $\frac{\alpha_{b1}}{\alpha_{m1}} - 1$ is always nonnegative.

which is true, because this inequality is an implication of equation (77). ∎

This lemma tells us that open market policy of the government, which means that the central bank buys bonds by means of issuing money ($dm = -db$), indeed has an expansionary effect on Tobin's q:

$$\frac{\partial q}{\partial m} dm + \frac{\partial q}{\partial b}(-dm) \quad > \quad 0 \tag{85}$$

Note finally that the effect of ρ^e on q can be related to the Rose effect in the working KMG model of Chiarella and Flaschel (2000a), while there is no longer a Mundell effect to the model as there is no influence of the real rate of interest on aggregate demand.

4 Local stability

In the following we assume that all assumptions stated in proposition 2 are met. What is left to analyze is the dynamic behavior of the system, when it does not rest in the steady state but is in a small neighborhood of the steady state. In the following we give propositions, which in sum imply that there must be a locally stable steady state, if some sufficient conditions are met.

We begin with an appropriate subsystem of the full dynamics for which the Routh–Hurwitz conditions can be shown to hold. Setting $\beta_p = \beta_w = \beta_{\pi_{ef}} = \beta_{\pi_{ec}} = \beta_n = \beta_\pi = 0$, $\beta_{y^e} > 0$, and keeping π, π_e, ω, ν thereby at their steady state values we get the following subdynamics of state variables m, b and y^e which is then independent from the rest of the system:[12]

$$\begin{aligned} \dot{m} &= m(\bar{\mu} - (\pi + i)) \\ \dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w \omega \frac{y}{x} - \bar{\mu} m - b(\pi + i) \\ \dot{y}^e &= \beta_{y^e} \left[c + i + \delta + \bar{g} - y^e \right] + y^e (i - n) \end{aligned} \tag{86}$$

Proposition 4 *The steady state of the system of differential equations (86) is locally asymptotically stable if β_{y^e} is sufficiently large, the investment adjustment speed i_2 concerning deviations of capital utilization from the normal capital utilization is sufficiently small and the partial derivatives of desired cash balances with respect to the interest rate $\partial f_m / \partial r$ and the rate of return on equities $\partial f_m / \partial r_e^e$ are sufficiently small.*

Proof: The proof makes use of the Routh–Hurwitz conditions (see Gantmacher (1971) for example) and is given in detail in the mathematical appendix. ∎

In other words the proposition asserts that local asymptotic stability at the steady state of the considered subdynamics is given, when the demand for cash is very little influenced by the rates of return on the financial asset markets (which corresponds to a strong Keynes effect in the corresponding working model of

[12] Note that l may vary, but does not feed back into the presently considered subdynamics.

Chiarella and Flaschel (2000a, ch. 6), the accelerating effect of capacity utilization on the investment behavior is sufficiently small, and the adjustment speed of expected sales towards actual demand is fast enough.

Next we consider the same system but allow β_p to become positive, though only small in amount. This means that ω which has entered the m, b, y^e–subsystem only by its steady state value so far, becomes now a dynamic variable, giving rise to a 4D dynamics now.

$$\begin{aligned}
\dot{m} &= m(\bar{\mu} - (\kappa\beta_p(\tfrac{y}{y^p} - U_c) + \pi + i)) \\
\dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w\omega\tfrac{y}{x} - \bar{\mu}m - b(\kappa\beta_p(\tfrac{y}{y^p} - U_c) + \pi + i) \\
\dot{y}^e &= \beta_{y^e}\left[c + i + \delta + \bar{g} - y^e\right] + y^e(i - n) \\
\dot{\omega} &= \omega\kappa(\kappa_w - 1)\beta_p(\tfrac{y}{y^p} - \bar{U}_c)
\end{aligned} \tag{87}$$

Proposition 5 *The interior steady state of dynamic system (87) is locally asymptotically stable if the conditions in proposition 4 are met and β_p is sufficiently small.*

Proof: The proof is left to the appendix. ∎

Enlarging the system (87) by letting β_w become positive we get the following subsystem:

$$\begin{aligned}
\dot{m} &= m(\bar{\mu} - (\kappa\left[\beta_p(\tfrac{y}{y^p} - U_c) + \kappa_p\beta_w(\tfrac{y}{xl} - \bar{V})\right] + \pi + i)) \\
\dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w\omega\tfrac{y}{x} - \bar{\mu}m - b(\kappa\left[\beta_p(\tfrac{y}{y^p} - U_c) + \kappa_p\beta_w(\tfrac{y}{xl} - \bar{V})\right] + \pi + i) \\
\dot{y}^e &= \beta_{y^e}[c + i + \delta + \bar{g} - y^e] + y^e(i - n) \\
\dot{\omega} &= \omega\kappa[(1 - \kappa_p)\beta_w(\tfrac{y}{xl} - \bar{V} + \kappa_w - 1)\beta_p(\tfrac{y}{y^p} - \bar{U}_c)] \\
\dot{l} &= l[-i_1(q - 1) - i_2(\tfrac{y}{y^p} - \bar{U}_c)]
\end{aligned} \tag{88}$$

Proposition 6 *The steady state of the dynamic system (88) is locally asymptotically stable if the conditions in proposition 5 are met and β_w is sufficiently small.*

Proof: The proof is left to the appendix. ∎

Again we enlarge the system by letting $\beta_n > 0$. Then we get the system

$$\begin{aligned}
\dot{m} &= m(\bar{\mu} - (\kappa\left[\beta_p(\tfrac{y}{y^p} - U_c) + \kappa_p\beta_w(\tfrac{y}{xl} - \bar{V})\right] + \pi + i)) \\
\dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w\omega\tfrac{y}{x} - \bar{\mu}m - b(\kappa\left[\beta_p(\tfrac{y}{y^p} - U_c) + \kappa_p\beta_w(\tfrac{y}{xl} - \bar{V})\right] + \pi + i) \\
\dot{y}^e &= \beta_{y^e}[c + i + \delta + \bar{g} - y^e] + y^e(i - n) \\
\dot{\omega} &= \omega\kappa[(1 - \kappa_p)\beta_w(\tfrac{y}{xl} - \bar{V} + \kappa_w - 1)\beta_p(\tfrac{y}{y^p} - \bar{U}_c)] \\
\dot{l} &= l[-i_1(q - 1) - i_2(\tfrac{y}{y^p} - \bar{U}_c)] \\
\dot{\nu} &= y - (c + i + \delta + \bar{g}) - \nu i
\end{aligned} \tag{89}$$

Proposition 7 *The steady state of the dynamic system (89) is locally asymptotically stable if the conditions in proposition 6 are met and β_n is sufficiently small.*

Proof: The proof is left to the appendix. ∎

Finally let β_π become positive. We then are back at the full differential equation system (though we are still neglecting the integral equation of the model and thus the dynamics of capital gain expectations).

$$\begin{array}{rcl}
\dot{\omega} &=& \omega\kappa[(1-\kappa_p)\beta_w(V-\bar{V})+(\kappa_w-1)\beta_p(U_c-\bar{U}_c)],\\
\dot{\pi} &=& \alpha\beta_\pi\kappa[\beta_p(U_c-\bar{U}_c)+\kappa_p\beta_w(V-\bar{V})]+(1-\alpha)\beta_\pi(\bar{\mu}-n-\pi),\\
\hat{l} &=& n-i=-i_1(q-1)-i_2(U_c-\bar{U}_c),\\
\dot{y}^e &=& \beta_{y^e}(y^d-y^e)+(n-i)y^e,\\
\dot{\nu} &=& y-y^d-i\nu,\\
\dot{b} &=& \bar{g}-\bar{t}_c^n-\tau_w\omega l^d-\bar{\mu}m\\
&& -b\left(\kappa[\beta_p(U_c-\bar{U}_c)+\kappa_p\beta_w(V-\bar{V})]+\pi+i\right),\\
\dot{m} &=& m\bar{\mu}-m(\kappa[\beta_p(U_c-\bar{U}_c)+\kappa_p\beta_w(V-\bar{V})]+\pi+i).
\end{array} \tag{90}$$

Proposition 8 *The steady state of the dynamic system (90) is locally asymptotically stable if the conditions in proposition 7 are met and β_π is sufficiently small.*

Proof: The proof is left to the appendix. ∎

We thus have in sum that fast sales expectations coupled with sluggish adjustments of wages, prices, inventories and inflationary expectations gives rise to local asymptotic stability if it is furthermore assumed that the investment accelerator term is weak and the real balance effect in the investment equation (via Tobin's q) sufficiently strong. We conjecture that slow adjustment of capital gain expectations will also preserve the stability of the interior steady state solution of the then really fully given original dynamical system.

Proposition 9 *The steady state of the dynamic system (90) always loses its stability by way of a Hopf bifurcation.*

Proof: The proof basically rests on the fact that the determinant of the Jacobian of steady state of the dynamic system (90) is always negative, see the proof of proposition 8, so that eigenvalues have to cross the imaginary axis (excluding zero) when stability gets lost. ∎

Note here that there are further conditions involved when showing the existence of either subcritical or supercritical Hopf bifurcations. There is first the positive speed condition when eigenvalues cross the imaginary axes and secondly the condition that the Liapunov coefficient must be nonzero then. Both condition are however purely technical in nature and will nearly always hold in a system with such nonlinear functional relationships as they are contained in the presently considered dynamics.

We expect that the above proposition also holds when capital gain expectations of chartists and fundamentalists are made endogenous and that in particular loss of stability can be obtained by increasing the adjustment speed of the backward looking part of the expectations mechanism we have based on the assumed

existence of these two groups of economic agents on the financial markets. Due to the difficulties of treating the 8D integro-differential system that represents the full dynamics of the present paper we do not go into a proof of this assertion here.

Let us finally assert without proof that the normal or adverse Rose effect of changing real wages leading to changing aggregate demand and thereby to further changes in money wages, the price level and the real wage, see Chiarella and Flaschel (2000a) and Chiarella, Flaschel, Groh, and Semmler (2000) in the case of the working KMG model, will also be present in the currently considered KMG dynamics, with their portfolio description of asset market behavior. Either wage or price flexibility will therefore be destabilizing and lead us to Hopf bifurcation, limit cycles and also purely explosive behavior eventually. Furthermore, the Mundell or real rate of interest effect is not obviously present in the considered dynamics as there is no longer a real rate of interest involved in investment (or consumption) behavior. Increasing expected price inflation does not directly increase aggregate demand, economic activity and thus the actual rate of price inflation. This surely implies that the model needs to be extended here in order to take account again of the role that is generally played by the real rate of interest. There are finally two accelerator effects involved in the dynamics, the Metzlerian inventory accelerator mechanism and the Harrodian fixed business investment accelerator (in level formulation). We therefore expect that increasing the parameters β_n, i_2 will also be destabilizing and lead us to Hopf bifurcations and more as well.

5 Extension 1: Including a transactions motive for money holdings

One might argue that the exposed model lacks one important feature which is known in literature as the transactions motive for money demand. Here we present an extended framework which allows us to incorporate the transactions motive of holding cash balances. The basic idea consists of the assumption, that primarily firms are concerned with liquidity consideration concerning payments to households, other firms and the government while these latter units nearly instantaneously return these payments to firms by buying the goods produced by them. For doing the daily transactions, firms thus use and need money.

Now suppose the firms do recognize some uncertainty with respect of monthly or weekly payments and there is an implicit punishment to firms, if they are not able to pay their obligations with money. Then it seems natural, that firms maintain a certain stock of cash. While we do allow the individual firms to have various ideas about their need to hold cash balance, we maintain in the aggregate as a first approach to an income dependent component of money demand the view, that the stock of money held by the firms can be modeled by a simple inventory dynamics similar to the case of inventories of finished goods already present in the dynamics of this paper. This implies as a first step towards such an extension of the model that the desired stock of cash balances is a constant fraction of the

expected sales.

The newly introduced equations describing the dynamics of the cash balances held by firms for income payments are

$$M_f^d = \beta_{m^d} p Y^e, \tag{91}$$

$$\mathcal{I}_m = \beta_m (M_f^d - M_f) + \bar{\mu} M_f^d, \tag{92}$$

$$\dot{M}_f = p(\mathcal{I} - \dot{N}) + \mathcal{I}_m. \tag{93}$$

In equation (91) we have defined the desired stock of cash balances M_f^d as a fixed proportion of expected nominal sales. The next equation gives the desired changes $\mathcal{I}_m$ in the stock of money. The first term in this dynamic law represents the adjustment due to the gap of desired cash balance and cash balance actually held, with an adjustment speed β_m. The second term refers to the growth rate of money supply in order to allow for a steady state solution of this adjustment process. Equation (93) finally states, that actual changes in money stocks are not only due to the desired changes $\mathcal{I}_m$ but also due to windfall profits or losses.

This modeling has in common with the model of the main part of the paper that the investment in business fixed capital must still be financed by issuing of new equities. But there is a difference in the financing of unintended changes in inventories now. They do not play any more a role in the issuing of equities, but the desired changes in money holdings do. The budget constraint therefore now gets the following appearance:

$$p_e \dot{E} = pI + \mathcal{I}_m. \tag{94}$$

This modeling is more satisfactory because it frees us from the need to explain the questionable implication that windfall profits or deficits influences immediately the issue of equities. The equation is not trivial, thus we go on to derive it explicitly now. The consequences of the cash balance holdings of firms can be seen in figure (3). The Production account, income account and accumulation account are not affected at all. But the financial account has been changed because the financial deficit plus the changes in the cash balance have to be financed by the issue of equities. The form of the financial deficit has changed: $FD = p_e \dot{E} - \dot{M}_f$. According to the accumulation account we can now write:

$$p_e \dot{E} = pI + p(\dot{N} - \mathcal{I}) + \dot{M}_f \tag{95}$$

From the definition of the desired changes in cash balance of firms we can then write $p_e \dot{E} = pI + \mathcal{I}_m$ which is exactly the budget constraint of the firm exposed in equation (94).

With the following proposition we show, that the Walras' law of flows is still valid in this extended framework of a portfolio approach to KMG growth.

Proposition 10 *If firms are allowed to hold money, and the changes of supply of money and bonds equal the changes of the corresponding demands, then the change in equity supply is exactly absorbed by the change in equity demand of the asset holders.*

Uses	Resources
Production Account of Firms:	
Depreciation $p\delta K$	Consumption pC
Wages wL^d	Gross investment $pI + p\delta K$
Gross accounting profits $\Pi = \rho^e pK + p\mathcal{I}$	Inventory investment $p\dot{N}$
	Consumption pG
Income Account of Firms:	
Dividends $\rho^e p_y K$	Gross accounting profits Π
Savings $p\mathcal{I}$	
Accumulation Account of Firms:	
Gross investment $pI + p\delta K$	Depreciation $p\delta K$
Inventory investment $p\dot{N}$	Savings $p\mathcal{I}$
	Financial deficit FD
Financial Account of Firms:	
Changes in money balance $\dot{M}_f$	Equity financing $p_e\dot{E}$
Financial deficit FD	

Figure 3: *Accounts of the firms' sector: firms with transaction motive for holding cash balances*

Proof: The savings of asset holder are $S_c = \dot{M}_c^d + \dot{B}^d + p_e\dot{E}^d$, of firms: $S_f = p\mathcal{I}$, and of the government $S_g = -\dot{M} - \dot{B}$. The assumptions of the proposition provide: $\dot{M}_f^d + \dot{M}_c^d = \dot{M}$ and $\dot{B}^d = \dot{B}$. Ex post investments must equal savings:

$$\begin{aligned} \Leftrightarrow \quad pI + p\dot{N} &= S_p^n + S_g^n + S_f^n \\ \Leftrightarrow \quad pI + p\dot{N} &= \dot{M}_c^d + \dot{B}^d + p_e\dot{E}^d + p\mathcal{I} - \dot{M} - \dot{B} \\ \Leftrightarrow \quad pI + p\dot{N} &= p_e\dot{E}^d + p\mathcal{I} - \dot{M}_f^d \end{aligned}$$

From equation (93) we obtain $pI + \mathcal{I}_m = p_e\dot{E}^d$ which due to equation (94) proves the proposition. ∎

5.1 Intensive form

Transferring the changes of the extensive form model to its intensive form, we first consider the intensive form of the desired changes in firms' cash balances, denoted by $\mathcal{J}_m$.

$$\mathcal{J}_m = \frac{\mathcal{I}_m}{pK} = (\bar{\mu}\beta_{m^d} + \beta_m\beta_{m^d})y^e - \beta_m m_f \tag{96}$$

The dynamic equation for the bond supply is almost the same as before, but we have to write the total amount of money in the economy now in the disaggregated

form $m = m_c + m_f$:

$$\begin{aligned}\dot{b} &= \bar{g} - \bar{t}_c^n - \tau_w \omega l^d - \bar{\mu}(m_c + m_f) \\ &\quad - b\left(\kappa[\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V})] + \pi + i\right), \end{aligned} \tag{97}$$

Important further changes concern the dynamic laws of motion for the money holdings. Now the supply has to be split between firms and asset holders.

$$\begin{aligned}\frac{\frac{\partial}{\partial t}(m_c + m_f)}{m_c + m_f} &= \hat{m} = \hat{M} - \hat{p} - \hat{K} \\ &= \bar{\mu} - (\kappa[\beta_p(U - \bar{U}) + \kappa_p\beta_w(V - \bar{V})] + \pi + i)\end{aligned}$$

From this we compute for the time rate of change of money supply per unit of capital:

$$\begin{aligned}\dot{m}_c + \dot{m}_f &= (m_c + m_f)\bar{\mu} \\ &\quad - (m_c + m_f)(\kappa[\beta_p(U - \bar{U}) + \kappa_p\beta_w(V - \bar{V})] + \pi + i)\end{aligned}$$

The law of motion of firms' cash balance is given by

$$\dot{m}_f = (y^d - y^e) + \mathcal{J}_m - m_f(\kappa\left(\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V})\right) + \pi + i) \tag{98}$$

From this we can compute the law of motion of the cash balances of the asset holders:

$$\begin{aligned}\dot{m}_c &= (\dot{m}_c + \dot{m}_f) - \dot{m}_f \\ &= (m_c + m_f)\bar{\mu} - (m_c + m_f)(\kappa[\beta_p(U - \bar{U}) + \kappa_p\beta_w(V - \bar{V})] + \pi + i) \\ &\quad - (y^d - y^e) - \mathcal{J}_m + m_f(\kappa\left(\beta_p(U - \bar{U}) + \kappa_p\beta_w(V - \bar{V})\right) + \pi + i) \\ &= (m_c + m_f)\bar{\mu} - (y^d - y^e) - \mathcal{J}_m \\ &\quad - m_c(\kappa[\beta_p(U - \bar{U}) + \kappa_p\beta_w(V - \bar{V})] + \pi + i)\end{aligned} \tag{99}$$

5.2 Steady state

The cash balances held by firms do not affect the steady state considerations of the variables $y^e, l^d, \nu, \pi, \omega$, and π_e, which are determined from the characteristics of the goods and the labor markets.

But on the financial part something has changed. First we determine the steady state of firms' cash balance and then we investigate its effect on the holdings of the other financial assets. The steady state value of the firms' cash balances is constructed by considering the time derivative of the term $M_f/(pK)$, setting it equal to zero, and by solving the resulting equation for m_f. We thereby obtain the result:

$$\check{m}_f = \beta_{m^d}\check{y^e}. \tag{100}$$

Note that this value for m_f is determined from the goods market. The steady state values of the asset holders' stocks of financial assets depends again solely

on the steady state interest rate on bonds, because Tobin's $q = 1$ is again implied through the investment behavior of firms in the steady state and thus through the real part of the economy. The computation of the steady state interest rate differs slightly from the approach of the main part of the paper, because the government budget constraint now also refers to the stock of money that is held by firms.

$$\begin{array}{rrcl} & \bar{\mu}(\check{m}_c + \check{m}_f + \check{b}) & = & \bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d \\ \Leftrightarrow & \bar{\mu}(\check{m}_c + \check{b}) & = & \bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d - \bar{\mu} \check{m}_f \end{array} \tag{101}$$

The steady state interest rate is the interest rate which equilibrates the supply of money and bonds and with the corresponding demands, again to be supplemented by the equations that are needed to determine these supplies. Of course, the sum of these supplies has to be the same as the sum of these demands. Multiplying this sum with $\bar{\mu}$ we obtain the equilibrium condition:

$$\bar{\mu}(\check{m}_c + \check{b}) \quad = \quad (f_m(\check{r}, \check{r}_e^e) + f_b(\check{r}, \check{r}_e^e))\bar{\mu}(\check{m}_c + \check{b} + 1) \tag{102}$$

Properties (47) and (49) tell us that $f_m(\check{r}, \check{r}_e^e) + f_b(\check{r}, \check{r}_e^e)$ is an increasing function in $\check{r}$. Using now the budget constraint we know that there must be a unique steady state interest rate that equilibrates the aggregate demand and supply of money and bonds and which satisfies the budget constraint:

$$(f_m(\check{r}, \check{r}_e^e) + f_b(\check{r}, \check{r}_e^e)) \quad = \quad \bar{\phi} \tag{103}$$

$$\text{with} \qquad \bar{\phi} \quad = \quad \frac{\bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d - \bar{\mu} \check{m}_f}{\bar{g} - \bar{t}_c^n - \tau_w \check{\omega} \check{l}^d - \bar{\mu} \check{m}_f + \bar{\mu}} \tag{104}$$

To exclude the situation that there is no market clearing interest rate we have to make sure that with an increasing interest rate the aggregate demand exceeds the debt of the public sector and falls short behind the supply when the interest rate becomes small enough. This condition is met according to the assumption made in the proposition 2.

Since we have deduced the steady state interest rate, it is easy to find the steady state holdings of cash balances in the asset holder sector, and the steady state stock of bonds.

We have to stress at this point, that the steady state growth rate of the price of equities is not equal to the price inflation of goods. This growth rate of share prices has to be newly calculated, which we are going to do now. In steady state Tobin's q must be one ($q = 1$). The time derivative of q furthermore is

$$\begin{aligned} \dot{q} \quad &= \quad \frac{(\dot{p}_e E + p_e \dot{E})pK - p_e E(\dot{p}K + p\dot{K})}{p^2 K^2} = \frac{\dot{p}_e E + p_e \dot{E}}{pK} - q(\hat{p} + i) \\ &= \frac{\dot{p}E}{pK} + \frac{p\dot{E}}{pK} - (\hat{p} + i) = \frac{\dot{p}_e}{p_e} q + i + \frac{\mathcal{I}_m}{K} - \hat{p} - i = \frac{\dot{p}_e}{p_e} + \mathcal{J}_m - \hat{p} \end{aligned}$$

Hence, in the steady state ($\dot{q} = 0$) we have:

$$\mathcal{J}_m \quad = \quad \hat{p} - \hat{p}_e \tag{105}$$

Uses	Resources
Production Account of Firms:	
Depreciation $p\delta K$	Consumption pC
Wages wL^d	Gross investment $pI + p\delta K$
Gross accounting profits $\Pi = \rho^e pK + p\mathcal{I}$	Inventory investment $p\dot{N}$
	Consumption pG
Income Account of Firms:	
Dividends $(1-\alpha_\rho)\rho^e p_y K$	Gross accounting profits Π
Savings $p\mathcal{I} + \alpha_\rho \rho^e pK$	
Accumulation Account of Firms:	
Gross investment $pI + p\delta K$	Depreciation $p\delta K$
Inventory investment $p\dot{N}$	Savings $p\mathcal{I} + \alpha_\rho \rho^e pK$
	Financial deficit FD
Financial Account of Firms:	
Changes in money balance $\dot{M}_f$	Equity financing $p_e \dot{E}$
Financial deficit FD	

Figure 4: *Accountings of the firms' sector when there are cash balances and retained earnings.*

Thus the inflation rate in good prices exceeds now the inflation rate in share prices. Empirical data however do not support this theoretical result. The problem seems to lie in the definition of Tobin's average q. The nominal value of equity holdings is divided by the nominal value of the productive capital stock pK solely and neglects the money holdings of the firms.

6 Extension 2: Retained earnings of firms

In the previous section we stressed the lack of empirical evidence for the implication that the transaction motive of firms lead us to inflation in equity prices less than the one of goods prices. Now we introduce another feature into our model with which we can overcome this problem. We allow the firms to retain a fixed proportion of their profits. These retained profits together with the issue of equities now serve to finance the investment of firms in business fixed capital and desired changes in their cash balances.

First we explicate the impact of this idea on the behavior of asset holders. They are concerned about the dividend payments of firms which are now of less amount compared to the earlier situation.

$$\rho^e = (Y^e - \delta K - \omega L^d)/K \tag{106}$$

$$C_c = (1-s_c)[(1-\alpha_\rho)\rho^e K + rB/p - T_c], \quad 0 < s_c < 1,\ 0 < \alpha_\rho < 1 \tag{107}$$

$$S_p = s_c[(1-\alpha_\rho)\rho^e K + rB/p - T_c] \tag{108}$$

$$= \quad (\dot{M}_c + \dot{B} + p_e\dot{E})/p, \tag{109}$$

$$W_c \quad = \quad (M_c + B + p_eE)/p, \quad W_c^n = pW_c. \tag{110}$$

Equation (106) repeats the unchanged definition of the expected rate of return on capital. But in equation (107) the change becomes obvious. The source of capital income, the dividend payments of firms enter only with the factor $(1-\alpha_\rho)$ the consumption function where $0 \leq \alpha_\rho \leq 1$. This modification is also taken into account in the formulation of the savings of asset holders in equation (108).

Additionally the economy is also changed by a different formulation of the markets for financial assets. The expected rate of return on equities must pick up the changes in the expected dividend payments:

$$r_e^e \quad = \quad \frac{(1-\alpha_\rho)\rho^e pK}{p_eE} + \pi_e. \tag{111}$$

In the firms' sector, the budget equation of firms now turns out to be

$$p_e\dot{E} \quad = \quad pI + \mathcal{I}_m - \alpha_\rho\rho^e pK, \tag{112}$$

which tells us that the residual of business fixed investment and planned changes in the cash balances of firms minus that part of expected profit, which is retained, has to be financed by the issue of new equities. The change in the financing policy of firms leads us therefore to a new definition of their savings:

$$S_f^n \quad = \quad p\mathcal{I} + \alpha_\rho\rho^e pK. \tag{113}$$

6.1 Intensive form and steady state considerations

The intensive form of the changed equations are now exposed. The intensive representation of the consumption of private households, which is an important part of the demand, is now given by

$$c \quad = \quad (1-\tau_w)\omega l^d + (1-s_c)[(1-\alpha_\rho)(y^e - \delta - \omega l^d) - \bar{t}_c^n]. \tag{114}$$

The steady state of the economy will also differ from the steady state without retained profits. But we focus here on those equations, where the newly introduced parameter α_ρ enters explicitly the steady state equations of the dynamic system. The steady state value of the real wage is to be rewritten as follows:

$$\breve{\omega} \quad = \quad \frac{\breve{y}^e - n - \delta - \bar{g} - (1-s_c)[(1-\alpha_\rho)(\breve{y}^e - \delta) - \bar{t}_c^n]}{[(1-\tau_w) - (1-s_c)(1-\alpha_\rho)]\breve{l}^d} \tag{115}$$

Now we come to the primary purpose of this section: The consideration of the steady state growth rate in equity prices. Building on the time derivative of q and using the fact that in the steady state $\dot{q} = 0$ and $q = 1$ must hold, we compute the following expressions:

$$\begin{array}{rrcl} & \dot{q} & = & 0 \\ \Leftrightarrow & \frac{(\dot{p}_eE + p_e\dot{E})pK - p_eE(\dot{p}K + p\dot{K})}{p^2K^2} & = & 0 \\ \Leftrightarrow & \hat{p}_eq + \dot{i} + \mathcal{J}_m - \alpha_\rho\rho^e & = & q(\hat{p} + i) \\ \Leftrightarrow & \hat{p}_e & = & \hat{p} - \mathcal{J}_m + \alpha_\rho\rho^e \end{array} \tag{116}$$

Thus an important finding of this section is, that if $\mathcal{J}_m$ is small enough, the growth rate of equity prices will exceed the growth rate of goods prices which may be an explanation of an important fact found in the data of modern market economies.

The question now is, whether we can achieve the same result by allowing firms to issue bonds for financing business fixed capital and desired changes in cash balances? An important problem in the consideration of bond financing of firms lies in the determination of the market value of the firms. So far we assumed that this market value of firms is given by the price of equities times the equities in existence. But when we allow for debt financing, we think that their debt must lower the firms value in some sense. The pure equity valuation thus seems inappropriate then, because of the claims of creditors that have then to be considered in addition.

A Mathematical Appendix

A.1 Deriving the law of motion for real wages

The growth rate of real wages is the growth rate of nominal wages minus price inflation:

$$\hat{\omega} = \left(\frac{d\frac{w}{p}}{dt}\right) / \left(\frac{w}{p}\right) = \left(\frac{\frac{dw}{dt}p - w\frac{dp}{dt}}{p^2}\right)\frac{p}{w} = \frac{dw}{dt}/w - \frac{dp}{dt}/p = \hat{w} - \hat{p}.$$

Plugging in the laws of motion for nominal wages and prices given in equations (40) and (41) we obtain

$$\begin{aligned}
\hat{p} &= \beta_p(U_c - \bar{U}_c) + \kappa_p\hat{w} + (1-\kappa_p)\pi \\
&= \beta_p(U_c - \bar{U}_c) + \kappa_p(\beta_w(V - \bar{V}) + \kappa_w\hat{p} + (1-\kappa_w)\pi) + (1-\kappa_p)\pi \\
&= \beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V}) + \kappa_p\kappa_w\hat{p} + \kappa_p(1-\kappa_w)\pi + (1-\kappa_p)\pi \\
&= \frac{\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V}) + (1-\kappa_w\kappa_p)\pi}{1-\kappa_w\kappa_p} \\
&= \kappa(\beta_p(U_c - \bar{U}_c) + \kappa_p\beta_w(V - \bar{V})) + \pi \\
\text{and } \hat{w} &= \kappa(\beta_w(V - \bar{V}) + \kappa_w\beta_p(U_c - \bar{U}_c)) + \pi
\end{aligned}$$

with $\kappa = (1 - \kappa_w\kappa_p)^{-1}$.

A.2 Computation of π_e

We assume that the average expected inflation rate of equity prices equals the actual inflation rate of equities $\pi_e = \hat{p}_e$. These expectations play an important role in the capital markets because they are one component of the expected rate of return on equities. As we have seen in equation (17) the actual overall expectation of equity price inflation is the weighted average of expectations held by fundamentalists and chartists. From the law of motion in equation (16) and

knowing the initial value of π_{ef} denoted by $\pi_{ef}(t_0)$ we can derive the definite solution

$$\pi_{ef}(t) = (\pi_{ef}(t_0) - \bar{\eta})e^{-\beta_{\pi_{ef}} t} + \bar{\eta}.$$

Now we give the definite solution for the expected equity price inflation held by chartists. From (15) we can derive the general solution:

$$\pi_{ec}(t) \quad = \quad \beta_{\pi_{ec}} \int_{t_0}^{t} \hat{p}_e(s) e^{-\beta_{\pi_{ec}}(t-s)} ds.$$

Note that this representation is equivalent to the exponential lag distribution if $t_0 = -\infty$. For this see Gandolfo (1996, ch. 12.4).

From the general solution one can easily derive the definite solution:

$$\pi_{ec}(t) \quad = \quad \pi_{ec}(t_0) e^{-\beta_{\pi_{ec}}(t-t_0)} + \beta_{\pi_{ec}} \int_{t_0}^{t} \hat{p}_e(s) e^{-\beta_{\pi_{ec}}(t-s)} ds,$$

where $\pi_{ec}(t_0)$ is the initial value of the expectations about growth the rate in equity prices performed by the chartists.

Building up the weighted sum of the definite solutions according to equation (17) we obtain equation (62).

A.3 Comparative statics

Defining F to be the difference between the money market clearing interest rate and the bond market clearing interest rate depending on q and other variables we get from (80) and (81):

$$\begin{aligned} F(\rho^e, q, \pi_e, m, b) \quad = \quad & \frac{\alpha_{m0} - \alpha_{m2}(\rho^e/q + \pi_e) - m/(m+b+q)}{\alpha_{m1}} \\ & - \frac{-\alpha_{b0} + \alpha_{b2}(\rho^e/q + \pi_e) + b/(m+b+q)}{\alpha_{b1}} \end{aligned}$$

We know that the steady state values of ρ^e, π_e, m, and b are positive and thus are also positive in a neighborhood around the steady state. The equilibrium condition is $F(\ldots) = 0$. Applying the implicit function theorem we can derive qualitative dependencies of q on changes of other variables. Therefore, we can compute the partial derivative of $F(\ldots)$ with respect to Tobin's q.

$$\partial F/\partial q = \left[\frac{\alpha_{m2}\rho^e}{\alpha_{m1} q^2} + \frac{m}{\alpha_{m1}(m+b+q)^2} + \frac{\alpha_{b2}\rho^e}{\alpha_{b1} q^2} + \frac{b}{\alpha_{b1}(m+b+q)^2} \right].$$

One can easily check that this term is positive.

The influence of expected rate of return on capital on q: The partial derivative of Tobin's q with respect to ρ^e is given by the following computation:

$$\begin{aligned} \frac{\partial q}{\partial \rho^e} \quad &= \quad -\frac{\partial F/\partial \rho^e}{\partial F/\partial q} \\ &= \quad \left(\frac{\alpha_{m2}}{\alpha_{m1} q} + \frac{\alpha_{b2}}{\alpha_{b1} q} \right) / \frac{\partial F}{\partial q} \end{aligned}$$

Both, numerator and denominator are positive. Hence $\partial q/\partial \rho^e > 0$ holds or in words: Tobin's q depends positively on the expected rate of return on capital.

The influence of expected equity price inflation on q: We compute

$$\begin{aligned}\frac{\partial q}{\partial \pi_e} &= -\frac{\partial F/\partial \pi_e}{\partial F/\partial q} \\ &= \left(\frac{\alpha_{m2}}{\alpha_{m1}} + \frac{\alpha_{b2}}{\alpha_{b1}}\right) / \frac{\partial F}{\partial q}.\end{aligned}$$

Again the numerator is positive and $\partial q/\partial \pi_e > 0$. Rising inflationary expectations with respect to equity prices leads to a rising q.

The influence of the cash balance on q:

$$\begin{aligned}\frac{\partial q}{\partial m} &= -\frac{\partial F/\partial m}{\partial F/\partial q} \\ &= -\left[\frac{1}{\alpha_{m1}}\left(-\frac{m+b+q-m}{(m+b+q)^2}\right) + \frac{1}{\alpha_{b1}}\left(-\frac{-b}{(m+b+q)^2}\right)\right] / \frac{\partial F}{\partial q} \\ &= \left[1/\alpha_{m1}\frac{b+q}{(m+b+q)^2} - 1/\alpha_{b1}\frac{b}{(m+b+q)^2}\right] / \frac{\partial F}{\partial q}\end{aligned}$$

Making use of the gross substitution property $\alpha_{m1} < \alpha_{b1}$ the last expression is positive and we get $\partial q/\partial m > 0$. An increase in cash balances leads to an increase in q.

The influence of the stock of bonds on q:

$$\begin{aligned}\frac{\partial q}{\partial b} &= -\frac{\partial F/\partial b}{\partial F/\partial q} \\ &= -\left[\frac{1}{\alpha_{m1}}\left(-\frac{-m}{(m+b+q)^2}\right) + \frac{1}{\alpha_{b1}}\left(-\frac{m+b+q-b}{(m+b+q)^2}\right)\right] / \frac{\partial F}{\partial q} \\ &= \left[1/\alpha_{m1}\frac{-m}{(m+b+q)^2} + 1/\alpha_{b1}\frac{m+q}{(m+b+q)^2}\right] / \frac{\partial F}{\partial q}\end{aligned}$$

which is positive if

$$\frac{m}{\alpha_{m1}} < \frac{m+q}{\alpha_{b1}} \quad \text{or equivalently} \quad q > m\left(\frac{\alpha_{b1}}{\alpha_{m1}} - 1\right)$$

holds. Here we obtain the ambiguity in (84). Because of the adding up constraint in (47) can be written by $-\alpha_{m1} + \alpha_{b1} - \alpha_{e1} = 0$ a necessary condition for $\frac{\partial q}{\partial b} > 0$ is that α_{e1} is sufficiently small.

A.4 Comparative statics, alternative calculations

One yields the qualitative influences of the exogenous variables on q and r also by applying a more general version of the implicit function theorem.

Defining F_1 to be the excess demand on money market and F_2 to be the excess demand on bond market we have the following equilibrium conditions for the financial markets:

$$
\begin{aligned}
F_1(r,q,m,b,\rho^e,\pi_e) = (\alpha_{m0} - \alpha_{m1} r - \alpha_{m2}(\frac{\rho_e}{q} + \pi_e))(m+b+q) - m &= 0 \\
F_2(r,q,m,b,\rho^e,\pi_e) = (\alpha_{b0} + \alpha_{b1} r - \alpha_{b2}(\frac{\rho_e}{q} + \pi_e))(m+b+q) - b &= 0
\end{aligned}
$$

In order to obtain the impact of the exogenous variables (m, b, ρ^e, π_e) on the endogenous variables we apply the implicit function theorem and make use of Cramer's rule (see for example Simon and Blume (1994)) in order to obtain the partial derivatives of q and r. In the case of the partial derivative of $\partial q/\partial m$ we have to compute

$$
\frac{\partial q}{\partial m} = -\frac{\det \frac{\partial(F_1,F_2)}{\partial(r,m)}}{\det \frac{\partial(F_1,F_2)}{\partial(r,q)}},
$$

where the Jacobian of the system with respect to the endogenous variables r and q, denoted by $\partial(F_1, F_2)/\partial(r, q)$, can be calculated by

$$
\frac{\partial(F_1,F_2)}{\partial(r,q)} = \begin{pmatrix} -\alpha_{m1}(m+b+q) & \alpha_{m2}\frac{\rho^e}{q^2}(m+b+q) + \frac{m}{m+b+q} \\ \alpha_{b1}(m+b+q) & \alpha_{b2}\frac{\rho^e}{q^2}(m+b+q) + \frac{b}{m+b+q} \end{pmatrix}.
$$

The determinant of the Jacobian above must be negative because the signs of the entries are positive except of the upper left entry, which is negative. Hence

$$
\det \frac{\partial(F_1,F_2)}{\partial(r,q)} < 0.
$$

The influence of the cash balance on q: The partial derivative of Tobin's q with respect to m is given by the following computation:

$$
\frac{\partial q}{\partial m} = -\frac{\det \frac{\partial(F_1,F_2)}{\partial(r,m)}}{\det \frac{\partial(F_1,F_2)}{\partial(r,q)}} \tag{117}
$$

This means that the sign of $\frac{dq}{dm}$ is the same as the sign of $\det \partial(F_1, F_2)/\partial(r, m)$ which is

$$
\begin{aligned}
\det \frac{\partial(F_1,F_2)}{\partial(r,m)} &= \begin{vmatrix} -\alpha_{m1}(m+b+q) & \frac{m}{m+b+q} - 1 \\ \alpha_{b1}(m+b+q) & \frac{b}{m+b+q} \end{vmatrix} \\
&= -\alpha_{m1} b + \alpha_{b1}(b+q) \\
&= (\alpha_{b1} - \alpha_{m1}) b + \alpha_{b1} q
\end{aligned}
$$

From gross substitution property we know that $\alpha_{m1} < \alpha_{b1}$. Thus we can conclude from positive stocks of assets and positive α's that $\det \partial(F_1, F_2)/\partial(r, m) > 0$ and therefore $\frac{\partial q}{\partial m} > 0$

The influence of the stock of bonds on q: The partial derivative of Tobin's q with respect to b is given by the following computation:

$$\frac{\partial q}{\partial b} = -\frac{\det \frac{\partial(F_1,F_2)}{\partial(r,b)}}{\det \frac{\partial(F_1,F_2)}{\partial(r,q)}} \tag{118}$$

The right hand side of this equation is much similar to that of the derivative of q with respect of m in the preceeding paragraph. Hence we know that the sign of the derivative $\frac{\partial q}{\partial b}$ again depends solely on the sign of the numerator.

$$\begin{aligned} & \det \begin{vmatrix} -\alpha_{m1}(m+b+q) & (\alpha_{m0} - \alpha_{m1} r - \alpha_{m2}(\frac{\rho_e}{q} + \pi_e)) \\ \alpha_{b1}(m+b+q) & (\alpha_{b0} + \alpha_{b1} r - \alpha_{b2}(\frac{\rho_e}{q} + \pi_e)) - 1 \end{vmatrix} \\ = & \qquad \alpha_{m1}(m+q) - \alpha_{b1} m \end{aligned}$$

Here the result displays an ambiguity. It tells us that

$$\frac{\partial q}{\partial b} \begin{matrix} < \\ = \\ > \end{matrix} 0, \quad \text{if} \quad q \begin{matrix} < \\ = \\ > \end{matrix} (\frac{\alpha_{b1}}{\alpha_{m1}} - 1)m.$$

So the condition that guarantees not to negative values of $\frac{\partial q}{\partial b}$ is a sufficiently small value of $\frac{\alpha_{b1}}{\alpha_{m1}}$, which is equivalent to a sufficient small value of $\frac{\partial f_e}{\partial r}$ by means of the adding up constraint (47).

The influence of the expected rate of return on capital on q: By the same argument as in the previous section the sign of the partial derivative of Tobin's q with respect to ρ^e is given by the sign of

$$\det \frac{\partial(F_1,F_2)}{\partial(r,\rho^e)} = \begin{vmatrix} -\alpha_{m1}(m+b+q) & -\frac{\alpha_{m2}}{q}(m+b+q) \\ \alpha_{b1}(m+b+q) & -\frac{\alpha_{b2}}{q}(m+b+q) \end{vmatrix} > 0 \tag{119}$$

Hence we know that $\frac{\partial q}{\partial \rho^e}$ is positive.

The influence of the expected growth rate of equity prices on q: Again we only have to check the sign of the determinant of the Jacobian $\partial(F_1,F_2)/\partial(r,\pi_e)$:

$$\det \tfrac{\partial(F_1,F_2)}{\partial(r,\pi_e)} = \begin{vmatrix} -\alpha_{m1}(m+b+q) & -\alpha_{m2}(m+b+q) \\ \alpha_{b1}(m+b+q) & -\alpha_{b2}(m+b+q) \end{vmatrix} > 0 \tag{120}$$

from which follows that $\frac{\partial q}{\partial \pi_e} > 0$.

The influence of the cash balance on r:

$$\frac{\partial(F_1,F_2)}{\partial(m,q)} = \begin{vmatrix} -\frac{b+q}{m+b+q} & \alpha_{m2}\frac{\rho^e}{q^2}(m+b+q) + \frac{m}{m+b+q} \\ \frac{m}{m+b+q} & \alpha_{b2}\frac{\rho^e}{q^2}(m+b+q) + \frac{b}{m+b+q} \end{vmatrix} < 0$$

From which we can conclude that $\frac{\partial r}{\partial m} < 0$.

The influence of the stock of bonds on r:

$$\frac{\partial(F_1,F_2)}{\partial(b,q)} = \begin{vmatrix} \frac{m}{m+b+q} & \alpha_{m2}\frac{\rho^e}{q^2}(m+b+q)+\frac{m}{m+b+q} \\ -\frac{b+q}{m+b+q} & \alpha_{b2}\frac{\rho^e}{q^2}(m+b+q)+\frac{b}{m+b+q} \end{vmatrix} > 0$$

from which follows that $\frac{\partial r}{\partial b} > 0$.

The influence of expected capital gains on equities on r:

$$\frac{\partial(F_1,F_2)}{\partial(\pi_e,q)} = \begin{vmatrix} -\alpha_{m2}(m+b+q) & \alpha_{m2}\frac{\rho^e}{q^2}(m+b+q)+\frac{m}{m+b+q} \\ -\alpha_{b2}(m+b+q) & \alpha_{b2}\frac{\rho^e}{q^2}(m+b+q)+\frac{b}{m+b+q} \end{vmatrix}$$

Without further assumptions we cannot specify the influence of the expected growth rate on equity prices on the interest rate.

The influence of the expected rate of return on capital on r: Again we only have to check the sign of the determinant of the Jacobian $\partial(F_1,F_2)/\partial(\rho^e,q)$:

$$\frac{\partial(F_1,F_2)}{\partial(\rho^e,q)} = \begin{vmatrix} -\frac{\alpha_{m2}}{q}(m+b+q) & \alpha_{m2}\frac{\rho^e}{q^2}(m+b+q)+\frac{m}{m+b+q} \\ -\frac{\alpha_{b2}}{q}(m+b+q) & \alpha_{b2}\frac{\rho^e}{q^2}(m+b+q)+\frac{b}{m+b+q} \end{vmatrix}$$

Again we cannot say anything about the the influence of the expected rate of return on capital so far.

Note that the findings in this subsection are totally in accordance with the findings of section and 3.2 and A.3.

A.5 Proofs of the stability propositions

Proof of Proposition 4: The Jacobian of the system (86) is

$$J = \begin{pmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} & -m\frac{\partial i}{\partial y^e} \\ -\bar{\mu}-bi_1\frac{\partial q}{\partial m} & -\bar{\mu}-bi_1\frac{\partial q}{\partial b} & -\tau_w\omega\frac{1}{x}\frac{\partial y}{\partial y^e}-b\frac{\partial i}{\partial y^e} \\ (\beta_{y^e}+y^e)i_1\frac{\partial q}{\partial m} & (\beta_{y^e}+y^e)i_1\frac{\partial q}{\partial b} & \beta_{y^e}\left(\frac{\partial c}{\partial y^e}+\frac{\partial i}{\partial y^e}-1\right)+y^e\frac{\partial i}{\partial y^e} \end{pmatrix}$$

Note that all other variables possessing a dynamic law are set to their steady state values.

The Routh–Hurwitz conditions for a 3×3–system are given by:

$$\det J < 0 \tag{121}$$
$$\text{tr } J < 0 \tag{122}$$
$$J_1+J_2+J_3 > 0 \tag{123}$$
$$(-\text{tr } J)(J_1+J_2+J_3)+\det J > 0 \tag{124}$$

where the J_i are the second order principal minors of J.

Beginning with condition (122) we have to calculate the trace:

$$\begin{aligned}\operatorname{tr} J &= -mi_1\frac{\partial q}{\partial m} - \bar{\mu} - bi_1\frac{\partial q}{\partial b} + (\beta_{y^e} + y^e)(i_1\frac{\partial q}{\partial y^e} + i_2\frac{1}{y^p}\frac{\partial y}{\partial y^e}) \\ &\quad +\beta_{y^e}\left(\frac{\partial c}{\partial y^e} - 1\right)\end{aligned}$$

The conditions for the trace to be negative are derived now by first showing that $-mi_1\frac{\partial q}{\partial m} - bi_1\frac{\partial q}{\partial b}$ and $\frac{\partial c}{\partial y^e} - 1$ are negative. Second we show that $(\beta_{y^e} + y^e)(i_1\frac{\partial q}{\partial y^e} + i_2\frac{1}{y^p}\frac{\partial y}{\partial y^e})$ is positive.

$$\begin{aligned}&-mi_1\frac{\partial q}{\partial m} - bi_1\frac{\partial q}{\partial b} \\ =\ & i_1\left(\det\frac{\partial(F_1,F_2)}{\partial(r,q)}\right)^{-1} m(\alpha_{b1}b - \alpha_{m1}b + \alpha_{b1}q) + b(\alpha_{m1}m + \alpha_{m1}q - \alpha_{b1}m) \\ =\ & i_1\left(\det\frac{\partial(F_1,F_2)}{\partial(r,q)}\right)^{-1} (\alpha_{b1}mq + \alpha_{m1}bq)\end{aligned}$$

Remember that the determinant was negative, hence the whole term is negative. Now we show that the term $\frac{\partial c}{\partial y^e} - 1$ is negative:

$$\begin{aligned}&\frac{\partial c}{\partial y^e} - 1 \\ =\ & (1-\tau_w)\omega\tfrac{1}{x}\tfrac{\partial y}{\partial y^e} + (1-s_c)(1-\omega\tfrac{1}{x}\tfrac{\partial y}{\partial y^e}) - 1 \\ =\ & (s_c - \tau_w)\omega\tfrac{1}{x}\tfrac{\partial y}{\partial y^e} - s_c\end{aligned}$$

which is in the rest point by means of the steady state value of ω:

$$\frac{-n - \bar{g} - s_c\delta + (1-s_c)\bar{t}^n}{y^e}$$

Knowing that in steady state the government runs a deficit, the only positively entering term $(1-s_c)\bar{t}^n$ must be smaller than $\bar{g}$, hence $\frac{\partial c}{\partial y^e} - 1$ must be negative.

The term $(\beta_{y^e} + y^e)(i_1\frac{\partial q}{\partial y^e} + i_2\frac{1}{y^p}\frac{\partial y}{\partial y^e})$ can be rewritten by:

$$\begin{aligned}&(\beta_{y^e} + y^e)(i_1\tfrac{\partial q}{\partial \rho^e}\tfrac{\partial \rho^e}{\partial y^e} + i_2\tfrac{1}{y^p}\tfrac{\partial y}{\partial y^e}) \\ =\ & (\beta_{y^e} + y^e)(i_1\tfrac{\partial q}{\partial \rho^e}\tfrac{\partial \rho^e}{\partial y^e} + i_2\tfrac{1}{y^p}(1 + n\beta_{n^d}))\end{aligned}$$

In section A.4 we have shown that $\frac{\partial q}{\partial \rho^e}$ is positive such that the whole term consists of positive entries. The term can be bounded to be sufficiently small by first, assuming a sufficiently small i_2, which makes the second part of the term in brackets small. Second, we assume sufficiently small values of $\frac{\partial f_m}{\partial r} = \alpha_{m1}$ and $\frac{\partial f_m}{\partial r_e^e} = \alpha_{m2}$ at the steady state, which let the first part of the sum in the brackets small enough as we will show now.

According to section A.4 we can write:

$$\frac{\partial q}{\partial \rho^e} = \frac{\frac{1}{q}(\alpha_{m1}\alpha_{b2} + \alpha_{b1}\alpha_{m2})(m+b+q)^2}{\frac{\rho^e}{q^2}(\alpha_{m1}\alpha_{b2} + \alpha_{b1}\alpha_{m2})(m+b+q)^2 + \alpha_{m1}b + \alpha_{b1}m}$$

From the right hand side one can easily see, that it could take sufficiently small values down to zero, when α_{m1} and α_{m2} are chosen small enough. In the following we will assume that the α_{m1}, α_{m2}, and i_2 are sufficiently small, such that

$$\frac{\partial c}{\partial y^e} + \frac{\partial i}{\partial y^e} - 1 \quad < \quad 0.$$

In addition with a sufficiently large β_{y^e} this assumption ensures a negative trace.

With respect to the Routh–Hurwitz condition 123 a sufficient condition for the sum of the principal minors to be positive is that all principal minors are positive.

$$J_3 \quad = \quad \begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} \\ -\bar{\mu} - bi_1\frac{\partial q}{\partial m} & -\bar{\mu} - bi_1\frac{\partial q}{\partial b} \end{vmatrix} = \bar{\mu}mi_1(\frac{\partial q}{\partial m} - \frac{\partial q}{\partial b})$$

According to Lemma 1 we know that this expression is positive.

$$\begin{aligned} J_2 \quad &= \quad \begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -m\frac{\partial i}{\partial y^e} \\ (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial m} & \beta_{y^e}\left(\frac{\partial c}{\partial y^e} + \frac{\partial i}{\partial y^e} - 1\right) + y^e\frac{\partial i}{\partial y^e} \end{vmatrix} \\ &= \quad \begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -m\frac{\partial i}{\partial y^e} \\ 0 & \beta_{y^e}(\frac{\partial c}{\partial y^e} - 1) \end{vmatrix} > 0 \end{aligned}$$

J_2 must be positive, because of the assumptions made in the proposition ensure that $(\partial c)/(\partial y^e) + (\partial i)/(\partial y^e) - 1 < 0$ holds.

$$J_1 = \begin{vmatrix} -\bar{\mu} - bi_1\frac{\partial q}{\partial b} & -\tau_w\omega\frac{1}{x}\frac{\partial y}{\partial y^e} - b\frac{\partial i}{\partial y^e} \\ (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial b} & \beta_{y^e}\left(\frac{\partial c}{\partial y^e} + \frac{\partial i}{\partial y^e} - 1\right) + y^e\frac{\partial i}{\partial y^e} \end{vmatrix} > 0$$

This last statement is true, because the assumption $(\partial f_e)/(\partial r)$ small enough implies that $(\partial q)/(\partial b) > 0$ holds, as we have found in appendix A.4.

Now we prove the Routh–Hurwitz condition (121): the determinant of J must be negative.

$$\begin{aligned} |J| \quad &= \quad \begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} & -m\frac{\partial i}{\partial y^e} \\ -\bar{\mu} - bi_1\frac{\partial q}{\partial m} & -\bar{\mu} - bi_1\frac{\partial q}{\partial b} & -\tau_w\omega\frac{1}{x}\frac{\partial y}{\partial y^e} - b\frac{\partial i}{\partial y^e} \\ (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial m} & (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial b} & \beta_{y^e}(\frac{\partial c}{\partial y^e} - 1) + (\beta_{y^e} + y^e)\frac{\partial i}{\partial y^e} \end{vmatrix} \\ &= \quad \begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} & -m\frac{\partial i}{\partial y^e} \\ -\bar{\mu} & -\bar{\mu} & -\tau_w\omega\frac{1}{x}\frac{\partial y}{\partial y^e} \\ 0 & 0 & \beta_{y^e}(\frac{\partial c}{\partial y^e} - 1) \end{vmatrix} \end{aligned}$$

which is negative due to $\beta_{y^e}(\frac{\partial c}{\partial y^e} - 1) < 0$. The last Routh–Hurwitz condition $(-\mathrm{tr}\, J)(J_1 + J_2 + J_3) + \det J > 0$ finally can be fulfilled by letting β_{y^e} to be large enough, because rising adjustment speeds lead to a decreasing trace and rising sum of principal minors and to decreasing determinant. But β_{y^e} enters

$(-\mathrm{tr}\, J)(J_1 + J_2 + J_3)$ quadratic and the determinant only linear. Hence, for sufficiently large adjustment speed the last Routh–Hurwitz condition is also fulfilled.

Thus all Routh Hurwitz conditions are fulfilled when i_2, α_{m1}, α_{m2} are sufficiently small and β_{y^e} sufficiently large. ∎

Proof of proposition 5: The Jacobian of the system (87) is given by

$$J^\star = \begin{pmatrix} J_{1,1} & J_{1,2} & J_{1,3} - m\kappa\beta_p \frac{1}{y^p}\frac{\partial y}{\partial y^e} & -mi_1\frac{\partial q}{\partial \omega} \\ J_{2,1} & J_{2,2} & J_{2,3} - b\kappa\beta_p \frac{1}{y^p}\frac{\partial y}{\partial y^e} & -\tau_w \frac{y}{x} - bi_1\frac{\partial q}{\partial \omega} \\ J_{3,1} & J_{3,2} & J_{3,3} & \beta_{y^e}(s_c - \tau_w)\frac{y}{x} + (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial \omega} \\ 0 & 0 & \omega\kappa(\kappa_w - 1)\beta_p \frac{1}{y^p}\frac{\partial y}{\partial y^e} & 0 \end{pmatrix}$$

With $\beta_p = 0$ we know that the system possesses three eigenvalues with negative real part and one eigenvalue of zero, the negative real part eigenvalues being identical to the eigenvalues of the Jacobian $J^\star$ of system (86). Let now β_p become positive, but small enough. Employing the fact that the eigenvalues are continuous in the entries of the Jacobian (see for example Sontag (1990)), we know that with sufficiently small perturbations of the entries of the Jacobian (small β_p) the negative real parts will stay negative. With three eigenvalues with negative real parts we can make use of the property that the product of the eigenvalues of a matrix equals the determinant of the matrix. There follows that the fourth eigenvalue must be negative if the determinant of the Jacobian is positive. Hence for proving proposition 5 it is equivalent to showing that the determinant of $J^\star$ shown below is positive:

$$|J^\star| = -\omega\kappa(\kappa_w - 1)\beta_p \frac{1}{y^p}\frac{\partial y}{\partial y^e} \begin{vmatrix} J_{1,1} & J_{1,2} & -mi_1\frac{\partial q}{\partial \omega} \\ J_{2,1} & J_{2,2} & -\tau_w \frac{y}{x} - bi_1\frac{\partial q}{\partial \omega} \\ J_{3,1} & J_{3,2} & \beta_{y^e}(s_c - \tau_w)\frac{y}{x} + (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial \omega} \end{vmatrix}$$

The term $-\omega\kappa(\kappa_w - 1)\beta_p \frac{1}{y^p}\frac{\partial y}{\partial y^e}$ is positive, because κ_w can only take values in the interval $[0, 1]$. All other terms are positive. This means that $|J^\star|$ will be positive if and only if the second component of the product, the determinant of the following 3×3 system is positive too:

$$\begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} & -mi_1\frac{\partial q}{\partial \omega} \\ -\bar{\mu} - bi_1\frac{\partial q}{\partial m} & -\bar{\mu} - bi_1\frac{\partial q}{\partial b} & -\tau_w\frac{y}{x} - bi_1\frac{\partial q}{\partial \omega} \\ (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial m} & (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial b} & \beta_{y^e}(s_c - \tau_w)\frac{y}{x} + (\beta_{y^e} + y^e)i_1\frac{\partial q}{\partial \omega} \end{vmatrix} > 0$$

Multiplying the first row by $(\beta_{y^e} + y^e)/m$ and adding this to the third row of the matrix and multiplying the first row by $-b/m$ and adding this to the second row we do not change the determinant and we can obtain thereby the expression:

$$\begin{vmatrix} -mi_1\frac{\partial q}{\partial m} & -mi_1\frac{\partial q}{\partial b} & -mi_1\frac{\partial q}{\partial \omega} \\ -\bar{\mu} & -\bar{\mu} & -\tau_w\frac{y}{x} \\ 0 & 0 & \beta_{y^e}(s_c - \tau_w)\frac{y}{x} \end{vmatrix} > 0$$

$$\beta_{y^e}(s_c - \tau_w)\frac{y}{x}\bar{\mu}mi_1\left(\frac{\partial q}{\partial m} - \frac{\partial q}{\partial b}\right) > 0$$

From lemma 1 we know that this inequality must hold true ∎

Proof of proposition 6: The Jacobian of the 5×5 system in proposition 6 can be written by

$$J^{\star\star} = \begin{pmatrix} J^{\star}_{1,1} & J^{\star}_{1,2} & J^{\star}_{1,3} - m\kappa\kappa_p \frac{\beta_w}{xl}\frac{\partial y}{\partial y^e} & J^{\star}_{1,4} & m\kappa\kappa_p\beta_w \frac{y}{xl^2} \\ J^{\star}_{2,1} & J^{\star}_{2,2} & J^{\star}_{2,3} - b\kappa\kappa_p \frac{\beta_w}{xl}\frac{\partial y}{\partial y^e} & J^{\star}_{2,4} & b\kappa\kappa_p\beta_w \frac{y}{xl^2} \\ J^{\star}_{3,1} & J^{\star}_{3,2} & J^{\star}_{3,3} & J^{\star}_{3,4} & 0 \\ J^{\star}_{4,1} & J^{\star}_{4,2} & J^{\star}_{4,3} + \frac{\omega\kappa(1-\kappa_p)\beta_w}{xl}\frac{\partial y}{\partial y^e} & J^{\star}_{4,4} & -\frac{\omega\kappa(1-\kappa_p)\beta_w y}{xl^2} \\ -li_1\frac{\partial q}{\partial m} & -li_1\frac{\partial q}{\partial b} & -l\frac{\partial i}{\partial y^e} & -li_1\frac{\partial q}{\partial \omega} & 0 \end{pmatrix}$$

where $J^{\star}_{i,j}$ are the entries of the Jacobian of the system (87). We follow the same idea as in the preceding proof. Hence, it is sufficient to show that $|J^{\star\star}| < 0$ holds, if the parameter β_w is sufficiently small. In a first step we do some row operations within the matrix which do not change its determinant: $-b/m$ times the first row and added to the second row gives the new second row, $(\beta_{y^e} + y^e)/l$ times the fifth row and added to the third row gives the new third row, $-l/m$ times the first row and added to the fifth row gives the new fifth row. The determinant is therefore equal to

$$|J^{\star\star}| = \begin{vmatrix} J^{\star\star}_{1,1} & J^{\star\star}_{1,2} & J^{\star\star}_{1,3} & J^{\star\star}_{1,4} & J^{\star\star}_{1,5} \\ -\bar{\mu} & -\bar{\mu} & -\frac{\tau_w}{x}\frac{\partial y}{\partial y^e} & -\tau_w\frac{y}{x} & 0 \\ 0 & 0 & \beta_{y^e}[-\tau_w\frac{\omega}{x}\frac{\partial y}{\partial y^e} - s_c(1-\frac{\omega}{x}\frac{\partial y}{\partial y^e})] & \beta_{y^e}(s_c-\tau_w) & 0 \\ 0 & 0 & \omega\kappa[(1-\kappa_p)\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e} + (\kappa_w-1)\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e}] & 0 & -\omega\kappa(1-\kappa_p)\beta_w\frac{y}{xl^2} \\ 0 & 0 & l\kappa[\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e} + \kappa_p\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e}] & 0 & -l\kappa\kappa_p\beta_w\frac{y}{xl^2} \end{vmatrix} < 0$$

We know that the upper left 2×2 submatrix has a positive determinant, thus in order to have a negative determinant of the full matrix, we need that the lower right 3×3 submatrix has a negative determinant. This submatrix is denoted by h_1 and we have to show that $|h_1| < 0$:

$$|h_1| = -\beta_{y^e}(s_c - \tau_w) \times \begin{vmatrix} \omega\kappa[(1-\kappa_p)\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e} + (\kappa_w-1)\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e}] & -\omega\kappa(1-\kappa_p)\beta_w\frac{y}{xl^2} \\ l\kappa[\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e} + \kappa_p\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e}] & -l\kappa\kappa_p\beta_w\frac{y}{xl^2} \end{vmatrix}.$$

In case of $0 < \kappa_p \leq 1$ the latter equals

$$|h_1| = -\beta_{y^e}(s_c - \tau_w) \begin{vmatrix} \omega\kappa[(\kappa_w-1) - \frac{1-\kappa_p}{\kappa_p}]\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e} & 0 \\ l\kappa[\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e} + \kappa_p\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e}] & -l\kappa\kappa_p\beta_w\frac{y}{xl^2} \end{vmatrix},$$

which is negative, because $s_c > \tau_w$ and κ_p is in $(0,1)$.

In the case $\kappa = 0$ we get

$$\begin{aligned} |h_1| &= -\beta_{y^e}(s_c - \tau_w) \\ &\quad \times \begin{vmatrix} \omega[\frac{\beta_w}{xl}\frac{\partial y}{\partial y^e} + (\kappa_w - 1)\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e}] & -\omega\beta_w\frac{y}{xl^2} \\ l\frac{\beta_p}{y^p}\frac{\partial y}{\partial y^e} & 0 \end{vmatrix}, \end{aligned}$$

which is negative too. ∎

Proof of proposition 7: The Jacobian of system (89) is

$$J^{\star\star\star} = \begin{pmatrix} J^{\star\star}_{1,1} & J^{\star\star}_{1,2} & J^{\star\star}_{1,3} & J^{\star\star}_{1,4} & J^{\star\star}_{1,5} & -m\kappa(\beta_p\frac{-\beta_n}{y^p} + \kappa_p\beta_w\frac{-\beta_n}{xl}) - m\frac{\partial i}{\partial \nu} \\ J^{\star\star}_{2,1} & J^{\star\star}_{2,2} & J^{\star\star}_{2,3} & J^{\star\star}_{2,4} & J^{\star\star}_{2,5} & J^{\star\star\star}_{2,6} \\ J^{\star\star}_{3,1} & J^{\star\star}_{3,2} & J^{\star\star}_{3,3} & J^{\star\star}_{3,4} & J^{\star\star}_{3,5} & J^{\star\star\star}_{3,6} \\ J^{\star\star}_{4,1} & J^{\star\star}_{4,2} & J^{\star\star}_{4,3} & J^{\star\star}_{4,4} & J^{\star\star}_{4,5} & J^{\star\star\star}_{4,6} \\ J^{\star\star}_{5,1} & J^{\star\star}_{5,2} & J^{\star\star}_{5,3} & J^{\star\star}_{5,4} & J^{\star\star}_{5,5} & 0 \\ J^{\star\star\star}_{6,1} & J^{\star\star\star}_{6,2} & J^{\star\star\star}_{6,3} & J^{\star\star\star}_{6,4} & 0 & J^{\star\star\star}_{6,6} \end{pmatrix},$$

where

$$\begin{aligned} J^{\star\star\star}_{2,6} &= \tau_w\omega\frac{\beta_n}{x} - b\kappa(\beta_p\frac{-\beta_n}{y^p} + \kappa_p\beta_w\frac{-\beta_n}{xl}) - b\frac{\partial i}{\partial \nu}, \\ J^{\star\star\star}_{3,6} &= \beta_{y^e}(\tau_w - s_c)\omega\frac{\beta_n}{x} + (\beta_{y^e} + y^e)\frac{\partial i}{\partial \nu}, \\ J^{\star\star\star}_{4,6} &= \omega\kappa[(1-\kappa_p)\beta_w\frac{-\beta_n}{xl} + (\kappa_w - 1)\beta_p\frac{-\beta_n}{y^p}], \\ J^{\star\star\star}_{6,1} &= -(\nu+1)i_1\frac{\partial q}{\partial m}, \\ J^{\star\star\star}_{6,2} &= -(\nu+1)i_1\frac{\partial q}{\partial b}, \\ J^{\star\star\star}_{6,3} &= \frac{\partial y}{\partial y^e} - \frac{\partial c}{\partial y^e} - (\nu+1)\frac{\partial i}{\partial y^e}, \\ J^{\star\star\star}_{6,4} &= (\tau_w - s_c)\frac{y}{x} - (\nu+1)i_1\frac{\partial q}{\partial \omega}, \\ J^{\star\star\star}_{6,6} &= -\beta_n - n - [(\tau_w - s_c)\omega\frac{\beta_n}{x}] - (\nu+1)\frac{\partial i}{\partial \nu}. \end{aligned}$$

If β_n is zero we obtain

$$|J^{\star\star\star}| = -n|J^{\star\star}|$$

because the last column would only consist of zeros, except of the entry $J^{\star\star\star}_{6,6} = -n$. Thus one Eigenvalue is $-n$. The other five Eigenvalues are those of the upper left 5×5 matrix, which are negative as we have shown in the proof of proposition 6. Letting β_n become positive but sufficiently small, the negativity of the Eigenvalues will be preserved. ∎

Proof of proposition 8 The Jacobian of the dynamic system (90) is given by

$$J^{\star\star\star\star} = \begin{pmatrix} J^{\star\star\star}_{1,1} & J^{\star\star\star}_{1,2} & J^{\star\star\star}_{1,3} & J^{\star\star\star}_{1,4} & J^{\star\star\star}_{1,5} & J^{\star\star\star}_{1,6} & -m \\ J^{\star\star\star}_{2,1} & J^{\star\star\star}_{2,2} & J^{\star\star\star}_{2,3} & J^{\star\star\star}_{2,4} & J^{\star\star\star}_{2,5} & J^{\star\star\star}_{2,6} & -b \\ J^{\star\star\star}_{3,1} & J^{\star\star\star}_{3,2} & J^{\star\star\star}_{3,3} & J^{\star\star\star}_{3,4} & J^{\star\star\star}_{3,5} & J^{\star\star\star}_{3,6} & 0 \\ J^{\star\star\star}_{4,1} & J^{\star\star\star}_{4,2} & J^{\star\star\star}_{4,3} & J^{\star\star\star}_{4,4} & J^{\star\star\star}_{4,5} & J^{\star\star\star}_{4,6} & 0 \\ J^{\star\star\star}_{5,1} & J^{\star\star\star}_{5,2} & J^{\star\star\star}_{5,3} & J^{\star\star\star}_{5,4} & J^{\star\star\star}_{5,5} & J^{\star\star\star}_{5,6} & 0 \\ J^{\star\star\star}_{6,1} & J^{\star\star\star}_{6,2} & J^{\star\star\star}_{6,3} & J^{\star\star\star}_{6,4} & J^{\star\star\star}_{6,5} & J^{\star\star\star}_{6,6} & 0 \\ 0 & 0 & J^{\star\star\star\star}_{7,3} & 0 & J^{\star\star\star\star}_{7,5} & J^{\star\star\star\star}_{7,6} & -(1-\alpha)\beta_\pi \end{pmatrix},$$

where

$$\begin{aligned} J^{\star\star\star\star}_{7,3} &= \alpha\beta_\pi\kappa(\frac{\beta_p}{y^p} + \frac{\kappa_p\beta_w}{xl})\frac{\partial y}{\partial y^e}, \\ J^{\star\star\star\star}_{7,5} &= -\alpha\beta_\pi\kappa\kappa_p\beta_w\frac{y}{xl^2}, \\ J^{\star\star\star\star}_{7,6} &= \alpha\beta_\pi\kappa(\beta_p\frac{-\beta_n}{y^p} + \kappa_p\beta_w\frac{-\beta_n}{xl}). \end{aligned}$$

Again we only have to show that the determinant is negative. Multiplying the first row by $\alpha\beta_\pi/m$ and adding this to the seventh row and adding to this new seventh row $-\alpha\beta_\pi/l$ times the fifth row we get a seventh row with zeros with the exception of the last element which is $-\beta_\pi$. We know that the determinant is the determinant of the upper left 6×6 matrix times $-\beta_\pi$ yielding a negative determinant, because we know from the proof of proposition 7 that the upper left 6×6 matrix has a positive determinant. ■

B Appendix: Notation

The models considered this paper are based on the following basically standard macroeconomic notation:

A. Statically or dynamically endogenous variables:

B, B^d	Bonds, bond demand
C_c	Consumption of asset holders
C_w	Consumption of workers
E, E^d	Equities, equity demand
G	Government expenditure
I	Fixed business investment
K	Capital stock
L	Labor supply
L^d	Level of employment
M, M^d	Money supply, money demand
M_f, M_c	Cash balance of firms and asset holders
N	Stock of inventories
N^d	Desired stock of inventories
$S = S_p + S_f + S_g$	Total savings
S_f, S_f^n	Savings of firms, nominal savings of firms
S_g, S_g^n	Government savings, nominal government savings
S_p, S_p^n	Private savings, nominal private savings
T	Total real taxes
$U = Y/Y^p$	Rate of capacity utilization
$V = L^d/L$	Rate of employment
W	Real wealth
Y	Output
Y^d	Aggregate demand $C + I + \delta K + G$
Y^e	Expected aggregate demand
Y^p	Potential output
$\nu = N/K$	Inventory–capital ratio
ω	Real wage ($u = \omega/x$ the wage share)
π	Expected rate of inflation (medium–run)
ρ, ρ^e	Rate of return on capital, expected rate of return on capital
p	Price level
p_e	Price of Equities
r	Nominal rate of interest (price of bonds $p_b = 1$)
r_e^e	Rate of return on equities
w	Nominal wages
$\mathcal{I}$	Desired inventory investment
$\mathcal{I}_m$	Firms' desired cash balance adjustment
$\mathcal{J}$	Firms' desired cash balance adjustment in intensive form

B. Parameters

$\bar{V}$	NAIRU-type normal utilization rate concept (of labor)
$\bar{U}$	NAIRU-type normal utilization rate concept (of capital)
δ	Depreciation rate
$\bar{\mu}$	Growth rate of the money supply
$\bar{g}$	Intensive government purchases (const.)
$\bar{\eta}$	Fundamentalists long run expectations equity price inflation
n	Natural growth rate
$i_{1,2} > 0$	Investment parameters
$\beta_w \geq 0$	Wage adjustment parameter
$\beta_p \geq 0$	Price adjustment parameter
$\beta_\pi \geq 0$	Inflationary expectations adjustment parameter

$\alpha \in [0,1]$	Weight of actual inflation on expected inflation
$\alpha_{ec} \in [0,1]$	Ratio of chartists to chartists and fundamentalists
$\alpha_\rho \in [0,1]$	share of retained earnings on profits
$\beta_{n^d} > 0$	Desired inventory – expected sales ratio
$\beta_n > 0$	Inventory adjustment parameter
$\beta_{m^d} > 0$	Desired cash balance – expected sales ratio
$\beta_m > 0$	Cash balance adjustment parameter
$\beta_{y^e} > 0$	Demand expectations adjustment parameter
$\kappa_{w,p} \in [0,1], \kappa_w\kappa_p \neq 1$	Weights for short– and medium–run inflation
κ	$= (1 - \kappa_w\kappa_p)^{-1}$
$y^p > 0$	Potential output–capital ratio ($\neq y$, the actual ratio)
$x > 0$	Output–labor ratio
$t (t_c^n = t - rb)$	Taxes (net of interest) per capital
$s_c \in [0,1]$	Savings–ratio (out of profits and interest)

C. Mathematical notation

$\dot{x}$	Time derivative of a variable x
$\widehat{x}$	Growth rate of x
l', l_w	Total and partial derivatives
$\breve{x}$	Steady state value of X
$y_w = y'(l)l_w$	Composite derivatives
$r_o, etc.$	Steady state values
$y = Y/K, etc.$	Real variables in intensive form
$m = M/(pK), etc.$	Nominal variables in intensive form

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