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Real Business Cycles with Disequilibrium in the Labor Market: A Comparison of the U.S. and German Economies

by

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Abstract

Real Business Cycles are often studied in the context of intertemporal general equilibrium models. This restrict the effective application of intertemporal models to the real world where disequilibria seem to be a wide spread phenomenon. If we regard the decision rules as a reflection of the agents' willingness to supply goods and labor effort, and thus represent only one side of the market forces, we might think of introducing the other side of the market as well. Disequilibria can occur if these two sides are not in balance. In this paper we consider different variants of labor market disequilibrium for the U. S. and German economies. Calibration for the U.S. economy shows that such model variants will produce a higher volatility in employment, and thus fit the data significantly better than the standard model. Although we do not find the same significant improvement for the German economy this does not mean that disequilibrium is not a relevant phenomenon in the German labor market. Instead, it reflects the special feature of the German labor market. Moreover, welfare analysis shows that the model variants with labor market disequilibrium are not necessarily inferior to the RBC benchmark model.

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1 Introduction

The real business cycle (RBC) model has become one of the major approaches in macroeconomics to explain the observed economic fluctuations. Despite its rather simple structure, it is, at least partially, successful in explaining the volatility of some key economic variables such output, consumption and capital stock. However, there are still substantial problems if one wants to employ the RBC model to explain employment fluctuations. It is well known that RBC models generally predict an excessive smoothness of labor effort in contrast to empirical data. Another problem is that the standard RBC model implies a high correlation between consumption and employment while empirical data do not indicate such a correlation.[1]

Both problems are related to the specification of the labor market. The standard RBC model only specifies one side of the market forces and thus one has to assume that the economy is always in equilibrium. For example, the moments of labor effort implied by the model result from the decision rule of the representative agent to supply labor.[2] In our view there is no restriction that one cannot also introduce the other side of the labor market, the demand for labor. Following this route intertemporal models can be enriched by accommodating disequilibrium phenomena. For such a model to effectively replicate empirical macroeconomic moments improvements have to be made upon labor market specifications.

This paper presents a standard intertemporal model augmented by labor market disequilibrium along the line of the above considerations. Attempts have been made that try to introduce non-Walrasian features into the labor market within an intertemporal framework. Most of them are based on the efficiency wage approach, see, for instance, Danthine and Donaldson (1990, 1995) and Uhlig and Xu (1996). Both approaches introduce an explicit labor demand function derived from the marginal product of labor. Our paper owes a substantial debt to this type of work. However, the decision rule with regard to labor supply in the above papers is dropped because the labor effort no longer appears in the utility function. Consequently, the moments of labor effort become purely demand-determined.

However, our above considerations has shown that it is not necessary to drop the decision on labor effort from the utility function.[3] Indeed, the decision rule of labor effort derived from utility maximization via dynamic optimization might be viewed as a natural reflection of the agent's willingness to supply labor. With the determination of labor demand, derived from the marginal product of labor, the two basic forces in the labor market could be formalized.[4]

On the other hand there are models, as in the non-Walrasian tradition, where prices do not move infinitely fast. These are sticky price models of the New Keynesian tradition. Yet markets are cleared when prices are sticky. To explain this stickiness of

[1] We will see this more clearly when we come to the calibration of the model in section IV.

[2] For a study regarding the two decision rules namely the agent's labor supply and consumption demand imperfectly competitive market and sluggish price adjustments, see Rotemberg and Woodford (1995:257), see also King and Wellman (1999).

[3] Another line of recent research on modeling unemployment in RBC models can be found on the work by Merz (1999) who employs search and matching theory to model the labor market in the RBC context. There labor effort is in the utility function.

[4] We want to remark that in current approaches using the efficiency wage theory labor effort might also be determined by other forces such as unemployment and social security, see Uhlig and Xu (1996).

prices monopolistic competition (Rotemberg and Woodford, 1995) or staggered prices are assumed as in Rotemberg and Woodford (1999) and King and Wollman (1999). Yet, in the end these are market clearing models. In the latter case, for example, markets are cleared by fast nominal wage adjustments although prices are sticky, see King and Wollman (1999).

The remainder of the paper is organized as follows. Section 2 will first provide an argument that if shocks are permitted, disequilibrium should occur even in a competitive Arrow-Debreu economy as long as the agents do not have perfect foresight of the shocks. This gives the main reason why one might want to consider the disequilibrium phenomena within an intertemporal framework. Section 3 discusses possible rules when disequilibria occur. Section 4 presents the calibration result of our different model variants for the U.S. economy, including the benchmark RBC model, and studies the welfare implications of the different model variants. Section 5 pursues the same exercise for the German economy. Section 6 concludes the paper. The appendix presents an improved approximation method for the stochastic dynamic optimization problem.

2 A Decentralized Competitive Economy

The RBC-theory assumes a representative agent who solves a resource allocation problem over the infinite time horizon via dynamic optimization. It is argued that "the solutions to planning problems of this type can, under appropriate conditions, be interpreted as predictions about the behavior of market economies."(Stokey and Lucas, 1989, pp. 22) Specifically, if households in the economy are identical, all with the same preference, and if firms are also identical, all producing a common output with the same constant returns to scale technology, the resource allocation problem can be viewed as the problem of maximizing a weighted average of households' utilities and the solution can be regarded as the Pareto optimum for the economy with many agents. This establishes the connection to the competitive equilibrium of the Arrow-Debreu economy. We want to argue that if shocks are permitted there is a strong reason to allow disequilibrium even in the Arrow-Debreu economy as long as we do not assume that the agents have perfect foresight with respect to the shocks.

We develop this argument based on the consideration of a decentralized Arrow-Debreu economy with identical firms and identical households. It is well known that one of the strong assumptions in the Arrow-Debreu economy is about the trading process in the economy. The following citation is again from Stokey and Lucas (1989). For the trading process of a deterministic model with finite time horizon they write:

> Finally, assume that all transactions take place in a single once-and-for-all market that meet in period 0. All trading takes place at that time, so all prices and quantities are determined simultaneously. No further trades are negotiated later. After this market has closed, in periods $t = 0, 1, ..., T$, agents simply deliver the quantities of factors and goods they have contracted to sell and receive those they have contracted to buy. (pp. 23)

Given this assumption of once-for-all markets, one thus can define a sequence of prices (including output price p_t, real wage w_t and rental price of capital r_t), at which the household maximizes utility and the firm maximizes profit over the finite horizon. The solution to these two optimization problems give rise to the two market forces (demand and supply) in output, labor and capital markets. However, before we formalize these two market forces, we shall first specify the ownership relations of this economy. We shall assume that the representative household owns all factors of production and all shares of the firm. Therefore, in each period the household sells factor services to the firm. We shall assume that the revenue from selling factors can only be used to buy the goods produced by the firm either for consuming or accumulating as capital. The representative firm owns nothing. In each period it simply hires capital and labor on a rental basis to produce output, sells the output and transfers any profit back to the household.

2.1 The Decision of the Household

Given the price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ the problem of the household is to choose the sequence of output demand and input supply $\left\{c_t^d, i_t^d, n_t^s, k_t^s\right\}_{t=0}^{\infty}$, that maximizes the discounted utility:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t^d, n_t^s)\right] \tag{1}$$

$$s.t. \;\; p_t(c_t^d + i_t^d) = p_t(r_t k_t^s + w_t n_t^s) + \pi_t \tag{2}$$

$$k_{t+1}^s = (1-\delta)k_t^s + i_t^d \tag{3}$$

Above δ is the depreciation rate; β is the discounted factor; π_t is the expected dividend; c_t^d and i_t^d are the demands for consumption and investment; and n_t^s and k_t^s are the supplies of labor and capital stock. Note that (2) can be regarded as a budget constraint. The equality holds due to the assumption $U_c > 0$. Next, we shall consider how the representative household calculates π_t. It is reasonable to assume that

$$\pi_t = p_t(\widehat{y}_t - w_t\widehat{n}_t - r_t\widehat{k}_t) \tag{4}$$

where $\widehat{y}_t$, $\widehat{n}_t$ and $\widehat{k}_t$ are the realized output, labor and capital expected by the household at given price sequence $\{p_t, w_t, r_t\}$. Thus assuming that the household knows the production function while expecting that the market will be cleared at the given price sequence $\{p_t, w_t, r_t\}$, (4) can be rewritten as

$$\pi_t = p_t\left[F(k_t^s, n_t^s, \widehat{A}_t) - w_t n_t^s - r_t k_t^s\right] \tag{5}$$

Above, $F(\cdot)$ is the production function and $\widehat{A}_t$ is the expected technology shock. We shall temporarily assume that the agent have perfect foresight concerning the shock, i.e., $\widehat{A}_t = A_t$ for all t's, $t = 0, 1, 2, ..., \infty$. Explaining π_t in (2) in terms of (5) and then substituting from (3) to eliminate i_t^d, we obtain

$$k_{t+1}^s = (1-\delta)k_t^s + F(k_t^s, n_t^s, \widehat{A}_t) - c_t^d \tag{6}$$

Note that (1) and (6) represent the standard RBC model, although it only specifies one side of the markets: output demand and input supply. Given the initial capital stock k_0^s, the solution of this model is the sequence of plans $\left\{c_t^d, i_t^d, n_t^s, k_t^s\right\}_{t=0}^{\infty}$, where k_t^s is implied by (6), and

$$c_t^d = G_c(k_t^s, \widehat{A}_t) \tag{7}$$

$$n_t^s = G_n(k_t^s, \widehat{A}_t) \tag{8}$$

$$i_t^d = F(k_t^s, n_t^s, \widehat{A}_t) - c_t^d \tag{9}$$

2.2 The Decision of the Firm

Given the same price sequence $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, and also the sequence of expected technology shocks $\left\{\widehat{A}_t\right\}_{t=0}^{\infty}$, the problem faced by the representative firm is to choose input demands and output supplies $\left\{y_t^s, n_t^d, k_t^d\right\}_{t=0}^{\infty}$ that maximizes the net discounted profit:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t p_t (y_t^s - r_t k_t^d - w_t n_t^d)\right] \tag{10}$$

$$s.t. \qquad y_t^s = F(k_t^d, n_t^d, \widehat{A}_t) \tag{11}$$

However, since the firm simply rents capital and hires labor on a period-by-period basis, its optimization problem is equivalent to a series of one-period maximizations (Stokey and Lucas, 1989, pp. 25). Hence the solutions n_t^d and k_t^d satisfy

$$r_t = F_k(k_t^d, n_t^d, A_t) \tag{12}$$

$$w_t = F_n(k_t^d, n_t^d, A_t) \tag{13}$$

while y_t^s is given by (11).

2.3 Competitive Equilibrium and Disequilibrium

A competitive equilibrium exists and can be described as a sequence of prices $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ at which the two market forces are equalized in all these three markets, i.e., $k_t^d = k_t^s$, $n_t^d = n_t^s$, $c_t^d + i_t^d = y_t^s$, for all t's, $t = 0, 1, 2, ..., \infty$. The economy is at the competitive equilibrium for $\{p_t, w_t, r_t\}_{t=0}^{\infty} = \{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$, which could be achieved via the famous tatonnement process. The solution to the household is the optimization problem, expressed by (6)-(9), represents the sequential realizations in output, labor and capital markets.

However, all these have been discussed upon the assumption that the sequence of technology shocks $\{A_t\}_{t=0}^{\infty}$ are all perfectly foreseen by both the household and the firm. Suppose we do not posit this assumption. Two possibilities can be considered. First, when the market is opened at the beginning of period 0, A_t is considered to be a random variable with a certain distribution. Then the quantities demanded and supplied should also be considered as random variables with certain distributions. The plans are not a sequence of numbers, but a sequence of contingency plans. Although this is a standard treatment of shocks, problems can arise if prices are considered. Indeed, in this stochastic case, the equilibrium price $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ can not be determined. The auctioneer can not find a way to adjust the prices because the quantities demanded and supplied are all random variables with certain distributions. It should be noted that the standard contingency treatment does not pose a problem for the standard non-decentralized RBC model as formulated by equation (1) and (6) because in that model prices are not introduced.

The second possibility is that when the market is open at the beginning of period 0, both agents, the firm and the household, have a point expectation $\left\{\widehat{A}_t\right\}_{t=0}^{\infty}$ on $\{A_t\}_{t=0}^{\infty}$. Thus, the plan can be interpreted as a sequence of numbers of quantities demanded and supplied. The equilibrium price can be determined and the tatonnement process can work out to ensure that prices are adjusted to the equilibrium corresponding to the point expection of the shocks $\left\{\widehat{A}_t\right\}_{t=0}^{\infty}$. However, the equilibrium prices $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ set at the beginning of period 0 are not necessarily the prices at which the demand and supply in period t can be equalized given that the point expectations are not fulfilled. This concludes our argument that disequilibrium should be allowed even in the Arrow-Debreu economy as long as we permit shocks and do not assume perfect foresight with respect to the shocks.

3 Disequilibrium Rules

Given the price sequence $\{p_t^*, w_t^*, r_t^*\}_{t=0}^{\infty}$ and given the point expectation of technology shocks $\left\{\widehat{A}_t\right\}_{t=0}^{\infty}$, we have argued that disequilibrium could occur as long as the agents do not perfectly foresee the shocks. When the agents find that A_t deviates from $\widehat{A}_t$, they have incentive to change their plans. Further, even if they do not want to change plans, due to their responsibility as a contractor, they still can not fulfil their plans. Therefore, adjustments have to be made either in terms of demands or supplies. This section discusses possible rules when disequilibria occur.

Suppose now that at the beginning of t, both the household and the firm find that the realization A_t deviates from its expectation, $\widehat{A}_t$, although in all previous periods the expectations have been successfully fulfilled. Given this deviation their first response is to change their willingness to demand and supply. Accordingly, we have for the household:

$$c_t^d = G_c(k_t^s, A_t) \tag{14}$$

$$n_t^s = G_n(k_t^s, A_t) \tag{15}$$

$$k_t^s = (1-\delta)k_{t-1} + F(k_{t-1}, n_{t-1}, A_{t-1}) - c_{t-1} \tag{16}$$

$$i_t^d = F(k_t^s, n_t^s, A_t) - c_t^d \tag{17}$$

and for the firm:

$$r_t^* = F_k(k_t^d, n_t^d, A_t) \tag{18}$$

$$w_t^* = F_n(k_t^d, n_t^d, A_t) \tag{19}$$

$$y_t^s = F(k_t^d, n_t^d, A_t) \tag{20}$$

Note that such adjusted plans are consistent with the so-called contingent plans and therefore they are the optimum plans at the new environment. However, there is no guarantee that the markets will still be cleared at the price sequence $\{p_t^*, w_t^*, r_t^*\}$ due to the deviation of A_t from $\widehat{A}_t$. Then what could be considered proper rules in these three markets to deal with the possible disequilibria? Note that k_t^s is determined by the realizations in the last period and hence it is given at the beginning of period t. It is, therefore, natural to think of the capital transaction as being the first transaction that needs to be carried out. The following is the first rule when disequilibrium has occurred:

Rule 1: The capital realized is equal to the capital supplied

$$k_t = k_t^s. \tag{21}$$

First, k_t^s is the maximum of available capital in period t. Therefore k_t has to be equal to k_t^s if the demand is larger than the supply. Second, k_t^s is not a determinant in the utility function of the household. This indicates that there is no welfare loss to the household even if it gives the excess supply to the firm as a free gift. These two properties[5] provide a strong reason to assume $k_t = k_t^s$.

The second transaction that needs to be carried out is with respect to employment. In this case, neither of the two properties of capital that we have just discussed can be applied. Therefore, we could consider two possible rules for the labor market:

Rule 2: When disequilibrium occurs in the labor market either one of the following rules will be applied

$$n_t = \min(n_t^d, n_t^s) \tag{22}$$

$$n_t = \omega n_t^d + (1-\omega) n_t^s \tag{23}$$

where $\omega \in (0,1)$.

Above, the first is the famous short-side rule when disequilibrium occurs.[6] It has been widely used in the literature for disequilibrium analysis (see for instance, Benassy 1984, among others). The second might be called compromising rule. The rule indicates that when disequilibrium exits in the labor market both firms and workers have to compromise. If there is excess supply, firms will employ more than what they wish to employ.[7] On the other hand, when there is excess demand, workers will have to offer

[5]Another way to bring into balance capital demanded and supplied would be introducing capital utilization rates as suggested by Burnside et al. (1993).

[6]One also may allow for wage reactions to disequilibria on the labor market. We have studied this case in a preliminary way. Yet, this approach still poses serious problems in an intertemporal approach (such as reopening of markets and renegotiations)

[7]This could also be realized by firms by demanding the same (or less) hours per worker but employing more workers than being optimal. This case corresponds to what is discussed in the literature as labor boarding where firms hesitate to fire workers during a recession because it may be hard to find new workers in the next upswing, see Burnside et al. (1993). Note that in this case firms may be off their marginal product curve and thus this might require wage subsidies for firms as has been suggested by Phelps (1998).

more labor effort than they wish to offer.[8] Such mutual compromises may be due to institutional structures and moral standards of the society.[9]

After the transactions in these two factor markets have been carried out, the firm will engage in its production activity. The result is the output supply, which, instead of (20), is now given by

$$y_t^s = F(k_t, n_t, A_t) \tag{24}$$

Then the transaction needs to be carried out with respect to y_t^s. In what follows, we shall assume the following rule for output:

Rule 3: The output realized is equal to the output supplied

$$c_t + i_t = y_t^s. \tag{25}$$

When the demand $c_t^d + i_t^d$ is larger than the supply y_t^s, the short side rule warrant that the realization should be equal to the supply, since it is the maximum available supply in that period t. On the other hand, when the demand is less than the supply there remains some output which can not be sold. However, according to the ownership relationship we have defined before the household owns the firm. Therefore it also owns those unsold products. There is no reason why the household will not utilize them either for consumption or for investment when it owns those products.

After the households obtain all the products supplied, it will distribute them between consumption and investment. The following is the rule for this distribution

Rule 4: the consumption realized is equal to the consumption demanded

$$c_t = c_t^d \tag{26}$$

This rule reflects only a matter that results from the household's decision. The rule may not hold if the supply of the output is so small that it is even less than the demand for consumption c_t^d. To simplify our analysis, we shall assume that such situation will not occur for all t's.

Given those above rules and equation (24) instead of (20), we thus complete our disequilibrium model by including the following:

$$c_t = G_c(k_t, A_t) \tag{27}$$

$$n_t^s = G_n(k_t, A_t) \tag{28}$$

$$w_t^* = F_n(k_t, n_t^d, A_t) \tag{29}$$

$$k_{t+1} = (1-\delta)k_t + F(k_t, n_t, A_t) - c_t \tag{30}$$

where n_t is given by either (22) or (23) and w_t^* is the given wage rate, which is assumed to be set at the beginning of period 0

[8]This could be achieved by employing the same number workers but each worker supplying more hours (varying shift length and overtime work); for a more formal treatment of this point, see Burnside et al. (1993).

[9]Note that if firms are off their supply schedule and workers off their demand schedule, a proper study would have to compute the firms cost increase and profit loss and the workers' welfare loss. If, however, the marginal cost for firms is rather flat (as empirical literature has argued, see Blanchard and Fischer, 1989) and the marginal disutility is also rather flat the overall loss may not be so high. A proper welfare analysis is given in section 4.

4 Calibration for the U.S. Economy

This section provides a calibration of different model variants. We consider three models: the standard RBC-model as the benchmark for comparison and two labor market disequilibrium models with the rules as expressed in (22) and (23) respectively. Specifically, we shall call the benchmark model the Model I; the disequilibrium model with short side rule (22) the Model II; and the disequilibrium model with the compromising rule (23) the Model III.

4.1 The Data Generating Processes

4.1.1 The Benchmark Model

The benchmark RBC model we employ here is the model by King et al. (1988). It includes two state equations:

$$A_{t+1} = a_0 + a_1 A_t + \varepsilon_{t+1} \tag{31}$$

$$K_{t+1} = (1-\delta)K_t + A_t K_t^{1-\alpha}(N_t X_t)^{\alpha} - C_t \tag{32}$$

where K_t is the capital stock, N_t per capita hours worked, A_t the temporary technology shock, and X_t the permanent shock that follows a growth rate γ. We remark that X_t includes both population and productivity growth, while ε_t is a typical *i.i.d.* innovation with standard deviation denoted by σ_ε. The model is nonstationary due to X_t. To transform the model into a stationary version we divide both sides of equation (2) by X_t:

$$k_{t+1} = \frac{1}{1+\gamma}\left[(1-\delta)k_t + A_t k_t^{1-\alpha}(n_t\ \bar{N}\ /0.3)^{\alpha} - c_t\right] \tag{33}$$

where $k_t = K_t/X_t, c_t = C_t/X_t$ and $n_t = 0.3N_t/\ \bar{N}$ with $\bar{N}$ to be the sample mean of N_t. Note that n_t is often regarded to be the normalized hours. The sample mean of n_t is equal to 30 %, which, as pointed out by Hansen (1985), is the average percentage of hours attributed to work. The objective function takes the form

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\log c_t + \theta \log(1-n_t)\right]\right] \tag{34}$$

Using an approximation method as discussed in Appendix I and in Gong (1998)[10], we solve the model, which gives rise to two linear decision rules:

$$c_t = G_{11}A_t + G_{12}k_t + g_1 \tag{35}$$

$$n_t = G_{21}A_t + G_{22}k_t + g_2 \tag{36}$$

The coefficients G_{ij} and $g_i (i = 1,\ 2$ and $j = 1, 2)$ are all complicated functions of the model's underlying parameters, α, β, δ, θ and γ. Given these parameters and the parameters in equation (31), including σ_ε, equations (31), (33), (35) and (36) can

[10]Our method is an improved approximation method as compared to Chow (1991, 1993).

be employed to generate stochastically simulated data. Those can then be used to compare the sample moments to the moments of the observed economy.

Obviously, (33) and (34) are similar to (1) and (6) while (35) and (36) are the linear approximations to (7) and (8). Therefore the benchmark model does not allow for labor market disequilibrium. The moments of the labor effort are solely reflected by the decision rule (36). Since this decision rule is quite similar in its structure to the other decision rule given by (35), i.e., they are both determined by k_t and A_t, one thus can expect that the volatility of labor effort can not be much different from the volatility of consumption, which generally appears to be smooth. However, they are likely to be highly correlated.

4.1.2 The Disequilibrium Models

Next we modify the benchmark model to obtain a labor market disequilibrium. For this purpose, we shall first write (36) as

$$n_t^s = G_{21}A_t + G_{22}k_t + g_2 \tag{37}$$

We need to derive the labor demand from the given production function $F(\cdot) = A_t K_t^{1-\alpha}(N_t X_t)^{\alpha}$. Let $X_t = Z_t L_t$, with Z_t to be the permanent shock resulting purely from productivity growth, and L_t from population growth. We shall assume that L_t has a constant growth rate μ and hence Z_t follows the growth rate $(\gamma - \mu)$. The production function can be written as $Y_t = A_t Z_t^{\alpha} K_t^{1-\alpha} H_t^{\alpha}$, where H_t equals $N_t L_t$ and can be regarded as total labor hours. Taking the partial derivative with respect to H_t and recognizing that the marginal product of labor is equal to the wage, we thus obtain

$$w_t^* = \alpha A_t Z_t k_t^{1-\alpha}(n_t^d \overline{N}_t/0.3)^{\alpha-1} \tag{38}$$

This equation is equivalent to (29). It generates the demand for labor as

$$n_t^d = (\alpha A_t Z_t / w_t^*)^{1/(1-\alpha)} k_t (0.3/\ \bar{N}). \tag{39}$$

Note that the per capita hours demanded n_t^d should be stationary if the real wage w_t^* and productivity Z_t grows at the same rate. This seems to be consistent with the U. S. experience that we will calibrate.

Thus, for the disequilibrium model with short side rule, Model II, the data generating process includes (22), (31), (33), (35), (37) and (39), with w_t^* given by the observed wage rate. We thereby do not attempt to give the actually observed sequence of wages a further theoretical foundation.[11] For our purpose it suffices to take the empirically observed series of wages. For Model III, we use (23) instead of (22).

4.2 Parameter Estimation

Before we calibrate the models we shall first specify the parameters. There are altogether 10 parameters in our three variants: $a_0, a_1, \sigma_\varepsilon, \gamma, \mu, \alpha, \beta, \delta, \theta$, and ω. We first specify α and γ respectively at 0.58 and 0.0045, which are standard. This allows us to

[11] One might apply here the efficiency wage theory or other theories that justify the wage rigidities over the business cycle.

generate the data series of the temporary shock A_t. With this data series, we estimate the parameters a_0, a_1 and σ_ε. We specify μ at 0.001, which is close to the average growth rate of the labor force in U.S. The next three parameters β, δ and θ are estimated with the GMM method by matching the moments of the standard RBC model generated by (33), (35) and (36). All these parameters are again used in the other model variants. The new parameter ω in Model III is estimated by minimizing the residual sum of square between actual employment and the model generated employment. The estimation by the GMM method is undertaken by a global optimization algorithm, called simulated annealing.[12] The estimation of ω are executed by a conventional algorithm of grid search. Table 1 illustrates these parameters:

Table 1: Parameters Used for Calibration

a_0	0.0333	σ_ε	0.0185	μ	0.0010	β	0.9930	θ	2.0189
a_1	0.9811	γ	0.0045	α	0.5800	δ	0.2080	ω	0.2465

4.3 The Data

The empirical studies of RBC models often require redefining the existing macroeconomic data accommodated to the definition of the variables as defined in the models. For example, it is suggested that not only private investment but also government investment and durable consumption increase the capital stock K_t . Consequently, the service generated from durable consumption goods and government capital stock should also appear in the definition of Y_t[13]. Since such data are not readily available one has to compute them based on some assumptions.

In this paper, we will use the data set constructed by Christiano. This data set has been used in Christiano (1988) and Christiano and Eichenbaum (1992)[14]. We are thus able to compare our estimation and calibration results with their papers. The wage series are obtained from Citibase. It is re-scaled to match the model's implication.[15]

4.4 Calibration Results

Table 2 provides our calibration results from 5000 stochastic simulations. The results in this table are confirmed by Figure 1, where a one time simulation with the observed innovation A_t are presented. All time series are detrended by the HP-filter. In Table 3, we further provide the means and variances of the residuals based on those one time simulations.

[12] For this estimation, we refer the reader to our technical paper, see Semmler and Gong (1997).

[13] For a discussion on data definitions in RBC models, see Cooley and Prescott (1995).

[14] We would like to thank them for making available to us their data set.

[15] Note that this re-scaling is necessary because we do not exactly know the initial condition of Z_t, which we set in this paper to be 1. We re-scaled the wage series in such a way that the first observation of employment is equal to the demand for labor as specified by equation (39).

Table 2: Calibration of the Model Variants
(numbers in parentheses are the corresponding standard deviations)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0081	0.0035	0.0165	0.0156
Model I Economy	0.0091	0.0036	0.0051	0.0158
	(0.0012)	(0.0007)	(0.0006)	(0.0021)
Model II Economy	0.0101	0.0024	0.0102	0.0210
	(0.0005)	(0.0002)	(0.0018)	(0.0014)
Model III Economy	0.0093	0.0047	0.0155	0.0210
	(0.0014)	(0.0010)	(0.0023)	(0.0028)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.1741	1.0000		
Employment	0.4604	0.2861	1.0000	
Output	0.7550	0.0954	0.7263	1.0000
Model I Economy				
Consumption	1.0000			
	(0.0000)			
Capital Stock	0.2043	1.0000		
	(0.1190)	(0.0000)		
Employment	0.9288	−0.1593	1.0000	
	(0.0203)	(0.0906)	(0.0000)	
Output	0.9866	0.0566	0.9754	1.0000
	(0.00332)	(0.1044)	(0.0076)	(0.0000)
Model II Economy				
Consumption	1.0000			
	(0.0000)			
Capital Stock	−0.0201	1.0000		
	(0.0556)	(0.0000)		
Employment	0.5533	0.2486	1.0000	
	(0.0369)	(0.0326)	(0.0000)	
Output	0.9348	0.1666	0.8064	1.0000
	(0.0100)	(0.0370)	(0.0226)	(0.0000)
Model III Economy				
Consumption	1.0000			
	(0.0000)			
Capital Stock	0.2280	1.0000		
	(0.1153)	(0.0000)		
Employment	0.7504	−0.0124	1.0000	
	(0.0687)	(0.0919)	(0.0000)	
Output	0.9411	0.0475	0.9188	1.0000
	(0.0239)	(0.0856)	(0.0235)	(0.0000)

Among our four key variables in the Sample Economy consumption and capital stock are stable variables while output and employment are relatively more volatile. However, these properties are not matched in the Simulated Model I Economy, the benchmark RBC model, where employment is excessively smooth.[16] The problem has been partially resolved in our Simulated Model II Economy and satisfactorily resolved in the Simulated Model III Economy representing the compromising rule. As we can see from the figures, the volatility of unemployment has been greatly increased for both Model II and Model III. In particular, the volatility in Model III Economy is quite close to the one in the Sample Economy. Indeed, if we look at Table 3, both residual means and variances from Model III are smallest for all the series except for consumption in terms of its residual mean.[17] We therefore might conclude that Model III is the best in terms of matching volatility.

Table 3: Comparison of Residual Means and Residual Variance

	Consumption	Capital	Employment	Output
Residual Means				
Model I Economy	2.14e-05	-2.39e-05	4.35e-05	2.14e-05
Model II Economy	1.59e-05	-2.00e-05	2.57e-05	1.87e-05
Model III Economy	2.07e-05	-1.99e-05	-4.78e-06	-1.39e-05
Residual Variances				
Model I Economy	3.81e-05	9.78e-06	0.00028	0.00010
Model II Economy	3.99e-05	6.66e-06	0.00030	0.00010
Model III Economy	3.51e-05	5.87e-06	0.00013	4.51e-05

Now let us look at the correlations. In the Sample Economy, there are basically two significant correlations. One is between consumption and output, and the other is between employment and output. Both of these correlations have also been found in all our simulated economies. However, in addition to these two correlations, consumption and employment in the Model I Economy are also significantly correlated. This is not surprising given that movements of employment as well as consumption reflect the movement in the state variables capital stock and the temporary shock. They, therefore, should be somewhat correlated. We remark here that such an excessive correlation has, to our knowledge, not been discussed in the RBC literature. Discussions have often focused on the correlation with output.

[16] This problem has been addressed in many recent papers, see for example Christiano and Eichenbaum (1992) and Gali (1999)

[17] Note that here the residual is defined to be the simulated series minus the observed series

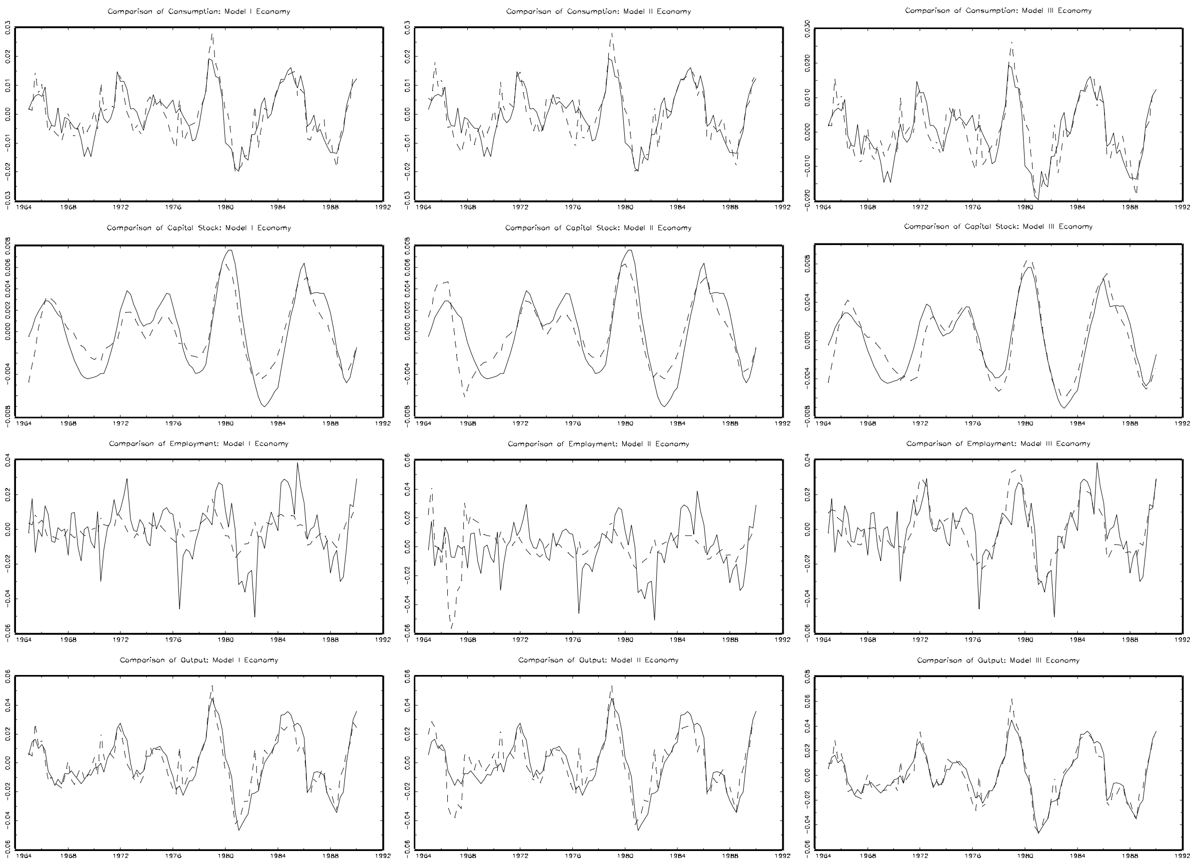

Figure 1: Simulated Economy versus Sample Economy: U.S. Case (solid line for sample economy, dotted line for simulated economy)

However, in our disequilibrium models, especially in Model II, employment is no longer significantly correlated with consumption. Apparently, this is because we have distinguished the demand and supply of labor, only the latter reflects the capital stock and the temporary shock (via one of the decision rules), and hence is expected to be correlated with consumption. Since actual employment is not necessarily the same as labor supply the correlation with consumption is no longer significant.

4.5 Welfare Comparison of the Model Variants

Next we want to undertake a welfare comparison of our different model variants. A likely conjecture is that the benchmark model should always be superior to the other two variants because the decisions on labor supply - which are optimal for the representative agent - are realized in all periods. On the other hand the compromising model should be the worst because in no period the optimal decisions on labor supply are realized.

However, we believe that this may not generically be the case. The point here is that the model specification is somewhat different due to the distinction between expected and actual moments with respect to our state variable, the capital stock. In the disequilibrium models, the representative agent may not rationally expect the moments of the sequence of the capital stock. The expected moments are represented by

$$k_{t+1} = (1-\delta)k_t + F(k_t, n_t^s, A_t) - c_t, \tag{40}$$

while the actual moments are expressed by (30). They are not necessarily equal unless n_t in (30) is equal to n_t^s in (40). Also, in addition to A_t, there is another external variable w_t^*, entering into the models, which will affect the labor employed (via demand for labor) and hence eventually impact the welfare performance. The welfare result due to these changes in the specification may therefore deviate from what one would expect.

Our exercise here is to compute the values of the objective function for all our three models, given the sequence of our two decision variables, consumption and employment. Note that for our disequilibrium model variants we use realized employment rather than the decisions on labor supply to compute the utility function. Specifically, we compute V, where

$$V \equiv \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \theta \log(1-n_t)]$$

The exercise here is conducted for different initial conditions of k_t denoted by k_0. We choose the different k_0's based on the grid search around the steady state of k_t. Obviously, the value of V for any given k_0 will also depend on the external variable A_t and w_t^* (though in the benchmark model, only A_t appears). We consider two different ways to treat these external variables. One is to set both external variables at their steady state levels for all t. The other is to employ their observed series entering into the computation. Figure 2 provides the welfare comparison of our model variants.

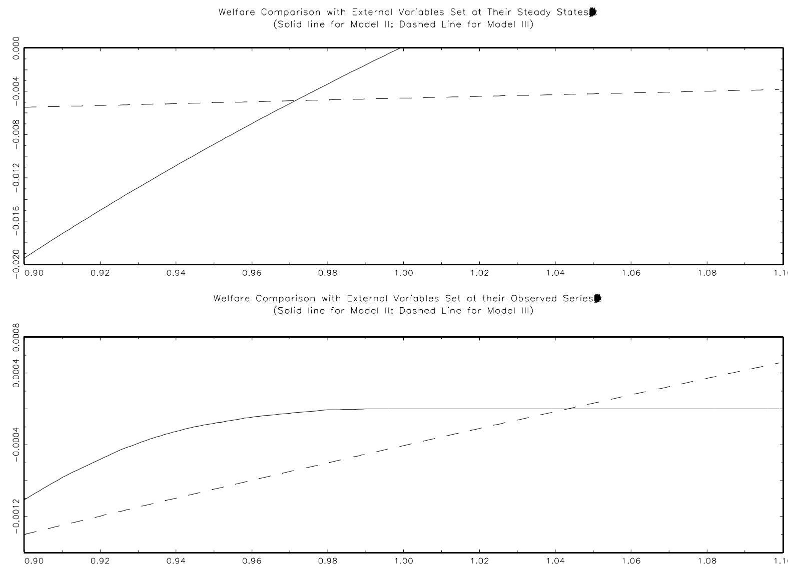

Figure 2: Welfare Comparison: U.S. Case

In Figure 2, the percentage deviations of V from the corresponding values of benchmark model are plotted for both Model II and Model III given the various k_0 around the steady states. The various k_0's are expressed in terms of deviation from percentage the steady state of k_t. It is not surprising to find that in most cases the benchmark model is the best in its welfare performance since most of the values are negative. However, it is important to note that the deviations from the benchmark model are very small. Meanwhile, not always is the benchmark model the best one. When k_0 is sufficiently high, close to or higher than the steady state of k_t, the deviations become 0 for the Model II, when the external variables are set at their steady states. Furthermore, in the case of using observed external variables, the Model III will be superior in its welfare performance when k_0 is larger than its steady state.

5 Calibration for the German Economy

We have studied the labor market disequilibrium in the U. S. economy. We have seen that one of the major reasons that the standard model can not appropriately replicate the labor market behavior is its lack of introducing the demand for labor. Next, we pursue a similar study of the German economy. For this purpose we shall discuss the data sources for the study on the German economy and summarize some stylized facts on the German economy compared to the U.S. economy.

5.1 The Data

In order to estimate parameters for the German economy and to undertake the calibration we use time series data from 1960.1 to 1992.1. We thus have included a short period after the unification of Germany (1990-1991). This way we can observe what outliers the unification might have created. We use again quarterly data. The time series data on GDP, consumption, investment and capital stock are OECD data, see OECD (1998a), the data on total labor force is from OECD (1998b). The time series data on total working hours is taken from Statistisches Bundesamt (1998). The time series on the hourly real wage index is from OECD (1998a).

5.2 Stylized Facts

Next, we want to compare some stylized facts. Figure 3 and 4 compare 6 key variables in the models for both German and U.S. economies. In particular, the data in Figure 4 are detrended by the HP-filter. The standard deviations of the detrended series are summarized in Table 4.

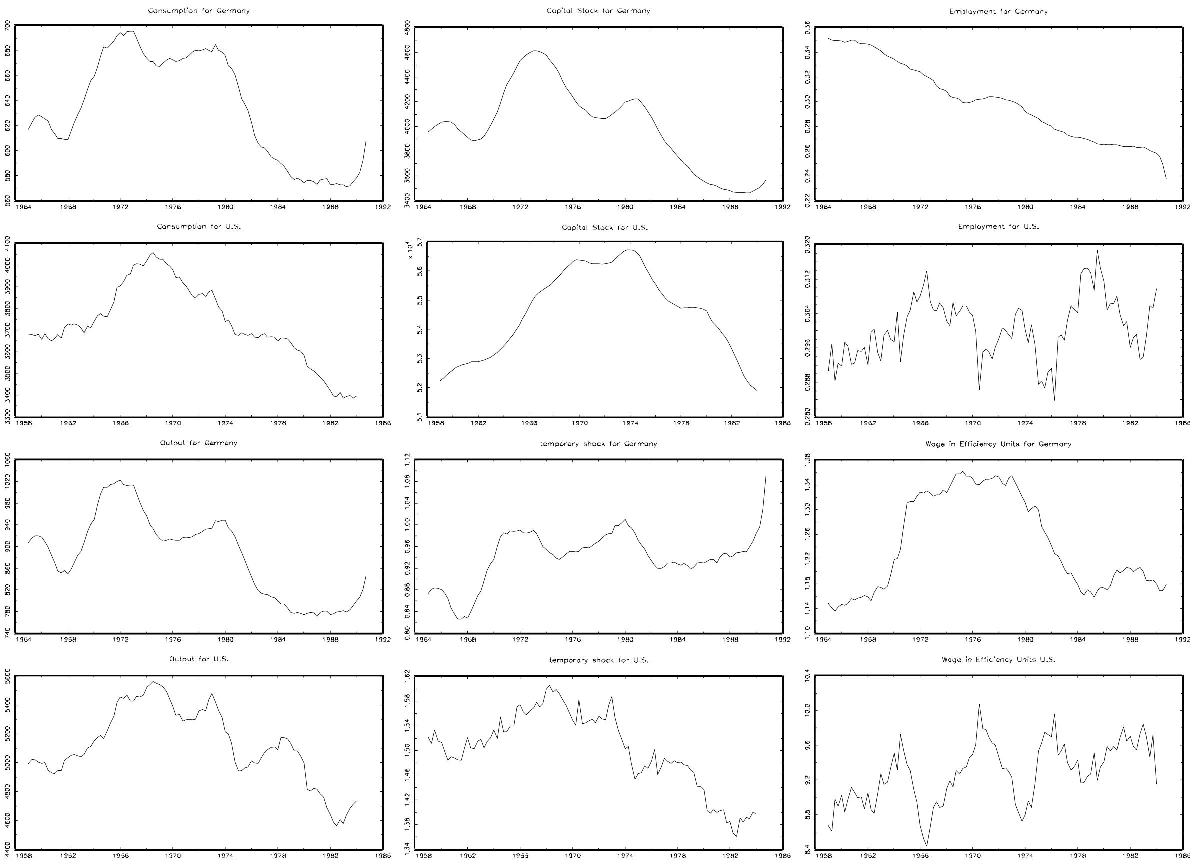

Figure 3: Comparison of Macroeconomic Variables U. S. versus Germany

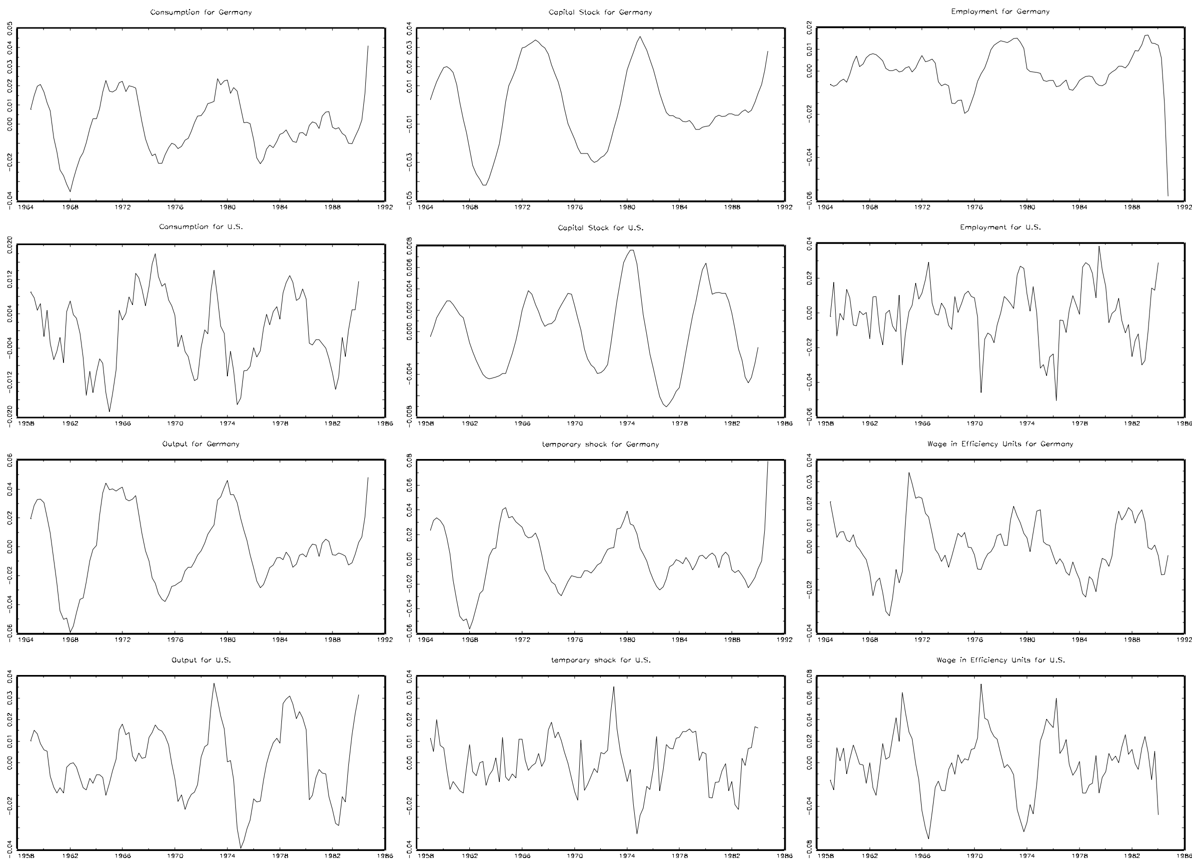

Figure 4: Comparison of Macroeconomic Variables: U. S. versus Germany (data series are detrended by the HP-filter)

Table 4: The Standard Deviations (U.S. versus Germany)

	$(detrended) Germany$	$(detrended) U.S.$
consumption	0.0146	0.0084
capital stock	0.0203	0.0036
employment	0.0100	0.0166
output	0.0258	0.0164
temporary shock	0.0230	0.0115
efficiency wage	0.0129	0.0273

Note that above we define the efficiency wage as the wage in efficiency units, thus as $\frac{w_t}{Z_t}$.

Several remarks should be provided here. First, employment and efficiency wage are among the variables with the highest volatility in the U. S. economy. However, in the German economy they are the smoothest variables. Second, the employment (measured in terms of per capita hours) are declining over time in Germany (see Figure 3 for the non-detrended series), while in the U.S. economy, the series is approximately stationary. Third, in the U. S. economy, the capital stock and temporary shock to technology are both relatively smooth. In contrast, they are both more volatile in Germany. These results might be due to our first remark regarding the difference in employment volatility. The volatility of output must be absorbed by some factors in the production function. If employment is smooth, the other two factors have to be volatile.

Should we expect such differences to lead to different calibration results of our model variants? This is explored next.

5.3 Parameters for Calibration

For the German economy, our investigation showed that an AR(1) process does not match well the observed process of A_t. Instead, we shall use an AR(2) process:

$$A_{t+1} = a_0 + a_1 A_t + a_2 A_{t-1} + \varepsilon_{t+1}$$

The parameters used for calibration are given in Table 5. All of these parameters are estimated in the same way as those for the U.S.

Table 5: Parameters used for Calibration (German Economy)

a_0	0.0044	γ	0.0083	δ	0.0538
a_1	1.8880	μ	0.0019	θ	2.1507
a_2	-0.8920	α	0.6600	ω	0.0578
σ_ε	0.0071	β	0.9876		

5.4 Calibration Results

As for the U.S. economy we provide in Table 6 for the German economy the calibration result from 5000 time stochastic simulations. In Figure 5 we again compare the one-time simulation with the observed A_t for our model variants. Note that here all time series are detrended by the HP-filter. In Table 7, we further present the means and variances of the residuals obtained from these one-time simulation.

Table 6: Calibration of the Model Variants
(number in parentheses are the corresponding standard deviation)

	Consumption	Capital	Employment	Output
Standard Deviations				
Sample Economy	0.0146	0.0203	0.0100	0.0258
Model I Economy	0.0292	0.0241	0.0107	0.0397
	(0.01066)	(0.00668)	(0.00235)	(0.01127)
Model II Economy	0.0242	0.0323	0.0403	0.0503
	(0.00061)	(0.00178)	(0.00337)	(0.00228)
Model III Economy	0.0228	0.0169	0.0119	0.0345
	(0.00050)	(0.00039)	(0.00032)	(0.00082)
Correlation Coefficients				
Sample Economy				
Consumption	1.0000			
Capital Stock	0.4360	1.0000		
Employment	0.0039	-0.3002	1.0000	
Output	0.9692	0.5423	0.0202	1.0000
Model I Economy				
Consumption	1.0000			
	(0.00000)			
Capital Stock	0.7208	1.0000		
	(0.0920)	(0.00000)		
Employment	0.5138	−0.1842	1.0000	
	(0.16403)	(0.13099)	(0.00000)	
Output	0.9473	0.4855	0.7496	1.0000
	(0.02000)	(0.10999)	(0.10283)	(0.00000)
Model II Economy				
Consumption	1.0000			
	(0.00000)			
Capital Stock	0.6232	1.0000		
	(0.02357)	(0.00000)		
Employment	0.7447	0.2445	1.0000	
	(0.01914)	(0.02355)	(0.00000)	
Output	0.9045	0.3332	0.9418	1.0000
	(0.00481)	(0.01546)	(0.00817)	(0.00000)
Model III Economy				
Consumption	1.0000			
	(0.00000)			
Capital Stock	0.6585	1.0000		
	(0.00867)	(0.00000)		
Employment	0.7679	0.0536	1.0000	
	(0.00498)	(0.00947)	(0.00000)	
Output	0.9615	0.4322	0.9120	1.0000
	(0.00059)	(0.01049)	(0.00272)	(0.00000)

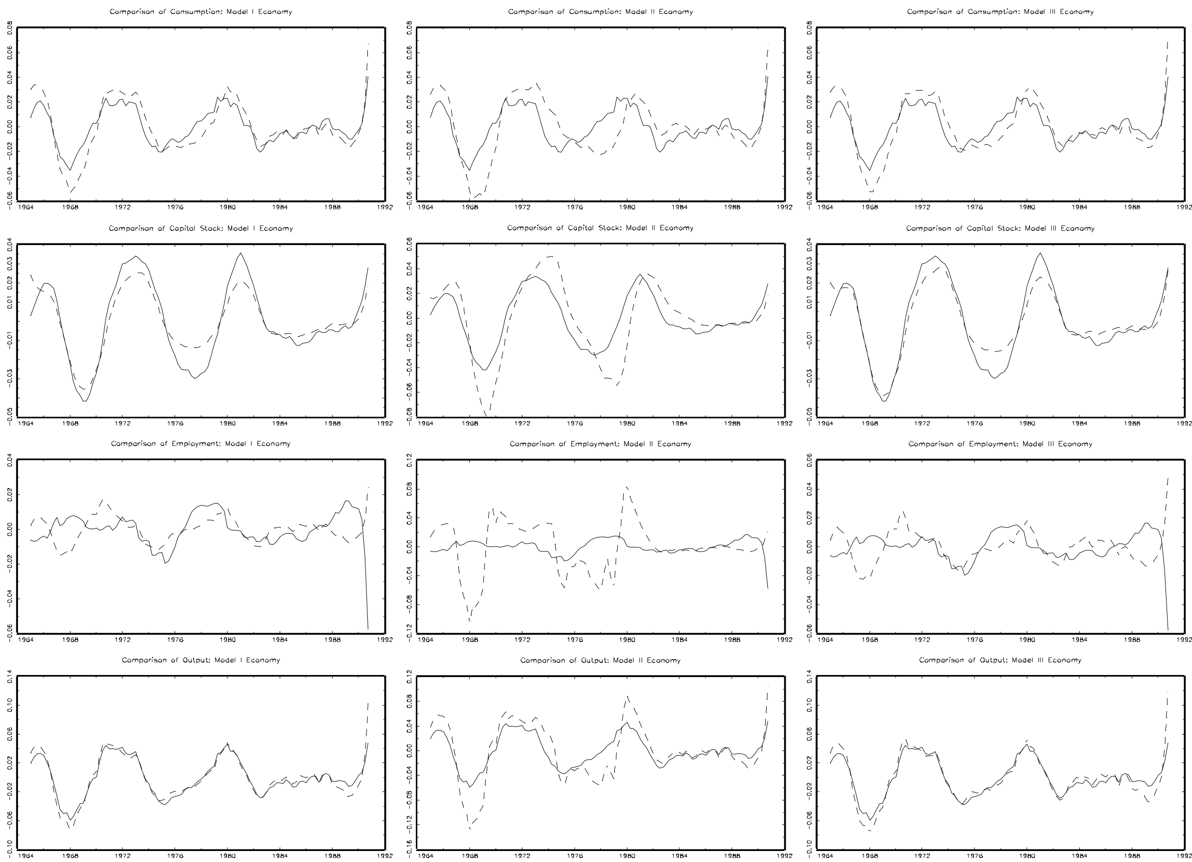

Figure 5: Simulated Economy versus Sample Economy: German Case (solid line for sample economy, dotted line for simulated economy)

Table 7: Comparison of Residual Means and Residual Variance

	Consumption	Capital	Employment	Output
Residual Means				
Model I Economy	-0.00010	6.88e-005	2.04e-005	-2.63e-005
Model II Economy	-0.00018	-0.00042	-0.00044	-0.00052
Model III Economy	-0.00010	5.91e-005	3.52e-006	-7.42e-005
Residual Variances				
Model I Economy	9.80e-005	5.29e-005	0.00016	7.67e-005
Model II Economy	0.00027	0.00032	0.00153	0.00068
Model III Economy	0.00011	3.75e-005	0.00028	0.00013

In comparison with the U. S. economy we can find some differences. First, the standard problem of excessive smoothness with respect to employment in the benchmark model no longer holds for the German economy. This might be due to the fact that employment itself is smooth in the German economy (see Table 4 and Figure 4). Second, in terms of predictive power there is no significant improvement in the disequilibrium models, Model II and Model III, over the equilibrium Model, Model I. In particular, the performance of the Model II economy is worse with respect to employment variation. On the other hand, there is no significant difference between Model I and Model III economies. The latter is certainly due to the small weighting parameter which is close to zero. Third, if we look at the labor demand and supply in Figure 6, we may find a real puzzle: the supply of labor is almost always the short side in the Germany economy whereas in the U.S. economy demand is dominating in most periods. This seems to be in contrast to the empirical evidence that unemployment in Germany is more severe than in U.S.

However, such differences should not lead us to conclude that a disequilibrium model is not a valid description of the German economy. Instead, we shall argue that all of the above results may be reasonably explained by the special features of the German labor market. In most labor market studies[18], the German labor market is often considered less flexible than the U.S. labor market. In particular, there are stronger influences of labor unions and various legal restrictions on firms' hiring and firing decisions. Such influences and legal restriction – which may also be viewed as a readiness to compromise as our Model III suggests – will give rise to the smoother employment series in contrast to the U.S.

The above established third result does not mean that there is always excess demand for labor in Germany and hence unemployment is not more severe. Yet, it reflects the dominance of the currently employed labor – who are often represented by labor unions and protected by legal restrictions – in the labor market. Note that here we must distinguish the supply that is actually provided in the labor market and the "supply" that is specified by the decision rule in the benchmark model. Due to the intertemporal optimization subject to the budget constraints it might reasonably be argued that the supply specified by the decision rule may only approximate the decisions from those households for which unvoluntary unemployment is not expected to pose a problem on their budgets. Such households are more likely to be currently employed and protected by labor unions and legal restrictions. In other words, the currently employed labor decides, through the optimal decision rule, about labor supply and not those who are currently unemployed. Although this might not be a very satisfying interpretation of the reduced labor supplied in comparison with demand, this is the only interpretation that our representative agent framework allows for. This difficulty could presumably be overcome by an intertemporal model with heterogenous households.[19]

[18]See, for example, Nickell (1997).

[19]See, for example, Uhlig and Xu (1996).

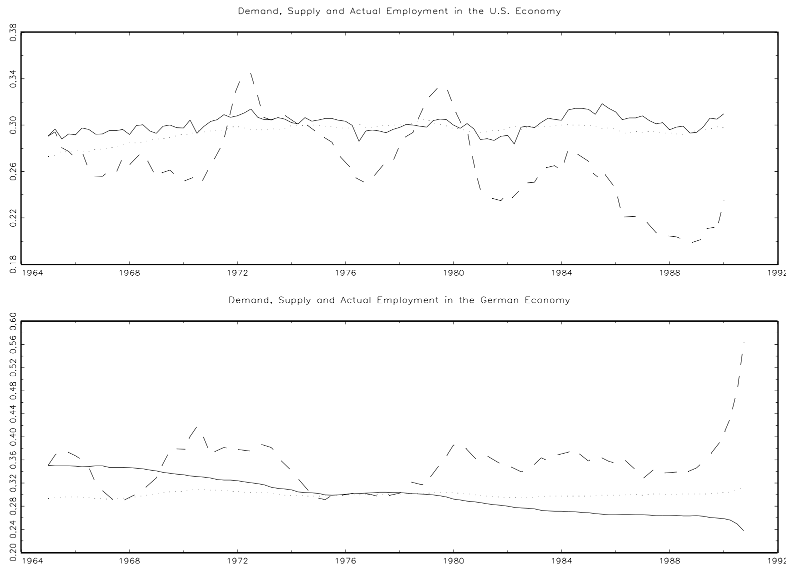

Figure 6: Comparison of demand and supply in the labor market (solid line for demand and dotted line for supply)

5.5 Welfare Comparison of the Model Variants

As in the case of U.S. economy we shall now compute the value of our welfare function for our model variants. In Figure 7, the percentage deviation of these values from the benchmark model are plotted for both Model II and Model III. As in the case of U.S., the disequilibrium model with short side rule can not be better than the benchmark model for all initial conditions of k_0 whether the external variable are set at their steady states or at the observed series. However, as in the case of the U.S. economy, it does not seem to be ruled out that the disequilibrium model with the compromising rule could exhibit a higher welfare. Nevertheless, the improvement we observe here is only marginal. Indeed, we are not able to see the difference if we look at Panel A and C where the percentage deviation for both Model II and III are plotted together. This is certainly due to the small weighting parameter so that the Model I and III are approximately the same.

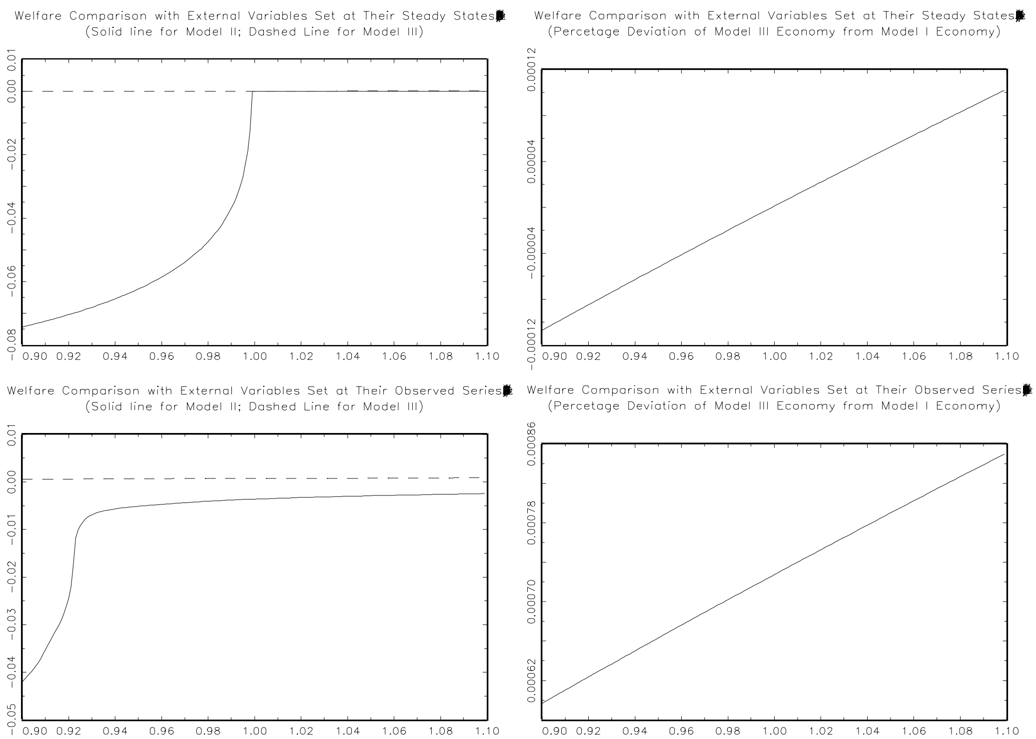

Figure 7: Welfare Comparison: German Case

6 Conclusion

The Real Business Cycle model is often presented intertemporal general equilibrium model that can predict the behavior of market economies. This is basically because the model often specifies only one side of market forces reflected by the model's decision rule. However, there is no restriction to do so, in fact, there is a strong temptation to introduce the other side of the market as well. With such considerations intertemporal models could be enriched by accommodating wide spread disequilibrium phenomena. In this paper a simple example has been provided that refers to the labor market disequilibrium and a comparison has been conducted for the U.S. and German economies. Calibration shows that the model's predictive power is improved compared to the benchmark model for the U.S. economy. However, for the German economy, no significant improvement over the benchmark model can be found. This does not mean that disequilibrium does not exist in the German labor market. Instead, it reflects the special feature of the labor market in Germany. This special feature in contrast to the U. S. economy is consistent with what has been found in many other empirical studies with regard to the German labor market. Finally, the welfare comparison of the benchmark model to the disequilibrium variants show that the latter are not necessarily significantly inferior.

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Appendix 1: An Approximation Method of Stochastic Dynamic Optimization

Suppose we can write a stochastic dynamic programming model as follows:

$$\{u_t\}_{t=0}^{\infty} \max E_0 \sum_{t=0}^{\infty} \beta^t U(x_t, u_t), A1 \tag{41}$$

subject to

$$x_{t+1} = F(x_t, u_t) + \varepsilon_{t+1}, A2 \tag{42}$$

where x_t is a vector of p state variables; u_t is a vector of q control variables; ε_{t+1} is a vector of disturbance terms. E_t is the mathematical expectation conditional on the information available at time t; and β is the discount factor. The Lagrangian of this model can be written as

$$L = E_0 \sum_{t=0}^{T} \left\{ \beta^t U(x_t, u_t) - \beta^{t+1} \lambda'_{t+1} \left(x_{t+1} - F(x_t, u_t) - \varepsilon_{t+1} \right) \right\}, A3 \tag{43}$$

where λ_t, the Lagrangian multiplier, is a px1 vector. Setting the partial derivatives of L to zero with respect to λ_t, x_t and u_t respectively will yield equation (A2) as well as

$$U_x(x_t, u_t) + \beta F_x(x_t, u_t) E_t \lambda_{t+1} = \lambda_t, A4 \tag{44}$$

$$U_u(x_t, u_t) + \beta F_u(x_t, u_t) E_t \lambda_{t+1} = 0, A5 \tag{45}$$

where $U_x(\cdot)$ is a px1 vector of $\partial U / \partial x; F_x(\cdot)$ is a pxp matrix of $\partial F / \partial x; U_u(\cdot)$ is a qx1 vector of $\partial U / \partial u$ and $F_u(\cdot)$ is a qxp matrix of $\partial F / \partial u$. They form the first-order conditions and can all be nonlinear. Suppose we linearize the first-order conditions as follows:

$$F_{11} x_t + F_{12} u_t + F_{13} E_t \lambda_{t+1} + f_1 = \lambda_t, A6 \tag{46}$$

$$F_{21} x_t + F_{22} u_t + F_{23} E_t \lambda_{t+1} + f_2 = 0, A7 \tag{47}$$

$$x_{t+1} = A x_t + C u_t + b + \varepsilon_{t+1}.A8 \tag{48}$$

From (A6), we obtain

$$E_t \lambda_{t+1} = F_{13}^{-1} \left(\lambda_t - F_{11} x_t - F_{12} u_1 - f_1 \right).A9 \tag{49}$$

Substituting (A9) into (A7) and then solving the equation for u_t, we get

$$u_t = N \lambda_t + M x_t + m, A10 \tag{50}$$

where

$$N = \left(F_{23} F_{13}^{-1} F_{12} - F_{22} \right)^{-1} F_{23} F_{13}^{-1}, A11 \tag{51}$$

$$M = \left(F_{23}F_{13}^{-1}F_{12} - F_{22}\right)^{-1}(F_{21} - F_{23}F_{13}^{-1}F_{11}), A12 \tag{52}$$

$$m = \left(F_{23}F_{13}^{-1}F_{12} - F_{22}\right)^{-1}(f_2 - F_{23}F_{13}^{-1}f_1), A13 \tag{53}$$

Substituting (A10) into (A8) and (A9) respectively, we obtain

$$x_{t+1} = (A + CM)x_t + CN\lambda_t + Cm + b + \varepsilon_{t+1}, A14 \tag{54}$$

$$E_t\lambda_{t+1} = F_{13}^{-1}(I - F_{12}N)\lambda_t - F_{13}^{-1}(F_{11} + F_{12}M)x_t - F_{13}^{-1}(f_1 + F_{12}m).A15 \tag{55}$$

We thus complete the dynamic system indicated by the first-order conditions. However, this is not satisfactory because of the inclusion of λ_t. What we want to derive is the linear decision rule:

$$u_t = Gx_t + g.A16 \tag{56}$$

For this purpose, we, as in Chow (1991, 1993b), first assume that

$$\lambda_{t+1} = Hx_{t+1} + h.A17 \tag{57}$$

We find that the assumed linear relation in (A19) can be justified once we take the expectation of λ_{t+1}. In this case, $E_t\lambda_{t+1} = HE_tx_{t+1}$. If we express x_{t+1} in terms of (A14), E_tx_{t+1} and hence $E_t\lambda_{t+1}$ will depend on λ_t and x_t, which is exactly consistent with the form of $E_t\lambda_{t+1}$ in (A15). Taking the expectation of (A17) and expressing E_tx_{t+1} in terms of (A8), we obtain

$$E_t\lambda_{t+1} = H(Ax_t + Cu_t + b) + h.A18 \tag{58}$$

Now substitute (A17) into (A14) and (A15):

$$x_{t+1} = (A + CM + CNH)x_t + CNh + Cm + b + \varepsilon_{t+1}, A19 \tag{59}$$

$$E_t\lambda_{t+1} = F_{13}^{-1}\left[(I - F_{12}N)H - (F_{11} + F_{12}M)\right]x_t + F_{13}^{-1}\left[(I - F_{12}N)h - (f_1 + F_{12}m)\right].A20 \tag{60}$$

Next, expressing u_t in terms of (A16) for equations (A18) and (A8) respectively, we get

$$E_t\lambda_{t+1} = H(A + CG)x_t + H(Cg + b) + h, A21 \tag{61}$$

$$x_{t+1} = (A + CG)x_t + Cg + b + \varepsilon_{t+1}.A22 \tag{62}$$

Comparing the above two equations with (A19) and (A20), we obtain

$$M + NH = G, A23 \tag{63}$$

$$Nh + m = g, A24 \tag{64}$$

$$F_{13}^{-1}\left[(I - F_{12}N)H - (F_{11} + F_{12}M)\right] = H(A + CG), A25 \tag{65}$$

$$F_{13}^{-1}\left[(I - F_{12}N)h - (f_1 + F_{12}m)\right] = H(Cg + b) + h.A26 \tag{66}$$

These four equations determine the solution of H, G, h and g. Substituting (A23) into (A25), we obtain the equation as to the solution of H:

$$F_{13}HCNH + F_{13}H(A + CM) + (F_{12}N - I)H + (F_{11} + F_{12}M) = 0A27 \tag{67}$$

This equation includes pxp nonlinear (quadratic) equations. Since all these equations are nonlinear, multiple (two) solutions may exist. One thus has to decide which solution is a proper solution. For example, in our prototypical model, there is one constraint, the capital stock, included in the Lagrangian. Therefore, H is a scalar and (A27) takes the form

$$a_1H^2 + a_2H + a_3 = 0, A28 \tag{68}$$

with the two solutions given by

$$H_{1,2} = (1/a_1)\left[a_2 \pm \left(a_2^2 - 4a_1a_3\right)^{1/2}\right].A29 \tag{69}$$

Thus if $a_1a_3 < 0$, one solution is positive and the other is negative. Given the meaning of the Lagrangian multiplier, λ_t is the shadow price of the resource and thus should be inversely related to quantity of the resource. Therefore, only the negative solution is a proper solution.

Once H is obtained, h, according to (A24) and (A26), is given by

$$h = -\left[(F_{13}HC + F_{12})N + F_{13} - I\right]^{-1}\left[(F_{13}HC + F_{12})m + F_{13}Hb + f_1\right].A30 \tag{70}$$

Then G and g are simply determined by (A23) and (A24).