Continuous Time, Period Analysis and Chaos from an Empirical Perspective *

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Abstract

We reconsider the issue of the (non-)equivalence of period and continuous time analysis. We start from the methodological precept that period and continuous time representations of the same macrostructure should give rise to the same qualitative outcome, i.e., that the results of period analysis should not depend on the length of the period. A simple example where this is fulfilled is given by the Solow growth model, while all chaotic period dynamics of dimension less than 3 are in conflict with this precept.

A basic empirical fact moreover is that the actual data generating process in macroeconomics is by and large a daily one (and the data collection frequency now also much less than a year). This suggests that empirically oriented macromodels should be iterated with a short period length as far as actual processes are concerned and will then in general provide the same answer as their continuous-time analogues.

We discuss a typical example from the literature, where chaos results from a 'too' stable continuous time approach when reformulated as a 'long-period' macro-model. The paper finally shows that baseline macromodels can give rise to complex dynamics in (quasi-)continuous time if they are sufficiently rich in structure and dimension.

Keywords: Continuous time, period models, (non-)equivalence, complex dynamics.
JEL CLASSIFICATION SYSTEM: **E24**, **E31**, **E32**.

Contents

1 Introduction

In this paper, we reconsider the issue of the (non-)equivalence of period and continuous time analysis. We stress here that period models – the now dominant model type in the macrodynamic literature – assume a single (uniformly applied) lag length for all markets, which therefore act in a completely synchronized manner. In view of this, we start in section 2 from the methodological precept that period and continuous time representations of the same macrostructure should give rise to the same qualitative outcome, i.e., that the qualitative results of period analysis should not depend on the length of the period, see Foley (1975) for an earlier statement of this precept and Medio (1991a), Sims (1998) for related observations. A simple example where this is fulfilled is given by the conventional Solow growth model, considered in section 3, while all chaotic period dynamics of dimension less than 3 are in conflict with this precept, see however Medio (1991a) for routes to chaos in such an environment.

A basic empirical fact moreover is that the actual data generating process in macroeconomics is by and large a daily one (and the data collection frequency now also much less than a year). This suggests that empirically oriented macromodels should be iterated with a short period length as far as actual processes are concerned and will then in general provide the same answer as their continuous-time analogues. Concerning expectations, the data collection process is however of importance and may give rise to certain (smaller) delays in the revision of expectations, which however may be overcome by the formulation of extrapolating expectation mechanisms and other ways by which agents smooth their expectation formation process. We do not expect here that this implies a major difference between period and continuous time analysis if appropriately modelled, a situation which may however radically change if proper delays as for example considered in Invernizzi and Medio (1991) are taken into account.

We discuss in section 4 a typical example from the literature (by far not the only one), where chaos results from a 'too' stable continuous time approach when reformulated as a 'long-period' macro-model, then exhibiting a sufficient degree of locally destabilizing overshooting. Shortening the period lengths in such chaotic macro models, i.e., iterating them with a finer step size, removes on the one hand 'chaos' from such model types, while it on the other hand (and at the same time) brings the model into closer contact with what happens in the data generating process of the real world.[1]

By contrast, the paper shows in section 5 that baseline macromodels can give rise to complex dynamics in (quasi-)continuous time if they are sufficiently rich in their dynamical structure and dimension. We conclude from this result that the investigation of complex dynamics is of a more fundamental type when restricted to higher dimensional continuous time macrodynamics, since such approaches avoid the mixture of locally destabilizing,

[1]Note in this respect again, that we focus in this paper on standard period models and therefore do not yet consider, as in Invernizzi and Medio (1991), Medio (1991a) the role of significant delays and exponential lags in economic activity.

strongly overshooting adjustment processes (which would converge in quasi-continuous time) with the dynamics that are typical for the larger models (with interacting real and financial markets) of advanced macrodynamic literature.

2 The J2-Status of Macrodynamic Period Analysis

We reconsider here the issue of the (non-)equivalence of period analysis (or for brevity discrete time) and continuous time macro modeling. Period analysis is now the dominant form for models in the macrodynamic literature and thus of interest in its own right, independently of the consideration of the existence of more complicated lags in more advanced macrosystems. Discrete time macro modeling is of course not restricted to the assumption of a single uniform and synchronized period length between all economic activities, on which this paper is focused. For a detailed consideration of the role of significant lags in macrodynamics the reader is referred to Invernizzi and Medio (1991).

We in this respect focus on the empirical fact that the actual data generating process in macroeconomics is of much finer step size than the corresponding data collection frequency available nowadays, at least in the real markets of the economy, and that the latter is nowadays also considerably finer than one year in general.[2] This implies that empirically applicable period macromodels (using annualized data) should be iterated with a much finer frequency (approximately with step size between '1/365 year' and '1/52 year' with respect to the actual performance of economy) in order for them to generate results that may then in general equivalent to the ones of their continuous time analogue (at least in dimensions one and two). Furthermore, models that contain expectational variables may be referring to the data collection process, yet are subject to expectational smoothing and thus also updated in shorter time intervals than the actually observed data.

These empirically applicable period models – which take account of the fact that macroeconomic (annualized) data are generally updated each day – will then not be able to give rise to chaotic dynamics in dimensions one and two, suggesting that the literature on such chaotic dynamics is of questionable empirical relevance (though mathematically often demanding and of interest from this point of view). To exemplify this we consider in this paper a 2D nonlinear monetarist baseline model that is known to be globally asymptoti-

[2]This discrepancy concerning the frequency between the data generating- and the data collection processes is ignored in the majority of empirical mainstream macroeconomic models, which, focusing on aggregate macroeconomic variables available in general at a quarterly basis (such as consumption or prices), simply assume for the time intervals of the theoretical framework the same periodicity as the data collection process. This strategy which is conditioned through the data collection technology available nowadays, can be misleading when the resulting dynamic properties of the calibrated theoretical model depend not on its intrinsic characteristics, but mainly on the length of the iteration intervals. This issue becomes particularly clear in discrete-time dynamic models of dimensions one or two which exhibit chaotic properties, whereas in the continuous time analogue the occurrence of such chaotic dynamics is simply impossible.

cally stable in continuous time and that has been used in a period framework to generate from its parameters a period doubling route to chaos.

Before doing so we however consider a simple case, the Solow growth model, where period and continuous analysis give qualitatively the same answer for any length of the period between zero and infinity. The clustering of production and investment activities at possibly very distant points in time thus does not raise in this case the question of which period length is the most appropriate one, though it may still be asked whether the assumed type of clustering of economic activities really makes sense from an applied macroeconomic point of view if periods longer than one week are considered.

In concluding, the paper therefore proposes that continuous time modeling (or period modeling with a short period length) is the better choice to approach macrodynamical issues compared to a period model where the length of the period remains unspecified, since it avoids the empirically uninterpretable situation of a uniform period length (with a length of one quarter, year or more) with an artificial synchronization of economic decision making. If discrete time formulations (not period analysis) are considered for macroeconomic model building they should represent averages over the day as the relevant time unit for complete models of the real-financial interaction on the macroeconomic level (interactions which in fact should be the relevant perspective for all partial macroeconomic model building). The stated dominance of continuous time modeling (or quasi-continuous modeling with a period length of one day) not only simplifies the stability analysis for macrodynamic model building, but also questions the relevance of period model attractors that differ radically from their continuous time analogue.

Chiarella and Flaschel (2000) argue that a fully specified Keynesian model of monetary growth exhibits at least the six state variables, namely wage share and labor intensity (the growth component), inflation and expected inflation (the medium-run component) and expected sales and actual inventories per unit of capital (the short-run dynamics), i.e., these models easily meet the 3D requirement for the existence of strange attractors in continuous time. Also the New Keynesian baseline model with both staggered wage and price setting is at least of dimension 4, so that even still simple models of a monetary economy (with only an interest rate rule of the central bank) can be used for routes to chaos analysis without running into the danger of synthesizing basically continuous-time ideas with radically synchronized (overshooting) discrete time adjustment processes (which when appropriately bounded produce chaos also in dimensions one or two). This suggests that all techniques developed for analyzing nonlinear dynamical systems represent unquestionably a useful stock of knowledge, to be applied now (in macroeconomics) to investigate strange attractors as they may come about in continuous or quasi-continuous time of high-order macrodynamics.

As a future research agenda we therefore propose to use existing higher dimensional macromodels with small (quasi-continuous, but still of period model type) iteration step size and basic parameters broadly in line with empirical magnitudes in order to investigate by the help of behavioral nonlinearities complex attractors that allow us to apply mathematical contributions like Guckenheimer and Holmes (1983) and Wiggins (1990) to macrodynamic model building. In mathematics, the step from 2D to 3D dynamical system is a truly significant and very interesting one that should now become the focus of interest, since complex 1D and 2D macrodynamics are fairly well exploited by now. Nevertheless such low dimensional models may be useful for testing the role of certain bounding mechanisms in 2D models in particular, but should then give rise to limit cycle behavior or at most attractors of the type shown in figure 3 in the conclusions of this paper (which by and large exhaust the possibilities of attracting sets for ordinary continuous time models in dimension 2).

Continuous vs. discrete time modeling, in macroeconomics, was discussed extensively in the 1970s and 1980s, sometimes in very confusing ways and often by means of highly sophisticated, but – as we shall show in this paper – also by an unnecessarily complicated mathematical apparatus. There are however some statements in the literature, old and new, which suggest that period analysis in macroeconomics, i.e. discrete-time analysis where all economic agents are forced to act in a synchronized manner (with a time unit that is usually left unspecified) can be misleading from the formal as well as from the economic point of view. Foley (1975, p.310) in particular states:

> The arguments of this section are based on a methodological precept concerning macroeconomic period models: *No substantive prediction or explanation in a well-defined macroeconomic period model should depend on the real time length of the period.*

Such a statement has however been completely ignored in the numerous analytical and numerical investigations of complex or chaotic macro-dynamics. Furthermore, from the view point of economic modeling, Sims (1998, p.318) states:

> The next several sections examine the behavior of a variety of models that differ mainly in how they model real and nominal stickiness ... They are formulated in continuous time to avoid the need to use the uninterpretable 'one period' delays that plague the discrete time models in this literature.

Tobin (1982, p.189), by contrast, states:

> Representation of economies as systems of simultaneous equations always strains credibility. But it takes extraordinary suspension of disbelief to imagine that the economy solves and re-solves such systems every microsecond. Even with modern computers the task of the Walrasian Auctioneer, and of the market participants who provide demand and supply schedules, would be impossible. Economic interdependence is *the* feature of economic life and

> we as professional economists seek to understand and explain. Simultaneous equations systems are a convenient representation of interdependence, but it is more persuasive to think of the economic processes that solve them as taking time than as working instantaneously.

Our response to such issues is that a macro-dynamic analysis that is intended to consider eventually real and financial markets simultaneously must consider period analysis with a very short time-unit ('1 day'), if a uniform and synchronized period length is assumed (with averaging of what happened during the day). But then, following Tobin (1982), real markets cannot be considered in equilibrium all of the time. Instead gradual adjustment of wages, prices and quantities occurs in view of labor and goods markets imbalances for which moreover convergence to real market equilibria cannot automatically be assumed, in particular if the economic fundamentals are changing in time. Real market behavior is therefore to be based on gradual adjustment processes, as suggested in Chiarella and Flaschel (2000) and extended in Chiarella, Flaschel and Franke (2005), and it can then be discussed whether, on this basis, financial markets should be modeled by equilibrium conditions (as Tobin (1982) proposes) or also by somewhat delayed responses as well, both in short period analysis as well as in continuous time.

Such implications may be the outcome of a reconsideration of discrete vs. continuous time dynamics. The present paper however focuses on a narrower point, namely, following Foley (1975), that discrete and continuous-time modeling should provide qualitatively the same results. We provide in this respect a positive example (the Solow model) and a negative one (the monetarist baseline model), but conclude with respect to both of them that an artificial clustering of macroeconomic activities with long intermediate intervals of inactivity should be avoided in empirically oriented macrodynamics.

In the linear case this can be motivated further by the following type of argument. We consider the economically equivalent discrete and continuous-time models[3]

$$x_{t+1} = Ax_t \quad \text{and} \quad \dot{x} = (A - I)x = Jx$$

which follow the literature by assuming an unspecified time unit 1.

Our above arguments suggest that we should generalize such an comparison and rewrite the discrete time model with a variable period length to compare it with the continuous version as follows:

$$x_{t+h} - x_t = hJx_t \quad \text{and} \quad \dot{x} = Jx.$$

This gives for their system matrices (and eigenvalues) the relationships

$$A = hJ + I, \qquad \lambda_i(A) = h\lambda_i(J) + I, i = 1, ..., n.$$

According to Foley's postulate both J and A should be (at least) stable matrices,[4] i.e., all eigenvalues of J should have negative real parts, while the eigenvalues of A should all

[3] I the identity matrix.

[4] and (even stricter) have the same number of real roots. We thank Laura Gardini, Anna Agliari, Gian-Italo Bischi, Roberto Dieci of making us aware of this additional restriction.

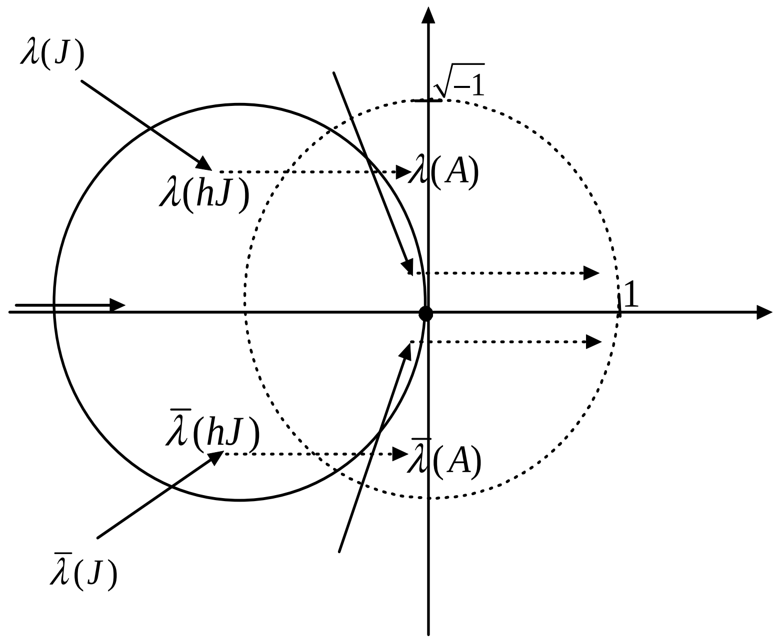

Figure 1: A choice of the period length that guarantees equivalence of continuous and discrete time analysis

be within the unit circle, if discrete and continuous time macromodels are supposed to have the same dynamic properties (as should be the case). Graphically this implies the situation shown in figure 1 (which shows that, if J's eigenvalues do not yet lie inside the unit circle shown, that they have to be moved into it by a proper choice of the time unit and thus the matrix hJ.)

If the eigenvalues of the matrix J of the continuous time case are such that they lie outside the solid circle shown, but for example within a circle of radius 2, the discrete time matrix $J+I$ would – in contrast to the continuous time case – have unstable roots (on the basis of a period length $h=1$ that generally is left implicit in such approaches). The system $x_{t+1} = Ax_t, A = J + I$ then has eigenvalues outside the unit circle (which is obtained by shifting the shown solid unit circle by 1 to the right (into the dotted one). Choosing $h = 1/2$ would however then already be sufficient to move all eigenvalues $\lambda(A) = h\lambda(J)+1$ of $A = hJ + I$ into this unit circle, since all eigenvalues of hJ are moved by this change in period length into the solid unit circle shown in figure 1, since $J's$ eigenvalues have all been assumed to have negative real parts and are thus moved towards the origin of the space of complex numbers when the period length h is reduced.

In view of this, we claim that sensible macro-dynamic discrete time models $x_{t+h} = (hJ + I)x_t = Ax_t$ have all to be based on a choice of the period length 'h' such that at least $||\lambda(A)|| < 1$ can be achieved (if the matrix J is stable). Since models of the real financial interaction suggest very small periods length and since the macroeconomy is updated at the least on a daily basis in reality, such a choice should always be available for the model builder. In this way it is guaranteed that linear period and continuous-time models give qualitatively the same answer, also when they are simulated numerically.[5]

As a generalizing statement and conclusion, related to Foley's (1975) observation, we would conclude that the empirical relevance of macroeconomic models specified with a uniform period length across all sectors and activities and with attractors whose dynamic properties differ substantially from their continuous-time analogue should be questioned.

This implies that a lot of mathematical simulations of typical macro-models should be evaluated as interesting and surely skilled mathematical exercises, but as questionable from the point of view of their empirical relevance. Period models thus in general depend on their continuous-time analogues for their results, if empirically meaningful, and thus exhibit, in terms of US migration policies, only a "J2 status" (dependent on a J1 visitor with work permission) in their macroeconomic implications.

Discrete time models (also empirically estimated ones, using annualized quarterly data in general) should be iterated approximately with step size of '1/100 year' at least as far as basic macroeconomic time series, like for example factual output levels and factual price inflation rates, are concerned and then will generally give rise to results that are equivalent to the ones of their continuous time limit. Thus, pure period models, as the one to be considered in section 4, iterated in this manner, in general will not give rise to chaotic dynamics in dimensions 1 and 2, suggesting that the literature on such chaotic dynamics is of no empirical relevance.

Section 3 will reconsider the Solow growth model from the perspective of the arguments of the present section. We will find that a baseline version of the 1D Solow growth model is not subject to a discrepancy between period and continuous time analysis, however large the period length is chosen. Section 4 will then however provide a 2D example, in fact the monetarist baseline model of inflation and unemployment dynamics, generally interpreted to provide global convergence, that has been extended into a synchronized uniform period setup by Soliman (1996). Here, such a discrepancy comes into being in a striking way, since the continuous-time version is globally asymptotically stable for all parameter choices, while the period version exhibits routes into chaotic dynamics if certain parameters values (representing longer period lengths) are sufficiently increased. Section 6 concludes and provides as an outlook an example of complex dynamics derived from an applicable macromodel of dimension 4.

3 1D Equivalence: The Solow Model

Solow's (1956) one-good model of economic growth is based on full employment throughout, with a natural rate of labor force growth that is exogenously given. The dynamics

[5]Note again that we may even be forced to demand that the number of real roots is the same in both types of analyses.

of Solovian growth are nonlinear due to its use of a neoclassical production function. In the usual continuous time formulation it implies a monotonic one-dimensional transition towards its steady state solution for all initial values of capital-intensity. It can be varied in many ways, including differentiated saving habits, endogenous saving rates, endogenous technological change.

The Solow model of neoclassical economic growth is usually based on the following set of assumptions on the supply side of a closed macroeconomy. In the form that is presented below we still ignore capital stock depreciation and technical change for expositional reasons, see Flaschel (1993) for this and further modifications of the Solow growth model.

$$\begin{aligned} Y &= F(K, L^d) \quad \text{the neoclassical production function} && (1)\\ S &= sY,\ s = const. \quad \text{Harrod type savings function} && (2)\\ \dot{K} &= S \quad \text{capital stock growth driven by household' savings decisions} && (3)\\ \dot{L} &= nL,\ n = const. \quad \text{labor force growth} && (4)\\ L^d &= L \quad \text{the full employment assumption} && (5)\\ \omega &= F_L(K, L) \quad \text{the marginal productivity theory of employment} && (6) \end{aligned}$$

The notation in these equations is fairly standard. We here use L^d to denote labor demand and $\omega = w/p$ to denote the real wage. Technology is described by means of a so–called neoclassical production function that exhibits constant returns to scale. There is only direct investment of savings in real capital formation in this model type, i.e., Say's Law is assumed to hold true in its most simple form:

$$I \equiv S = sY$$

with savings being strictly proportional to output and income Y. Labor is growing at a given natural rate n and is fully employed, i.e., this model simply bases economic growth on actual factor growth without any demand side restriction on the market for goods. The last of the above equations is only added to justify the full employment assumption and it does not play a role in the quantity dynamics to be considered below. These dynamics are obtained from the following reduced form representation of the above model:

$$\begin{aligned} \dot{K} &= sF(K, L) && (7)\\ \dot{L} &= nL && (8) \end{aligned}$$

Since the state variables of these dynamics exhibit an exponential trend the model is generally only analyzed in intensive form, i.e., in terms of the variable k.[6] In intensive form the above Solow model reads:

[6] See however Pampel (2005) for a recent discussion of problems that characterize its actual growth path $(K(t), L(t))$.

$$\dot{k} = sF(k,1) - nk = sf(k) - nk \tag{9}$$

and thus gives rise to a single differential equation in the state variable k which is nonlinear due to the strict concavity of the function f.

In the form of a period model with period length h this form of the Solow model can be represented by

$$\begin{aligned} Y_{t+h} &= hF(K_t, L_t) && (10) \\ S_{t+h} &= sY_{t+h} && (11) \\ K_{t+h} &= K_t + S_{t+h} && (12) \\ L_{t+h} &= (1+nh)L_t && (13) \end{aligned}$$

We note here that the literature generally sets h equal to 1 and considers instead

$$\begin{aligned} Y_t &= F(K_t, L_t) \\ S_t &= sY_t \\ K_{t+1} &= K_t + S_t \\ L_{t+1} &= (1+n)L_t \end{aligned}$$

i.e., it assumes that output and savings occur instantaneously and that there is a uniform gestation lag in investment only (that is synchronized over the whole set of firms). This however is misleading, since production Y (a flow) grows the longer the stocks capital K (the number of machines)and L (the number of workers) are employed, i.e., output Y must vary with h. Our discrete model can be reduced to the two equations

$$\begin{aligned} K_{t+h} &= K_t + shF(K_t, L_t) && (14) \\ L_{t+h} &= (1+nh)L_t && (15) \end{aligned}$$

which for $h = 1$ are identical to the ones implied by the case where the role of the period h length is neglected. Using the identity $K_{t+h}/L_{t+h} = (K_{t+h}/L_t)(L_t/L_{t+h})$, this model can again be reduced to the state variable k now given by $k_t = K_t/L_t$ and gives then rise to:

$$k_{t+h} = (k_t + shf(k_t))/(1+nh) \tag{16}$$

At first sight, this law of motion of the period version of the Solow model looks quite different compared to the one in continuous time

$$\dot{k} = sF(k,1) - nk = sf(k) - nk$$

and its discretization by way of difference quotients

$$k_{t+h} = k_t + h(sf(k_t) - nk_t) = k_t + shf(k_t) - nhk_t$$

Yet, since this last difference equation is (for small period lengths h) but an approximation to the continuous time case we have to check here whether this can also be stated with respect to $k_{t+h} = (k_t + shf(k_t))/(1 + nh)$, the law of motion of the period model. Indeed, this law of motion can be reformulated as

$$\frac{k_{t+h} - k_t}{h} = \frac{(k_t + shf(k_t))/(1 + nh) - k_t}{h} = \frac{(k_t - (1 + nh)k_t + shf(k_t))}{(1 + nh)h} = \frac{sf(k_t) - nk_t}{1 + nh}$$

For small period lengths h this expression is close to the period analogue of the intensive form continuous time case, i.e., the original extensive form period model and the original extensive form continuous time model provide nearly the same dynamics on the intensive form level for small periods h. Yet, with respect to large period lengths, we have to compare the outcome of the continuous time case with the properties of the period case directly and not via the latter approximations, which indeed depart from their original forms when the parameter h is increased.

Moreover, all versions of the Solow model of this section share the same qualitative property of global monotonic convergence to the unique interior steady state of the model. This is exemplified by means of figure 2 where the mapping H of the period version of the Solow model is always strictly increasing and strictly concave and thus must cut the 45 degree line as shown in this figure if the Inada conditions are assumed to hold.[7] Note that the steady state, to be calculated from $k_o = (k_o + shf(k_o))/(1 + nh)$, is independent from the period length h.

We thus have two Solow growth model versions, using continuous time and period analysis respectively, that not only give rise to closely related reduced form dynamics, but that always share the same qualitative feature of not only convergence, but even monotonic convergence, independently of the period length that is assumed to underlie the period model. The period model may therefore assume as radical a clustering or bunching of economic activities, with huge amounts of idle time in between, but does give us the same qualitative results in a stricter sense than we demanded it to be the case in section 2. The Solow growth model is therefore an ideal example for the fulfillment of Foley's (1975) quotation that we have given in section 2. Nevertheless we would argue that iteration step size for this model type (with annualized capital-output ratios) should be chosen as small as one day, since in reality annualized output, investment etc. is changing every day due to the huge number of firm activities that are here aggregated.

One might however argue here that there are significant gestation lags in investment behavior in reality, between investment orders and actual production increases, and this is indeed a relevant observation. This idea was indeed already put forward in Kalecki

[7] We have to thank T. Pampel of making us aware of the fact that this indeed holds for all positive period lengths h.

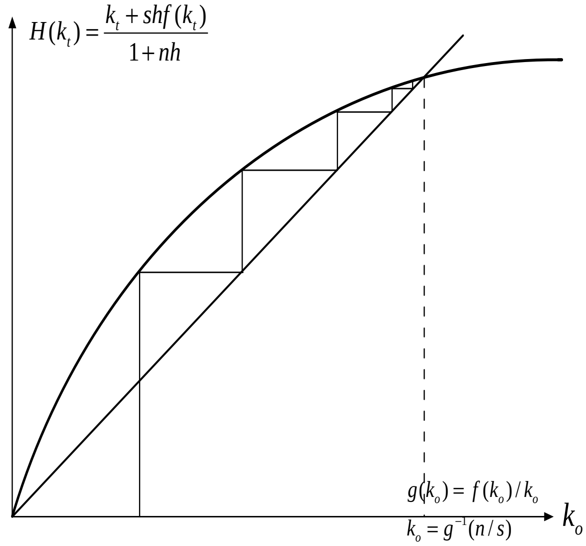

Figure 2: Monotonic convergence in the h-period Solow growth model

(1935) in an important 'post-Keynesian' approach that even preceded the General Theory of Keynes (1936). But this does not question our above empirical observation on nearly continuous output and investment changes, but only extends the (quasi-)continuous formulation of an empirically oriented Solow growth model towards its restatement as a delayed differential equation, a situation from which interesting results may be expected, but that is here beyond what can be and has to be treated in this paper.[8] Such delays are of empirical relevance, but not the clustering of activities that period analysis with large period lengths is suggesting.

In closing, we briefly observe that the ideal convergence properties of the Solow growth model get lost in discrete time when there is real wage rigidity as in the Goodwin (1967) growth cycle model, to be formalized by a real wage Phillips curve $\hat{\omega} = \beta_\omega(e-1)$. We then get a synthesis of the Solow and the Goodwin model, see e.g. Flaschel (1993) for its investigation, and can recover the original Solow model as the limit case $\beta_\omega = \infty$ (if appropriately interpreted). In the continuous-time formulation we get faster and faster convergence, at first cyclical and then monotonic, as the adjustment speed β_ω of real wages is increased, i.e., the Solow model is a meaningful limit case of the Goodwin-Solow model. Yet, for period modeling – where the increase in β_ω can be related to an increase in the period length h – we get, since one eigenvalue in the continuous version is approaching $-\infty$, sooner or later instability and thus a model that must be excluded from those that are of empirical relevance.

From an empirical perspective we finally must even conclude that not all period versions of the law of motion of the Solow growth model should be used from an applied perspective, since it makes no sense to assume in this growth model that aggregate investment behavior exhibits a significant degree of clustering in time, as is assumed by period analysis with

[8]See Chen (1988) for a delayed feedback model of economic growth.

period lengths higher than a day or week. Such a statement must however be carefully distinguished from the assumption of significant gestation periods in single investment projects which – though not clustered – imply the need for using difference or differential equations with a pronounced time delay. Apart from this however, continuous or quasi-continuous time (small periods) is all that is needed from the perspective of applicable macro modeling.

Since the 2D Solow-Goodwin model (just sketched) has not yet been investigated in period form with respect to the complex dynamics it may generate under appropriate assumptions, we turn in the next section to a typical example of the literature on macroeconomic chaos where the empirically motivated postulate of the equivalence of period and continuous time analysis of section 2 finds useful application.

4 2D Monetarist Baseline Analysis. Chaotic Attractors?

In a 2D period model, Soliman (1996) considers a small model of inflation dynamics of textbook type with two laws of motion, one for the expected inflation rate π_t^e and one for the rate of unemployment U_t. The model is formulated in discrete time (in a form that is directly analogous to the familiar continuous-time versions) and makes use of a uniform period length (of one year, see Soliman (1996, p.143)) in describing the adjustment of expected inflation and unemployment based on labor market disequilibrium and a vertical LM curve (the monetarist case of IS-LM equilibrium). This latter relationship implies that real growth g_t is given (approximately) by the discrepancy between nominal money supply growth μ and the actual inflation rate π_t, to be inserted in to Okun's law as shown below. The model is a nonlinear one due to its use of a nonlinear Phillips curve in the determination of actual inflation.

The Phillips curve of this approach to inflation dynamics is indeed given by

$$\pi_t = f(U_t) + \alpha \pi_t^e, \quad 0 < \alpha \leq 1, f' < 0,$$

and inflationary expectations π_t^e are adjusted adaptively according to

$$\pi_{t+1}^e = \pi_t^e + c(\pi_t - \pi_t^e), \quad 0 < c < \infty.$$

The final equation of the model is given by Okun's Law in the form

$$U_{t+1} = U_t - bg_t = U_t - b(\mu - \pi_t), \quad b > 0$$

which due to its specific form assumes that the steady state value of g_t is zero.

Soliman (1996) uses this monetarist model of inflation dynamics (where the long- run Phillips curve need not be vertical) in order to explore numerically transitions from stable equilibrium points to finally chaotic attractors. Since such a result is simply impossible in continuous time, see Hirsch and Smale (1974, p.240) for a classification of the limit sets for two-dimensional differential equation systems, the model is formulated in discrete time, using a uniform length for the period of the model, i.e., by using period analysis. In figure 2 of Soliman it is then shown for example that the dynamics give rise to a period doubling sequence that finally leads to chaos when the parameters μ and b, the growth rate of money supply and the Okun parameter, in the parameter space (b, μ) (jointly or separately) cross a certain critical line.

In continuous time the above model gives rise to the system of 2 differential equations:[9]

$$\dot{U} = b(\pi - \mu) = b(f(U) + \alpha\pi^e) - \mu), \quad \mu \quad \text{given} \tag{17}$$

$$\dot{\pi}^e = c(\pi - \pi^e) = c(f(U) - (1-\alpha)\pi^e) \tag{18}$$

in the state variables U, π^e. The steady state of this system is given by

$$\pi^o = \mu, \pi^{eo} = \mu, U^o = f^{-1}((1-\alpha)\pi^{eo}) \tag{19}$$

and is thus uniquely determined. For the Jacobian of these dynamics we get in $\Re^2$

$$J = \begin{pmatrix} bf' & b\alpha \\ cf' & -(1-\alpha)c \end{pmatrix} = \begin{pmatrix} - & + \\ - & - \text{ or } 0 \end{pmatrix}$$

if $\alpha \leq 1$ holds. According to Olech's theorem, see Flaschel (1993, ch. 4), one obtains from this sign structure in the Jacobian J that the dynamics are globally asymptotically stable in $\Re^2$. Note however that the above system must be modified if the rate of unemployment U assumes negative values or values larger than 1.[10] We conclude that this monetarist baseline model is always asymptotically stable in the large, but may need some extra qualifications if the unemployment rate approaches its boundary conditions. Be that as it may, the eigenvalues of the considered Jacobian at the steady state always have negative real parts in the continuous time case.

As already indicated above, one should use in continuous time - as in discrete time - annualized data and variables from an empirical point of view (and for better comparison)

[9] Note that both systems are assumed to use annualized data for reasons of comparability, i.e., yearly rates of growth which are updated in the period case the stronger, the longer the assumed period h where the discrepancy that is driving them is working.

[10] This can be done by either directly imposing the side conditions $1 \geq U \geq 0$ or by assuming an appropriate slope of the function f at $U = 0, 1$. A third possibility is to take note of a more appropriate form of Okun's law, derived from its level formulation $e = au^b$ with e, u the employment rate of labor and capital, respectively. This form is then given by

$$\hat{e} = \frac{-\dot{U}}{1-U} = b\hat{u} \quad \text{or} \quad \dot{U} = -bg(1-U) = -b(\mu - \pi)(1-U).$$

This formulation avoids the situation that U can become 1, but still needs the side condition $e \leq 1$, i.e. $U \geq 0$.

and thus for example annual inflation rates that are in principle updated 'every second' and thus can lead only to smoothly changing annualized inflation rates as time moves on. Inflation rates thus mirror a period over which they are calculated (quarters or years), but are in principle available at any 'second' if the data updating process is that fast (a 'day' and averages generated over the day would already normally sufficient in this respect). The variables π, π^e are therefore of the same order of magnitude, independently of the assumed period length h of the discrete-time model that is used, and they are moving nearly continuously with time under normal conditions (no hyperinflation).

In discrete time, the dynamics (1), (2) therefore read with respect to a period of length h :

$$\begin{aligned} U_{t+h} &= U_t + bh(\pi_t - \mu) \\ \pi^e_{t+h} &= \pi^e_t + ch(f(U_t) - (1-\alpha)\pi^e_t) \end{aligned}$$

These expression give rise to the formal relationship between the period model and its continuous time representation:

$$A(z_t) = hJ(z_t) + z_t, \quad z_t = (U_t, \pi^e_t)'$$

where the expressions $A(\cdot), J(\cdot)$ are viewed as nonlinear functions here. Such a translation from continuous time is necessary (and here of particularly simple type) from the applied perspective, since it is necessary to know in applied macromodels how the parameter values (and which ones) are changing if one uses for example monthly data in place of quarterly ones and wants to check the comparability of the obtained parameter estimates. In the present case, this is of the simplest type, since only the parameters b, c are changing (proportional to h) with the lengths of the considered period of the data collection process, implying larger reactions of the state variables the longer the time period that passes by until their annualized values are measured again.

Turning to local comparisons around the steady state now, we obtain from the above that the system matrix A (of partial derivatives) of the period model is again related to the Jacobian of the continuous time case as follows:

$$A = hJ + I$$

which gives for the eigenvalues $\lambda(A), \lambda(J)$ of the matrices J, A the relationship

$$\lambda(A) = h\lambda(J) + I.$$

For asymptotic stability we need in the discrete time case $|\lambda(A)| < 1$, i.e., $h < \bar{h}$ where $\bar{h}$ is given by $\lambda(J) = a \pm bi, i = \sqrt{-1}$:

$$|\bar{h}\lambda(J) + 1|^2 = 1 = \bar{h}^2(a^2 + b^2) + 2a\bar{h} + 1.$$

This gives (note that $a < 0$ holds in continuous time):

$$\bar{h} = \frac{2|a|}{a^2 + b^2}$$

We thus get that the discrete time case is asymptotically stable (in line with the continuous time case) if the period length of the period model satisfies $h < \bar{h}$. In an appendix we consider the bifurcation value $\bar{h}$, that separates stability from instability, in more detail and obtain in particular the result that the speed of adjustment of inflationary expectations is the most important contributor to such bifurcations (see also the numerical example below).

We provide an example that this local result should be the case for all empirically relevant parameter sizes of the model, so that there is no period doubling route and the like to chaotic attractors in this parameter range. The parameter b in Okun's Law is approximately 1/3 (Okun's rule of thumb), while the slope of the Phillips curve at the NAIRU has often been estimated as being not too far away from '1' (for annualized data). For the parameter α one generally assumes '1' (the monetarist accelerator case). We thus are left here with the adjustment speed c of inflationary expectations for which no easy estimates can be provided. Loss of local asymptotic stability occurs here solely due an adjustment of inflationary expectations that is chosen sufficiently fast. Such a loss of stability cannot occur in the continuous time case, since there is not yet a Mundell or real interest rate effect present in the Monetarist baseline model (see section 5 for its presence in a Keynesian baseline model). Loss of stability therefore only occurs in the period version in particular due to the fact that the eigenvalues of the continuous time case become too negative in their real parts, i.e., the continuous time case becomes 'too stable'.

The eigenvalues of the matrix J are in this case given by ($\alpha = 1$):

$$\lambda_{1,2} = \frac{bf'}{2} \pm \sqrt{\left(\frac{bf'}{2}\right)^2 + bcf'} = 1/6 \pm \sqrt{(1/6)^2 - c/3} = a \pm bi$$

For $\bar{h}$ we then get by the above derivations ($c > 1/12$):

$$\bar{h} = \frac{\frac{2}{6}}{\frac{1}{36} + \left(\frac{c}{3} - \frac{1}{36}\right)} = \frac{1}{c}.$$

For c <1 we thus would get that the model could be iterated with a year as period length without a change in its stability properties (i.e., the one of its continuous time limit). But the actual macroeconomy is factually updated at least every day, i.e., the iteration step size (which is independent of the currently used timing of the data collection process) should be close to 1/365. Thus the parameter c can assume any value in the interval (1/12,365) in such a situation without loss of asymptotic stability of the daily iterated or updated rates of unemployment and inflation.

The basic problem with the numerical simulations shown in Soliman (1996) thus is that the parameter ranges that are used in the various figures are simply much too high from an empirical point of view. A macroeconomy is updated (in terms of annualized data) every day and thus moving according to ($\alpha = 1$):[11]

$$\begin{aligned} U_{t+h} &= U_t + bh\,(\mu - f(U_t) - \pi^e_t) \\ \pi^e_{t+h} &= \pi^e_t + chf(U_t) \end{aligned}$$

with $h = 1/100$ approximately. Soliman uses $h = 1$ and indicates by the choice of μ, etc. that this period length has to be interpreted as '1 year'. Parameter values $c \in (0, 36.5), b \in (1/5, 1)$ added to her empirical Phillips curve (which has approximately -0.2 as slope at the steady state) give for the iteration parameters bh the maximum $1/365$ and for ch the maximum value $1/10$. This suggests with respect to her figure 2 for example that we are so close to the vertical axes that only the white = stable domain is relevant from the empirical point of view. The period doubling transition to chaos is thus of a purely hypothetical nature. Similar observations apply to the other simulation studies shown in Soliman (1996), as for example to her figure 5, where b, c are chosen much too high in order to provide an empirically relevant study of basins of attraction (which would be totally white if constrained to empirically meaningful parameter values).

We conclude again that the data generating process is to be considered to be of a much finer step size than the data collection and data processing process (to be performed such that annualized data are established on a quarterly or maybe even monthly basis) and that this leads us to a dynamical system in discrete time (even if period synchronization is assumed) that in general (for most empirically relevant macrodynamic models) is not qualitatively distinguishable in its dynamic properties from its continuous time analogue.

Such an assertion, of course, needs further investigation by means of other applied models from the macrodynamic literature. In particular with respect to expectations formation one may then argue against our conclusions, since they have to rely on the data collection process, because the data generating process becomes only visible through such an activity. Yet, even then we would expect not much difference from such a perspective, since the data collection process has meanwhile been improved very much in many areas and since there may exist means by which individuals smooth their observations. Nevertheless, it may be sensible to investigate this further, for example in differential equation systems which exhibit expectational delays.[12]

As a future research agenda we consequently propose to go beyond what has been investigated in this section and to use existing higher dimensional macromodels with small (quasi-continuous) iteration step size, still of a period model type, however updated each 'day', with basic parameters broadly in line with empirical magnitudes, in order to investigate by the help of appropriate behavioral nonlinearities their possibly complex attractors, by applying mathematical studies as Guckenheimer and Holmes (1983), Wiggins (1990) and more recent work to such high order macrodynamic model building. In mathematics, the step from 2D to 3D dynamical system is a truly significant and very interesting one

[11]Note here that all rates in Soliman (1996) are calculated in %.

[12]See Invernizzi and Medio (1991) for a detailed discussion of lags and chaos in economic dynamic models and Medio (1991b) for a discussion of continuous-time models of chaos in economics.

that should now become the focus of interest, since complex 1D and 2D macrodynamics are fairly well exploited. Nevertheless such low dimensional models may be useful for testing the role of certain bounding mechanisms in lower dimensions, but should then give rise to limit cycle behavior or at most of attractors of the type shown in figure 3 (which by and large exhaust the possibilities of attracting sets for ordinary continuous time models in dimension 2). In section 5 we shall briefly discuss as an example a basically linear 4D continuous time model which allows for complex dynamics if a typical nonlinearity of the macrodynamic literature is added to it. Such models should become the focus of interest in the future investigation of complex dynamics, i.e., models with parameter dependent strange attractors that can, with this parameter range and small period lengths h, be applied to actual economies.

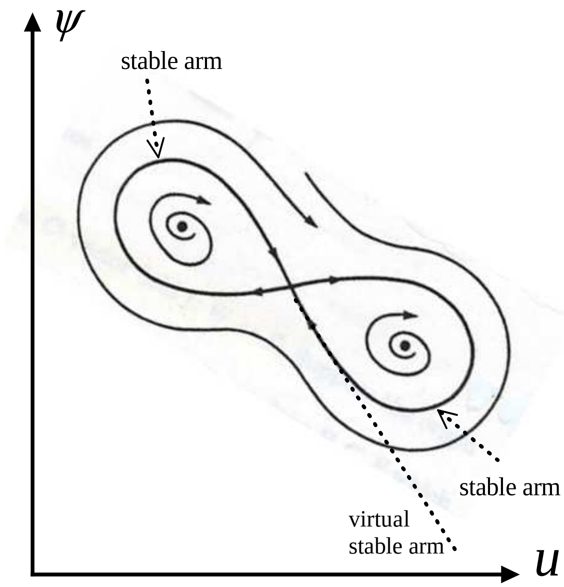

Figure 3: Limit sets in continuous time planar systems that are not closed orbits.[13]

5 4D Complex Keynesian Macrodynamics

In this section, we consider as an example for the emergence of complex dynamics in (quasi-)continuous time, the 4D Keynesian macro model formulated, estimated and simulated in Asada et al. (2007). The presentation of the model is slightly simplified here, in that Okun's law is assumed to be a 1:1 relationship between the employment gap on the labor market and the capacity utilization gap on the goods market, both measured as deviation from the corresponding steady state values, in place of an Okun coefficient

[13]In linearized rational expectations models, as they are the subject of a companion paper Asada et al. (2007) to the present one, the virtual stable arm shown in figure 3 is in fact used in place of the true one, in order to solve these models, a fact that may lead to forward-looking reactions of economic agents that differ from those that would happen in the true model, see also Asada et al. (2003, p.214) for details.

of 1/3. It can be further simplified to a 3D system if a static interest rate policy rule is assumed (no interest rate smoothing) in place of the dynamic one shown below (or even an interest rate peg by the central bank). The four laws of motion of the model are given as follows:

$$
\begin{aligned}
\hat{u} &= -\alpha_{uu}(u-u_o) - \alpha_{ui}(i-\hat{p}-(i_o-\bar{\pi})) + \alpha_{u\omega}(\ln\omega - \ln\omega_o) \\
\hat{\omega} &= \kappa[(1-\kappa_p)(\beta_{we}(u-u_o) - \beta_{w\omega}(\ln\omega - \ln\omega_o)) - (1-\kappa_w)(\beta_{pu}(u-u_o) + \beta_{p\omega}(\ln\omega - \ln\omega_o))] \\
\dot{\pi}^c &= \beta_{\pi^c}(\hat{p} - \pi^c) \\
\dot{i} &= -\gamma_{ii}(i-i_o) + \gamma_{ip}(\hat{p}-\bar{\pi}) + \gamma_{iu}(u-u_o)
\end{aligned}
$$

with

$$[\hat{p} = \kappa[\beta_{pu}(u-u_o) + \beta_{p\omega}(\ln\omega - \ln\omega_o) + \kappa_p(\beta_{we}(u-u_o) - \beta_{w\omega}(\ln\omega - \ln\omega_o))] + \pi^c]$$

Note that the $\hat{p}-$equation has to be inserted in some of the other equations in order to arrive at an autonomous system of differential equations in the four state variables u, the rate of capacity utilization of firms, ω, the real wage, π^c, the inflationary climate and i, the nominal rate of interest. Note also that the system is close to being linear, since $d\ln\omega/dt = \hat{\omega} = \dot{\omega}/\omega$ (the use of logs of real wages is derived in Blanchard and Katz (1999) for the money wage Phillips curve and here also applied to the price Philips curve and the goods market dynamics $\hat{u}$, the growth rate of capacity utilization).

Here we only briefly explain the contents of the above dynamical system and refer the reader to Asada et al. (2007) for more detailed explanations and to Asada et al. (2006) for an alternative dynamic macro approach which allows for complex dynamics in continuous time through the inclusion of typical behavioral nonlinearities far off the steady state.[14] The meaning of the above laws of motion is in fact an intuitively simple one, despite some lengthy derivation procedures concerning underlying wage and price Phillips curve in their structural form. The $\hat{u}$ equation states that the growth rate of capacity utilization depends negatively on its level (the dynamic multiplier assumption), as usual negatively on the real rate of interest and positively on the real wage (in a wage-led economy). We have reduced form price (the one in brackets) and wage Phillips curves where demand pressure in the labor and the goods market act positively (directly or indirectly) on these inflation rates, and also the inflationary climate π^c into which wage and price inflation are embedded. The difference between reduced form wage inflation and price inflation then provides the law of motion for real wages $\omega = w/p$ which then depends positively on demand pressure in the labor and negatively on demand pressure in the goods market (here both measured by the rate of capacity utilization u). The inflationary climate π^c is revised adaptively and thus follows price inflation with a certain delay. Finally, the law of motion for the nominal rate of interest is of conventional Taylor rule type. The above

[14]Note that these models also provide baseline Keynesian alternatives to the oversimplistic monetarist model we considered in section 4 (with its trivial quantity-theory driven explanation of upper and lower turning points in economic activity and inflation rates in the continuous time case).

dynamics mirror the building blocks of New Keynesian models with staggered wages and prices, but is only built on Neoclassical model consistent expectations formation (in place of their forward looking expectations) which moreover are supplied with sufficient inertia due to their combination with the inflationary climate into which these expectations are embedded.

We know from the literature that the real rate of interest channel is destabilizing if inflationary expectations are formed sufficiently fast. Moreover, the real wage channel of the above dynamical system is unstable in a wage-led economy if the growth rate of real wages depends positively on utilization (by and large: if real wages are moving procyclically), since real wages then stimulate economic activity which in turn leads to further real wage increases and so on. The above nearly linear system is therefore likely to be governed by destabilizing forces around the steady state and thus in general purely explosive. This is the case for the base parameter set underlying the simulations shown below which is the following:

$$\beta_{pu} = 1, \quad \beta_{p\omega} = 0.4, \quad \kappa_p = 0.3, \quad \beta_{we} = 0.8, \quad \beta_{w\omega} = 0.4, \quad \kappa_w = 0.7, \quad \beta_{\pi^c} = 0.5,$$
$$\alpha_{uu} = 0.22, \quad \alpha_{u\omega} = 0.1, \quad \alpha_{ui} = 0.25, \gamma_{ii} = 0.1, \quad \gamma_{ip} = 0.5, \quad \gamma_{iu} = 1,$$

This parameter set is broadly in line with empirical observations, see Asada et al. (2007) and is here used for illustrative purposes solely. The model, when based on these parameters, is not a viable one, not even in the medium run. It is fairly well known however, see Keynes (1936) for the initial statement of this fact, that nominal wages are downwardly rigid to a certain degree. Again for illustrative purposes we assume in the following simulations that they can rise, but will not fall to a significant degree. This simple modification has significant consequences since it implies that the explosive dynamics are tamed thereby and becomes bounded or viable, since in a wage led regime we then get in depressions that real wages will start rising (due to falling prices) and thus will stimulate aggregate demand and economic activity. Yet, due to the explosive nature in the boom this will happen in an irregular way in the case of strong centrifugal forces around the steady state. We thus get the outcome that a single and very basic nonlinearity in a fully fledged Keynesian dynamical system can generate complex dynamics in 4D continuous time, a situation where bounding mechanisms are not as easy to design as in dimensions 1 or 2.

Figure 4 presents two bifurcation diagrams around the given set of parameter values which show the plot of local maxima and minima (in the vertical direction) plotted against one typical parameter on the horizontal axis. The first plot shows the implications of an increase in the adjustment speed of wages with respect to demand pressure on the labor market for the real wage (after a certain transient period has been passed). We see that the fluctuations in real wages become complex from 1 onwards, a value that in our empirical studies (for the USA) is however higher than the actually observed one. In theory however even infinite adjustment speeds are allowed for and indicate that not too narrowly chosen values may be of interest here. Be that as it may, the floor in the money wage Phillips curve is indeed already sufficient to make the economy viable even up to $\beta_{we} = 8$!

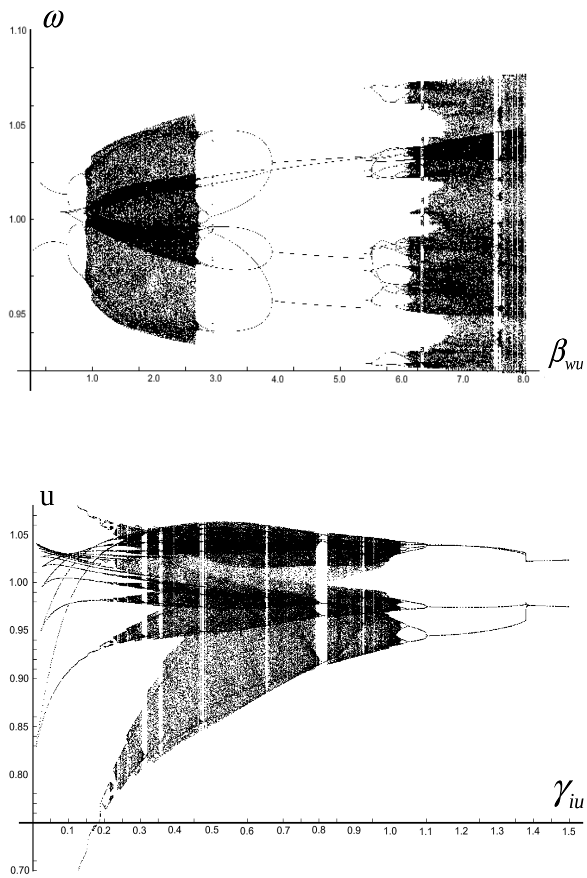

Figure 4: Bifurcation diagrams for selected adjustment speeds

In the second plot we consider the role of monetary policy and clearly see its stabilizing role as the reaction of the central bank to the capacity utilization gap is increased. It may be out of reach however for the central bank to indeed establish convergence to the steady state in the considered situation. Complex mathematical dynamics (though maybe not really complex from an economic point of view where irregularity is part of the observations that are actually made) may thus be an issue with which the central banks in the world have to deal.

6 Concluding Remarks

In this paper, we have reconsidered the issue of the (non-)equivalence of discrete and continuous time macro modeling or more exact period vs. continuous time analysis. We started from the empirical fact that the actual data generating process in macroeconomics is (in the real markets) of much finer step size (daily) than the corresponding data collection process (a month / a quarter), which in turn is often much finer than the periods implicit in macromodels with chaotic attractors. This implies that empirically applicable period macromodels should be iterated at least with step size of '1/52 year' (if not even 1/365) as far as actual processes (concerning production, investment, inflation etc.) are concerned and will then in general (at least in dimensions 1 and 2) generate results that are equivalent to the ones of their continuous time analogue (assumed to exist). In our view, therefore, such empirically constrained applicable macromodels will in general not be able to give rise to chaotic dynamics in dimensions 1 and 2, if their parameter ranges are broadly in line with empirical observations, suggesting that the literature on such chaotic dynamics is of questionable empirical relevance.

Such a statement however needs further qualification if models with dynamic expectation formation, discretionary economic policy making or gestation lags in investment are considered. In the first example, the data collection process (which is generally much cruder than the data generating process) may need extrapolating behavior of economic agents in order to allow for a smooth representation of expected magnitudes. In the second case, there is generally a smoothing process involved that transforms for example government expenditure programs from discretionary policy decisions to fairly smooth factual policy performance, due to implementation lags. The third example may be the most significant one, as already the early article by Kalecki (1935) exemplifies. This case however leads to the consideration of continuous or period models with behaviorally specific and significant realization delays (time to build), to be justified by economic reasoning, that cannot be uniformly applied to all evolving updating processes in a period model and that will also occur in an unsynchronized, non-clustered fashion across the micro-structures of the economy. Theory may assume representative firms, etc., but it must take account of the empirical fact that such agents do not act in a synchronized manner, but that unsynchronized staggered and thereby smoothed decision making must be added when going from the micro to the macro level. Such a procedure will however make macroeconomic discrete time analysis in general quasi-continuous and thus by and large equivalent to continuous time modeling if available.

However, we have to leave such considerations for future research. We have considered in this paper only a first step towards such a discussion, by way of a mapping from ordinary differential equation systems into systems of difference equations that replaces differential quotients simply through difference quotients with a given period length or time-interval h, as in the model of Soliman (1996). Not all systems of difference equations, however, will be captured in this way (but have to be investigated then as to why they are not well-suited for small iteration steps). The conclusions of the paper are thus for the moment only applicable to a subset of period models of the considered dimensionality. But with respect to this subset we have argued, that – from the empirical perspective – we should only allow for period lengths which give the generated period model a 'quasi continuous-time' outlook. Of course, there may be exceptions to this rule for special choices of the difference and differential equation system, but we expect that macrodynamic models of conventional type will not be of such an exceptional nature.

We also conclude that applied macrodynamic period models are unlikely to give rise to chaotic dynamics in dimensions one and two and thus suggest that the extensive literature on such chaotic dynamics is of no empirical relevance, as exemplified in section 4 by means of a monetarist baseline model. This implies as strategy for future research that one should concentrate investigations on macrodynamic models, which when represented by ordinary differential equation systems, have at least 3 state variables in order to find economically well motivated reasons for the occurrence of complex macrodynamics. But here too one needs to keep in mind that the finding of complex macrodynamics in period representations that is not present in their continuous time analogue may question such a finding from the empirical perspective.

Chiarella and Flaschel (2000) have for example argued that a fully specified balanced model of Keynesian monetary growth exhibits at the least the state variables wage share and labor intensity (the growth component), inflation and expected inflation (the medium-run component) and expected sales and actual inventories per unit of capital, i.e., meets the dimensionality condition easily. And also the New Keynesian baseline model with both staggered wage and price setting is at least of dimension 4, see Asada et al. (2007), i.e., even still simple models of a monetary economy can be used for a bifurcation analysis as considered in section 4 without running into the danger of synthesizing basically continuous-time ideas with radically synchronized (overshooting) discrete time adjustment processes, which when appropriately bounded can produce chaos also in dimensions one and two. This implies that all developed techniques for analyzing nonlinear dynamical systems and the experience related with them represent a useful stock of knowledge by which to investigate the strange attractors that may come about in continuous or quasi-continuous time of high order macro-systems.

As a future research agenda we therefore propose to use existing higher dimensional macromodels with small (quasi-continuous, but still of period model type) iteration step size and basic parameters broadly in line with empirical magnitudes, in order to investigate by the help of appropriate behavioral nonlinearities their possibly complex attractors, applying mathematical approaches such as in Guckenheimer and Holmes (1983), Wiggins (1990) and more recent work to such higher order macrodynamic model building. In section 5 of this paper we have briefly considered as an outlook an example for such a research strategy, intended to show the relevance of the occurrence of complex dynamics in the realm of applicable macromodels.

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Appendix A

Loss of Stability for Large Periods h.

We consider the linear approximation of the discrete time dynamics investigated in section 3 (with $\alpha \leq 1$), which is given by:

$$A = hJ + I = \begin{pmatrix} 1 + bf'(\cdot)h & b\alpha h \\ cf'(\cdot)h & 1-(1-\alpha)ch \end{pmatrix} \tag{A.1}$$

and want to determine the size (in its the dependence on the model's parameters) of the period length h where the considered system loses its local stability property (implying global instability unless the Phillips curve $f(\cdot)$ is made nonlinear as in Soliman (1996)).

Note that $f' < 0$ in Eq. (A1) is evaluated at the equilibrium point. The characteristic equation of this system becomes

$$\Delta(\lambda) \equiv \lambda^2 + a_1\lambda + a_2 = 0 \tag{A.2}$$

where

$$\begin{aligned} a_1 &= -trace\, A = -2 + \{(1-\alpha)c - b \underset{(-)}{f'}\}h, && \text{(A.3)} \\ a_2 &= \det A = (1+bf'h)\{1-(1-\alpha)ch\} - bc\alpha f'h^2 \\ &= -bc \underset{(-)}{f'} h^2 + \{b \underset{(-)}{f'} -(1-\alpha)c\}h + 1. && \text{(A.4)} \end{aligned}$$

Then, we have

$$\begin{aligned} A_1 &\equiv 1 + a_1 + a_2 = -bc \underset{(-)}{f'} h^2 > 0 \quad \text{for all } h > 0, && \text{(A.5)} \\ A_2 &\equiv 1 - a_2 = b c\underset{(-)}{f'} h^2 + \{-b \underset{(-)}{f'} +(1-\alpha)c\}h = A_2(h), && \text{(A.6)} \\ A_3 &\equiv 1 - a_1 + a_2 = -bc \underset{(-)}{f'} h^2 + 2\{b \underset{(-)}{f'} -(1-\alpha)c\}h + 4 = A_3(h). && \text{(A.7)} \end{aligned}$$

The characteristic roots of Eq. (A2) are given by

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}, \tag{A.8}$$

and the discriminant of these roots is given by

$$D \equiv a_1^2 - 4a_2 = [(1-\alpha)c\{(1-\alpha)c - 2b \underset{(-)}{f'}\} + b \underset{(-)}{f'}(b \underset{(-)}{f'} +4c)]h^2. \tag{A.9}$$

The equilibrium point of this system of difference equations is locally stable if and only if the inequalities $|\lambda_j| < 1$ $(j = 1, 2)$ are satisfied, and it is well known that this local stability condition is equivalent to the following set of inequalities, which is called the 'Cohn-Schur condition' in case of the two dimensional discrete time dynamic system(cf. Gandolfo 1996 p. 58).

$$A_j > 0 \quad (j = 1, 2, 3) \tag{A.10}$$

Proposition A1.

The equilibrium point of this system is (i) locally asymptotically stable for all sufficient small values of $h > 0$, and (ii) it is unstable for all sufficiently large values of $h > 0$ irrespective of the values of the parameters $b > 0, \quad c > 0, \quad f' < 0$, and $\alpha \in (0, 1]$.

Proof:

(i) We have $A_2(0) = 0$, $\left.\frac{dA_2}{dh}\right|_{h=0} = -b \underset{(-)}{f'} + (1 - \alpha)c > 0$, and $A_3(0) = 4 > 0$ from equations (A6) and (A7), which means by continuity that we have $A_2 > 0$ and $A_3 > 0$ for all sufficiently small values of $h > 0$. On the other hand, it follows from Eq. (A5) that the inequality $A_1 > 0$ is always satisfied. In this case, all of the local stability conditions (A10) are satisfied.

(ii) It is easy to see from Eq. (A6) that we have $A_2 < 0$ for all sufficiently large values of $h > 0$. In this case, one of the local stability conditions (A10) is violated.

Proposition A2.

(i) Suppose that $4c > b\,|f'|$ and α is sufficiently close to 1 (including the case of $\alpha = 1$). Then, the characteristic equation (A2) has a set of complex roots irrespective of the value of $h > 0$.

(ii) Suppose that the characteristic equation (A2) has a set of complex roots. Then, we have $|\lambda| = 1$ for $h = \bar{h}$, $|\lambda| < 1$ for all $h \in (0, \bar{h})$, $|\lambda| > 1$ for all $h \in (\bar{h}, +\infty)$, and $\left.\frac{d|\lambda|}{dh}\right|_{h=\bar{h}} > 0$, where $\bar{h}$ is defined as $\bar{h} \equiv \frac{1}{c} + \frac{1-\alpha}{b|f'|} > 0$, and $|\lambda|$ is the modulus of the characteristic roots.

Remark A1.

In the case where $\alpha = 1$ holds (the 'Friedman' limit case) the bifurcation value $\bar{h}$ depends only on the parameter c that determines the strength of the adaptively revised expectations mechanism (in reciprocal form). Loss of local asymptotic stability occurs therefore the earlier, the faster inflationary expectations are adjusted. Such a loss of stability – we repeat – cannot occur in the continuous time case, since there is not yet a Mundell or real interest rate effect present in the Monetarist baseline model (see section 4 for its presence in a Keynesian baseline model). Loss of stability therefore only occurs in the period version in particular due to the fact that the eigenvalues of the continuous case become too negative in their real parts, i.e., the continuous time case becomes 'too stable'.

Proof:

(i) Suppose that $4c > b|f'|$ and $\alpha = 1$. Then, it follows from Eq. (A9) that $D = b \underset{(-)}{f'}(-b|f'| + 4c)h^2 < 0$ for all $h > 0$. It is obvious from continuity that we have $D < 0$ even if $\alpha < 1$, as long as α is sufficiently close to 1 and $4c > b|f'|$.

(ii) Suppose that $D < 0$. Then, we have $a_2 > 0$ and

$$|\lambda| = \sqrt{(\frac{a_1}{2})^2 + (\frac{\sqrt{-a_1^2 + 4a_2}}{2})^2} = \sqrt{a_2} = \sqrt{a_2(h)}. \tag{A.11}$$

It is easy to see from Eq. (A4) that $a_2(\bar{h}) = 1, \quad a_2(h) < 1$ for all x$h \in (0, \bar{h}), \quad a_2(h) > 1$ for all $h \in (\bar{h}, +\infty)$, and $\frac{da_2}{dh}\big|_{h=\bar{h}} > 0$.

Remark A2.

Proposition A2 (i) means that the large values of $c > 0$ and $\alpha \in (0, 1]$ are conductive to cyclical fluctuations.

Remark A3.

The critical value $\bar{h}$ in Proposition A2 (ii) is decreasing function of the parameters $c, \quad b, \quad |f'|$, and α. This means that the increases of these parameter values have destabilizing effects.

Remark A4.

The point $h = \bar{h}$ in Proposition A2 (ii) is in fact the Hopf bifurcation point of the two dimensional discrete time system if the additional technical conditions $\lambda_j^n(\bar{h}) \neq \pm 1$ $(n = 1, 2, 3, 4)$ are satisfied, where $\lambda_j(h) \quad (j = 1, 2)$ are the characteristic roots (cf. Gandolfo 1996 p. 492). In this case, there exist some non-constant closed orbits at some parameter values h that are sufficiently close to $\bar{h}$.