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The dynamic behaviour of an endogenous growth model with public capital and pollution

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Abstract

In this paper we present an endogenous growth model with productive public capital and pollution. As to pollution we assume that it is a by-product of aggregate production and that it negatively affects utility of the household but not production possibilities directly. The paper studies the dynamics of the model and demonstrates that there exists either a unique balanced growth path which is a saddle point or there exist two balanced growth paths with one being locally saddle point stable and one being asymptotically stable.

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1 Introduction

One strand in endogenous growth theory assumes that public spending or a public capital stock generates sustained per capita growth in the long run. For example, Barro (1990) presents a simple model where public spending as a flow variable shows productive effects. Futagami et al. (1993) analyze a more elaborate model with public capital as a stock which we take as a starting point of our model. We do so because empirical studies have shown that the stock of public capital is dramatically more important than the flow of public government spending as concerns aggregate productivity (see e.g. Aschauer (1989) and for a survey of these studies Sturm et al. (1998)). However, this assumption implies that the model has transition dynamics which does not hold for the model when public spending as a flow variable shows productive effects. In the latter case the economy immediately jumps on the balanced growth path. The model presented by Futagami et al. is characterized by a unique balanced growth path which is a saddle point. Although the question of whether the long run balanced growth path is unique and whether it is stable is an important issue this question is not frequently studied in this type of research. Most of the contributions study growth and welfare effects of fiscal policy for the model on the balanced growth path.

In another line of research endogenous growth models are presented where it is assumed that economic activities cause pollution which negatively affects the environment. Examples of that type of research are the papers by Bovenberg and Smulders (1995), Gradus and Smulders (1993), Bovenberg and de Mooij (1997) or Hettich (1998). In most of these models it is assumed that pollution or the use of resources influences production possibilities either through affecting the accumulation of human capital or by directly entering the production function. The goal of these studies, then is to analyze how different tax policies affect growth, pollution and welfare in an economy. But, as with the approaches mentioned above, most of these models do not have transition dynamics or the analysis is limited to the balanced growth path. An explicit analysis of the dynamics often is beyond the scope of these contributions. An exception is provided by the paper by Koskela et al. (2000) who study an overlapping generations model with a renewable resource whihc serves as a store of value and as an input factor in the production of the consumption good. They find that indeterminacy and cycles may result in their model, depending on the value of the intertemporal elasticity of consumption.

In this paper we will combine the two approaches mentioned above where we assume that pollution is an inevitable by-product of production and can be reduced by abatement activities but not completely as frequently assumed in the economics literature (see e.g. the survey by Smulders, 1995, or Hettich, 2000). Thus, our paper is closely related to the contributions by Smulders and Gradus (1996) and Bovenberg and de Mooij (1997) who are interested in growth and welfare effects of fiscal policy but do not explicitly study the dynamics of their models. As concerns the structure, our model is basically the same as the one presented by Bovenberg and de Mooij with the exception that we assume that public capital as a stock enters the aggregate production function while Bovenberg and de Mooij assume that public investment as a flow has positive efects on aggregate production. However, in contrast to the aforementioned contributions, our goal with this paper is not to study growth and welfare effects of public policies but to give a complete characterization of the dynamics of the model we present.

Thus, our paper contributes to the literature on the dynamics of competitive economies with externalities. Examples of these studies are the contributions by Benhabib and Farmer (1994) and by Benhabib et al. (2000). The difference of our paper to these studies is twofold. First, we consider negative external effects of production, i.e. pollution as a by-product of production, in contrast to the aforementioned papers which assume positive externalities. Second, we do not assume that these externalities affect production in our economy but instead they have negative repercussion on the utility of the household.

The rest of the paper is organized as follows. In section 2 we present the structure of our model. Section 3 studies the dynamics of the model and in section 4 we summarize and conclude the paper.

2 Structure of the model

We consider a decentralized economy which comprises three sectors: the household sector, a productive sector, and the government.

2.1 The household

Our economy consists of one representative household which maximizes its discounted stream of utility subject to its budget constraint:

$$J(\cdot) \equiv \max_{C(t)} \int_0^\infty e^{-\rho t} V(t) dt, \tag{1}$$

with $V(t)$ the instantaneous subutility function which depends positively on the level of consumption, $C(t)$, and negatively on effective pollution, $P_E(t)$. $V(t)$ takes the following form

$$V(t) = (C(t) P_E(t)^{-\xi})^{1-\sigma}/(1-\sigma),$$

where $\xi > 0$ gives the disutility arising from effective pollution.[1] $1/\sigma > 0$ gives the intertemporal elasticity of substitution of private consumption between two points in time for a given level of effective pollution and ln is the natural logarithm. ρ in (1) is the subjective discount rate and there is no population growth.

The budget constraint is given by[2]

$$\dot{K} = -C + wL + rK, \tag{2}$$

with L denoting labour which is supplied inelastically. The budget constraint (2) states that the individual has, as usual, to decide how much to consume and how much to

[1] For a survey of how to incorporate pollution in the utility function see Smulders (1995), p. 328-29.

[2] In what follows we will suppress the time argument if no ambiguity arises.

save, thus increasing consumption possibilities in the future. The depreciation of physical capital is assumed to equal zero. w in the budget constraint is the wage rate and r is the return to capital K.

Assuming that a solution to (1) subject to (2) exists we can use the current-value Hamiltonian to describe that solution. The Hamiltonian function is written as

$$\mathcal{H}(\cdot) = (CP_E^{-\xi})^{1-\sigma}/(1-\sigma) + \lambda(-C + wL + rK),$$

with λ the costate variable. The necessary optimality conditions are given by

$$\lambda = C^{-\sigma} P_E^{-\xi(1-\sigma)}, \tag{3}$$

$$\dot{\lambda}/\lambda = \rho - r, \tag{4}$$

$$\dot{K} = -C + wL + rK. \tag{5}$$

Since the Hamiltonian is concave in C and K jointly, the necessary conditions are also sufficient if in addition the transversality condition at infinity $\lim_{t\to\infty} e^{-\rho t}\lambda(t)(K(t) - K^\star(t)) \geq 0$ is fulfilled with $K^\star(t)$ denoting the optimal value. Moreover, strict concavity in C and L also guarantees that the solution is unique (cf. Seierstad and Sydsaeter (1987), pp. 234-235).

2.2 The productive sector

The productive sector in our economy can be represented by one firm which chooses inputs in order to maximize profits and which behaves competitively. As to pollution, we suppose that it is the result of aggregate production. In particular, we assume that pollution $P(t)$ is a by-product of output $Y(t)$, i.e. $P(t) = \varphi Y(t)$, with $\varphi = const. > 0$. Thus, we follow the line invited by Forster (1973) and worked out in more details by Luptacik and Schubert (1982).

The production function is given by,

$$Y = K^\alpha L^{1-\alpha} H^{1-\alpha}, \tag{6}$$

with H denoting the stock of productive public capital which raises efficiency of labour and $\alpha \in (0,1)$ gives the capital share. Pollution is taxed at the rate $\tau_p > 0$ and the firms take into account that one unit of output causes φ units of pollution for which they have to pay $\tau_p \varphi < 1$ per unit of output.

The optimization problem of the firm then is given by

$$\max_{K,L} K^{\alpha} L^{1-\alpha} H^{1-\alpha} (1 - \varphi \tau_p) - rK - wL \tag{7}$$

Assuming competitive markets and taking public capital as given optimality conditions for a profit maximum are obtained as

$$w = (1 - \tau_p \varphi)(1-\alpha) L^{-\alpha} K^{\alpha} H^{1-\alpha}, \tag{8}$$

$$r = (1 - \tau_p \varphi) \alpha K^{\alpha - 1} H^{1-\alpha} L^{1-\alpha}. \tag{9}$$

2.3 The government

The government in our economy receives tax revenue from the taxation of pollution. The tax revenue is spent for abatement activities $A(t)$ which reduce total pollution and for the formation of public capital, $H(t)$. Abatement activities are determined by $A(t) = \eta \tau_p P(t)$, with $\eta < 1$. $\eta < 1$ means that not all of the pollution tax revenue is used for abatement activities and the remaining part is used for public investment in the public capital stock I_p, $I_p > 0$.

However, pollution cannot be eliminated completely. We call that part of pollution which remains in spite of abatement activites the effective pollution $P_E(t)$. In particular, we follow Gradus and Smulders (1993) and Lighthart and van der Ploeg (1994) and take the following specification

$$P_E = \frac{P}{A^{\beta}}, \; 0 < \beta \leq 1. \tag{10}$$

The limitation $\beta \leq 1$ assures that a positive growth rate of aggregate production goes along with an increase in effective pollution, $\beta < 1$, or leaves effective pollution unchanged,

$\beta = 1$. We make that assumption because it is realistic to assume that higher production also leads to an increase in pollution, although at a lower rate because of abatement. Looking at the world economy that assumption is certainly justified.

Moreover, the government in our economy runs a balanced budget at any moment in time. Thus, the budget constraint of the government[3] is written as

$$I_p + A = \tau_p P \leftrightarrow I_p = \tau_p P(1-\eta). \tag{11}$$

The evolution of public capital is described by

$$\dot{H} = I_p, \tag{12}$$

where for simplicity we again neglect depreciation of public capital.

3 The dynamics of the model

In the following, labour is normalized to one, i.e. $L \equiv 1$. An equilibrium allocation in the economy, then, is given if $K(t)$ and $L(t)$ maximize profits of the firm, $C(t)$ maximizes (1) and the budget of the government is balanced.

Profit maximization of the firm implies that the marginal products of capital and of labour equal the interest rate and the wage rate. This implies that in equilibrium the growth rate of physical capital is given by

$$\frac{\dot{K}}{K} = -\frac{C}{K} + \left(\frac{H}{K}\right)^{1-\alpha}(1-\varphi\tau_p),\ K(0) = K_0. \tag{13}$$

Using the budget constraint of the government the growth rate of public capital is

$$\frac{\dot{H}}{H} = \left(\frac{H}{K}\right)^{-\alpha}\varphi\tau_p(1-\eta),\ H(0) = H_0. \tag{14}$$

[3]The budget constraint is the same as in Bovenberg and de Mooij (1997) except that these authors also impose a tax on output.

Utility maximization of the household yields the growth rate of consumption as

$$\frac{\dot{C}}{C} = -\frac{\rho}{\sigma} + \sigma^{-1}(1-\varphi\tau_p)\alpha\left(\frac{H}{K}\right)^{1-\alpha} - \xi(1-\beta)\frac{1-\sigma}{\sigma}\left(\alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{H}}{H}\right). \tag{15}$$

Equations (13), (14) and (15) completely describe the economy in equilibrium. The initial conditions $K(0) = K_0$ and $H(0) = H_0$ are given and fixed and $C(0)$ can be chosen freely by the economy. Further, the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda(t)(K(t) - K^\star(t)) \geq 0$ must be fulfilled, with $K^\star(t)$ denoting the optimal value and λ determined by (3).[4]

Before we study the dynamics of our model we have to define a balanced growth growth (BGP). This is done in the following theorem.

Definition 1 *A balanced growth path (BGP) is path on which the ratios $c \equiv C/K$ and $h \equiv H/K$ are constant.*

This definition implies that C, H, K and Y grow at the same rate on a BGP.

With definition 1, the system describing the dynamics around a BGP is written as

$$\begin{aligned}\frac{\dot{c}}{c} &= -\frac{\rho}{\sigma} + \frac{\alpha h^{1-\alpha}(1-\varphi\tau_p)}{\sigma} - (1-\alpha)\xi(1-\beta)\frac{1-\sigma}{\sigma}h^{-\alpha}\varphi\tau_p(1-\eta) + \\ &\quad \left(1 + \alpha\xi(1-\beta)\frac{1-\sigma}{\sigma}\right)(c - h^{1-\alpha}(1-\tau_p\varphi)) \qquad (16)\\ \frac{\dot{h}}{h} &= c - h^{1-\alpha}(1-\varphi\tau_p) + h^{-\alpha}\varphi\tau_p(1-\eta). \qquad (17)\end{aligned}$$

Concerning a rest point of system (16) and (17) it should be noted that we only consider interior solutions. That means that we exclude the economically meaningless stationary point $c = h = 0$ such that we can consider our system in the rates of growth.[5] As to the existence and stability of a BGP we can state proposition 1.

Proposition 1 *If $1 + \xi(1-\beta)(1-\sigma)/\sigma \geq 0$ there exists a unique BGP which is a saddle point.*

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[4]Note that only in the formulation of the transversality condition we use the $\star$ to denote optimal values.

[5]Note also that h is raised to a negative power in (16).

Proof: See appendix.

That proposition gives conditions for our model to be locally and globally determinate, i.e. there exists a unique value for $c(0)$ such that the economy converges to the BGP in the long run.[6] A prerequisite for that outcome is $1+\xi(1-\beta)(1-\sigma)/\sigma \geq 0$. So, it can be stated that a unique BGP is the more likely the larger β and the smaller ξ for a given value of σ. From an economic point of view this means that an effective abatement technology, i.e. a large β, makes a unique BGP more likely.[7] Further, a small ξ also favours that outcome. A small ξ implies that the effect of pollution on instantaneous utility is small. Thus, we can summarize that the smaller the negative external effect of production, either because abatement is very effective or because the household does not attach much value to a clean environment, the less likely is the emergence of global and local indeterminacy. So, it is the externality which gives rise to possible indeterminacy of equilibrium paths.

For given values of β and ξ, $1+\xi(1-\beta)(1-\sigma)/\sigma \geq 0$ always holds for $\sigma \leq 1$. The intertemporal elasticity of substitution, $1/\sigma$, often plays an important role as to the question of whether the model is global indeterminate (see e.g. Benhabib and Perli, 1994 or Benhabib, Perli and Xie, 1994). For our model we see from proposition 1 that $\sigma > 1$ is a necessary condition for multiple BGPs to be feasible. In other papers a small value for σ, i.e. a high intertemporal elasticity of substitution, is a necessary condition for multiple BGPs (see again for example Benhabib and Perli, 1994). The different outcome in our model compared to other contributions in the economics literature is due to the fact that in our model utility does not only depend on consumption but also on effective pollution which is a by-product of aggregate production.

Therefore, the outcome stated in proposition 1 makes sense from an economic point of view: Global indeterminacy means that the economy may either converge to the BGP with the high balanced growth rate or to the BGP with the low balanced growth rate in

[6]For a definition of local and global determinacy see e.g. Benhabib and Perli (1994) or Benhabib, Perli and Xie (1994).

[7]For $\beta = 1$ the inequality is always fulfilled.

the long run. So it may either choose a path with a higher initial consumption level (but lower initial investment) or a path with a lower level of initial consumption (but higher initial investment). In the latter case, the household must be willing to forgo current consumption and shift it into the future. If production and, thus, consumption do not have negative effects in form of pollution then the household will do that only if he has a high intertemporal elasticity of substitution of consumption. However, if production and, thus, consumption do have negative repercussions because they lead to a rise in effective pollution (if $\beta < 1$) then the household is willing to forgo current consumption even with a low intertemporal elasticity of substitution because renouncing to consumption also has a positive effect since effective pollution is then lower, too, which raises current utility. It should be noted that our result is in line with the outcome in Koskela et al. who find that indeterminacy and cycles occur for relatively small values of the intertemporal elasticity substitution of consumption. But is must be recalled that their model is quite different from ours because it considers a renewable resource and no externalities and is formulated in discrete time.

Next, we consider the case $1+\xi(1-\beta)(1-\sigma)/\sigma < 0$. Proposition 2 gives the dynamics in this case.

Proposition 2 *If* $1+\xi(1-\beta)(1-\sigma)/\sigma < 0$, $\alpha \leq 0.5$ *is a sufficient but not necessary condition for the existence of two BGPs. The BGP yielding the lower growth rate is saddle point stable and the BGP giving the higher growth rate is asymptotically stable.*

Proof: See appendix.

This theorem states that two BGPs can be observed in our model depending on the parameter values and that one equilibrium is locally determinate while the other is indeterminate. For $1+\xi(1-\beta)(1-\sigma)/\sigma < 0$ to hold, σ must be larger 1. $\sigma > 1$ implies that $\xi(1-\beta) > 1$ must hold so that the inequality $1+\xi(1-\beta)(1-\sigma)/\sigma < 0$ can be fulfilled. From an economic point of view, this means that pollution has a strong effect on utility, ξ is large, and abatement is not very effective, β is small.

4 Conclusion

In this paper we have analyzed the dynamics of an endogenous growth model with productive public spending and pollution. As to the financing of public spending we assumed that the tax on pollution is used for both abatement and for public investment, with the latter leading to sustained per capita growth.

We could demonstrated, without resorting to numerical examples, that the parameters determining the negative effect of pollution on utility is crucial as to the question of whether indeterminate equilibrium paths may exist besides the intertemporal elasticity of substitution. The stronger the effect of pollution the more likely are multiple BGPs provided that the intertemporal elasticity of substitution is sufficiently small.

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Appendix

Proof of proposition 1: To prove proposition 1 we first calculate c^∞ on a BGP which is obtained from $\dot{h}/h = 0$ as

$$c^\infty = h^{1-\alpha}(1-\varphi\tau_p) - h^{-\alpha}\varphi\tau_p(1-\eta).$$

Inserting c^∞ in $\dot{c}/c$ gives

$$f(\cdot) \equiv \dot{c}/c = -\rho/\sigma + (1-\varphi\tau_p)\alpha h^{1-\alpha}/\sigma - h^{-\alpha}\varphi\tau_p(1-\eta)(1+\xi(1-\beta)(1-\sigma)/\sigma).$$

For $(1+\xi(1-\beta)(1-\sigma)/\sigma) \geq 0$ we have

$$\lim_{h\to 0} f(\cdot) = -\infty \text{ and } \lim_{h\to\infty} f(\cdot) = \infty \text{ and}$$

$$\partial f(\cdot)/\partial h = (1-\varphi\tau_p)(1-\alpha)\alpha h^{-\alpha}/\sigma + \alpha h^{-\alpha-1}\varphi\tau_p(1-\eta)(1+\xi(1-\beta)(1-\sigma)/\sigma) > 0.$$

This shows that there exists a finite $h^\infty > 0$ such that $f(\cdot) = 0$ holds and, thus, a unique BGP.

Saddle point stability is shown as follows. Denoting with J the Jacobian of $\dot{c}$ and $\dot{h}$ evaluated at the rest point we first note that det $J < 0$ is a necessary and sufficient condition for saddle point stability, i.e. for one negative and one positive eigenvalue. The Jacobian in our model can be written as

$$J = \begin{bmatrix} c^\infty(1+\alpha\xi(1-\beta)(1-\sigma)/\sigma) & c^\infty\phi \\ h^\infty & h^\infty\upsilon \end{bmatrix},$$

with ϕ given by $\phi = (1-\varphi\tau_p)(1-\alpha)(h^\infty)^{-\alpha}(-1+\alpha/\sigma) - (\xi(1-\beta)(1-\sigma)/\sigma)(1-\alpha)\alpha(h^\infty)^{-\alpha-1}[(h^\infty)(1-\varphi\tau_p)-\varphi\tau_p(1-\eta)]$ and $\upsilon = -\alpha(h^\infty)^{-\alpha-1}\varphi\tau_p(1-\eta)-(1-\alpha)(h^\infty)^{-\alpha}(1-\varphi\tau_p)$. c^∞ and h^∞ denote the values of c and h on the BGP. The determinant can be calculated as det $J = c^\infty h^\infty(-\alpha(1-\alpha)(1-\varphi\tau_p)(h^\infty)^{-\alpha}/\sigma) - \alpha(h^\infty)^{-\alpha-1}\varphi\tau_p(1-\eta)(1+\xi(1-\beta)(1-\sigma)/\sigma)) < 0$, for $(1+\xi(1-\beta)(1-\sigma)/\sigma) \geq 0$.

Proof of proposition 2: To prove proposition 2 we recall from the proof of proposition 1 that a h^∞ such that $f(\cdot) \equiv \dot{c}/c = 0$ holds gives a BGP.

For $(1+\xi(1-\beta)(1-\sigma)/\sigma) < 0$ we have

$$\lim_{h\to 0} f(\cdot) = \infty \text{ and } \lim_{h\to\infty} f(\cdot) = \infty \text{ and}$$

$$\partial f(\cdot)/\partial h >=< 0 \Leftrightarrow h >=< h_{min} \text{ and } \lim_{h\to 0} \partial f(\cdot)/\partial h = -\infty,\ \lim_{h\to\infty} \partial f(\cdot)/\partial h = 0,$$

with $h_{min} = (-1)\alpha\varphi\tau_p(1-\eta)(1-\alpha)^{-1}(1+\xi(1-\beta)(1-\sigma)/\sigma)^{-1}$.

This implies that $f(h,\cdot)$ is strictly monotonic decreasing for $h < h_{min}$, reaches a minimum for $h = h_{min}$ and is strictly monotonic increasing for $h > h_{min}$. This implies that there exist two BGPs (two points of intersection with the horizontal axis) if $f(h,\cdot)$

crosses the horizontal axis. This is guaranteed if $f(h_{min}, \cdot) < 0$ holds. Inserting h_{min} in $f(\cdot)$ gives

$$f(h_{min}, \cdot) = -\rho/\sigma + h_{min}^{-\alpha} \varphi \tau_p (1 - \eta) \left(1 + \xi(1 - \beta)(1 - \sigma)/\sigma\right) \left(1 - \alpha/(1 - \alpha)\right).$$

A sufficient condition for $f(h_{min}, \cdot) < 0$ is $(1 - \alpha/(1 - \alpha)) \geq 0 \Leftrightarrow \alpha \leq 0.5$.

To analyze stability we note that the determinant of the Jacobian can be written as $\det J = -c^{\infty} h^{\infty} \partial f(\cdot)/\partial h$. This shows that the first intersection point of $f(\cdot)$ with the horizontal axis (smaller h^{∞} and, thus, larger balanced growth rate, see (14)) cannot be a saddle point since $\partial f(\cdot)/\partial h < 0$ holds at this point. This point is asymptotically stable (only negative eigenvalues or eigenvalues with negative real parts) if the trace is negative, i.e. if $trJ < 0$ holds. The trace of the Jacobian can be calculated as $trJ = -h^{\infty} \varphi \tau_p + c^{\infty} \alpha (1 + \xi(1 - \beta)(1 - \sigma)/\sigma)$ which is negative for $(1 + \xi(1 - \beta)(1 - \sigma)/\sigma) < 0$.

The second intersection point of $f(\cdot)$ with the horizontal axis (lower h^{∞} and, thus, higher balanced growth rate, see (14)) is a saddle point since $\partial f(\cdot)/\partial h > 0$ holds at this point implying $\det J < 0$.