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Monetary Policy with Nonlinear Phillips Curve and Endogenous NAIRU

by

Willi Semmler and Wenlang Zhang

University of Bielefeld
Department of Economics
Center for Empirical Macroeconomics
P.O. Box 100 131

33501 Bielefeld, Germany

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Willi Semmler[*]   Wenlang Zhang[†]

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Abstract

The recent literature on monetary policy has questioned the shape of the Phillips curve and the assumption of a constant NAIRU. In this paper we explore monetary policy considering nonlinear Phillips curves and an endogenous NAIRU which can be affected by the monetary policy. We first study monetary policy with different shapes of the Phillips curve: Linear, convex and convex-concave. We find that the optimal monetary policy changes with the shape of the Phillips curve, but there exists a unique equilibrium no matter whether the Phillips curve is linear or nonlinear. We also explore monetary policy with an endogenous NAIRU, since some researchers, Blanchard (2003) for example, have proposed that the NAIRU may be influenced by monetary policy. Based on some empirical evidence and assuming that monetary policy can influence the NAIRU, we find that there may exist multiple equilibria in the economy, different from the results of models presuming a constant NAIRU.

JEL: E0, E5
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1 Introduction

Recently an important topic in studies of monetary policy has been the shape of the Phillips curve. The shape of the Phillips curve plays an important role in monetary policy and has important implications for policy-makers, because the IS- and Phillips curves have been the core model for monetary policy studies from a Keynesian perspective.

In earlier studies the Phillips curve has been proposed to be linear. But most of the recent literature has casted doubt on the linearity of the Phillips curve. Dupasquier and Ricketts (1998a), for example, survey several models that explore why the Phillips curve may be nonlinear. It has been argued that the Phillips curve may have three shapes. The first group of researchers propose that the Phillips curve is convex. This possibility has been considered by Clark, Laxton and Rose (1996), Schaling (1999) and Bean (2000) for example. A convex Phillips curve appears in an economy subject to capacity constraints. The second group of researchers propose that the Phillips curve is concave. Eisner (1997), for example, reports some results concerned with this possibility. A concave Phillips curve may exist in an economy where firms are not purely competitive. Besides these two shapes of the Phillips curve, Filardo (1998), on the basis of some empirical research with U.S. data, proposes that the Phillips curve is not purely convex or concave, but instead convex-concave. He finds that the Phillips curve is convex if the output gap is positive and concave if the output gap is negative. Therefore he points out that the supporters of a convex or concave Phillips curve have studied only one case and overlooked the other. The shape of the Phillips curve has crucial implications for central banks, since the optimal monetary policy may change with the shape of the Phillips curve.

Another important topic of monetary policy is whether the NAIRU is constant. In the 1960s Friedman and Phelps proposed a vertical long-run Phillips curve. The traditional view is that money can not influence the unemployment rate in the long run and therefore the unemployment rate returns to the natural rate or the NAIRU over time, which is presumed to be constant. The recent literature has put this view into question and proposes that the long-run Phillips curve may be non-vertical and the NAIRU nonconstant. That is, the NAIRU can be affected by the inflation rate and monetary policy. Gordon (1997), for example, estimates the time-varying NAIRU for the U.S. with and without supply shocks. Some other researchers have also tried to estimate a time-varying NAIRU. An earlier survey on the estimation of the NAIRU is given by Staiger, Stock and Watson (1996). Blanchard (2003), moreover, remarks that the natural rate of unemployment has been affected by the real interest rate in Europe in the 1970s and 1980s in different directions. He further proposes several mechanisms in which the real interest rate may influence the natural rate of the unemployment. On the other hand, some researchers, Stiglitz (1997) for example, maintain that the NAIRU may have feedback effects on policies in the sense that it might produce a moving target for monetary policy. Therefore, an economic model with an exogenous NAIRU might not explore monetary policy properly.

The remainder of this paper is organized as follows. In the second section we explore monetary policy in a traditional model with a linear Phillips curve and a constant NAIRU. In Section 3 we study monetary policy with different shapes of the nonlinear Phillips curve, maintaining the assumption of a constant NAIRU. In the fourth section, however, we study the monetary policy with an endogenous NAIRU which can be affected by the monetary policy and the last section concludes the paper.

2 Monetary Policy in a Traditional Model: Linear Phillips Curve with and without Expectation

The traditional IS curve reads as

$$\dot{\mu}(t) = \alpha r(t) + \vartheta \mu(t), \quad \alpha > 0,$$

where $\mu(t)$ denotes the gap between the actual unemployment $u(t)$ and the NAIRU $u_n(t)$, namely $u(t) - u_n(t)$, $r(t)$ is the monetary policy instrument. Assuming $\vartheta = 0$ for simplicity, we obtain

$$\dot{\mu}(t) = \alpha r(t). \tag{1}$$

In the traditional model the NAIRU is assumed to be an exogenous variable which remains constant because of the neutrality of money. "·" denotes the derivative with respect to time t. The equation above implies

$$\dot{\mu}(t) = \dot{u}(t) - \dot{u}_n(t) = \dot{u}(t) = \alpha r(t), \tag{2}$$

since the NAIRU is constant. Let π denote the deviation of the actual inflation rate from its target (assumed to be zero here). Following Walsh(1999), Ball (1999) and Hall (2000) who assume that the inflation rate is affected by the real interest rate as well as the unemployment gap, we write the Phillips curve without expectation as[1]

$$\pi(t) = -\beta r(t) - \theta \mu(t), \quad \beta, \theta > 0. \tag{3}$$

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[1]In fact, the Phillips curve above is equivalent to an open economy Phillips curve with

$$\pi(t) = -\tau e(t) - \theta \mu(t),$$

with $e(t)$ denoting the real exchange rate and τ the share of imports in domestic spending. A higher $e(t)$ means appreciation. Such a Phillips curve can be found in Walsh (1999) and Guender (2001). Ball (1999) employs a similar equation except that the change of $e(t)$ is included in the open-economy Phillips curve. Moreover, following Ball (1999) we can link the real exchange rate and the real interest rate by

$$e(t) = \omega r(t) + \nu(t),$$

with $\nu(t)$ being a white noise which captures other effects on the exchange rate. The Phillips curve in eq. (3) is then obtained by substituting the deterministic version of the exchange rate equation into the above Phillips curve. Hall (2000) also includes the real interest rate in the Phillips curve. Furthermore, we follow Ball (1999) and take the real interest rate as the

Suppose the central bank has the following loss function

$$L(t) \equiv \pi^2(t) + \lambda\mu(t)^2, \tag{4}$$

where $\lambda(> 0)$ denotes the weight of unemployment stabilization. We will drop the notation "t" in variables in the remainder of the paper just for simplicity. Suppose the goal of the monetary policy is to minimize the loss function with an infinite horizon, the central bank's problem turns out to be

$$\underset{r}{Min} \int_0^\infty e^{-\rho t} L dt$$

subject to

$$\dot{\mu} = \alpha r, \tag{5}$$

where ρ $(0 < \rho < 1)$ is the discount factor and L defined by (4). The current-value Hamiltonian of the above problem reads as

$$H_c = [\beta r + \theta\mu]^2 + \lambda\mu^2 + \gamma\alpha r, \tag{6}$$

where γ is a costate variable. The optimal conditions for this problem turn out to be

$$\frac{\partial H_c}{\partial r} = 0, \tag{7}$$

$$\dot{\gamma} = -\frac{\partial H_c}{\partial \mu} + \rho\gamma, \tag{8}$$

with the following transversality condition

$$\lim_{t\to\infty} \gamma e^{-\rho t} = 0. \tag{9}$$

From (7) we get the optimal monetary policy

$$r = -\frac{\alpha\gamma}{2\beta^2} - \frac{\theta}{\beta}\mu, \tag{10}$$

from which we know

$$\dot{\gamma} = -\frac{2\beta}{\alpha}(\beta\dot{r} + \theta\alpha r). \tag{11}$$

Some rearrangement of (8), (10) and (11) gives us the following dynamic system of r and μ:

$$\dot{r} = \rho r + [\frac{\theta\rho}{\beta} + \frac{\alpha}{\beta^2}(\theta^2 + \lambda^2)]\mu, \tag{12}$$

$$\dot{\mu} = \alpha r. \tag{13}$$

policy instrument. Although McCallum casts doubt on Ball (1999) for the use of the real rate as a policy instrument, Ball (1999, footnote 2), however, correctly argues that policy makers can move the real interest rate to their desired level by setting the nominal rate equal to the desired real rate plus the inflation rate. Ball (1999) also considers the exchange-rate effect in the IS equation. Given the exchange rate equation above, eq. (1) can also be considered as the IS equation in the open economy.

Setting $\dot{r} = \dot{\mu} = 0$, we obtain the unique equilibrium

$$r^* = 0, \quad \mu^* = 0 \ (u^* = u_n).$$

The Jacobian matrix of the dynamic system evaluated at the equilibrium is

$$J = \begin{pmatrix} \rho & \frac{\theta\rho}{\beta} + \frac{\alpha}{\beta^2}(\theta^2 + \lambda^2) \\ \alpha & 0 \end{pmatrix}.$$

If x_1 and x_2 are two characteristic roots of J, it is obvious that

$$x_1 + x_2 = \rho > 0,$$
$$x_1 x_2 = -\alpha[\frac{\theta\rho}{\beta} + \frac{\alpha}{\beta^2}(\theta^2 + \lambda^2)] < 0.$$

We find that x_1 and x_2 are both real with opposite signs. This indicates that the unique equilibrium $(0, 0)$ is a saddle point.

Above we have explored monetary policy in a traditional model with a linear Phillips curve without expectation considered. Next we explore monetary policy with expectation in the Phillips curve. The expectation-augmented Phillips curve is written as

$$\pi = -\beta r - \theta\mu + \pi_e, \tag{14}$$

with π_e denoting the expectation of the inflation rate. Following the traditional literature we assume that the expectation evolves in an adaptive way[2]

$$\dot{\pi}_e = \kappa(\pi - \pi_e), \quad \kappa > 0. \tag{15}$$

The loss function L now looks like

$$L = [\pi_e - \beta r - \theta\mu]^2 + \lambda\mu^2. \tag{16}$$

After some rearrangement the problem of a central bank turns out to be

$$\underset{r}{Min} \int_0^\infty e^{-\rho t} L \, dt$$

subject to

$$\dot{\pi}_e = -\kappa[\beta r + \theta\mu], \tag{17}$$
$$\dot{\mu} = \alpha r. \tag{18}$$

The current-value Hamiltonian of this problem reads

$$H_c = [\pi_e - \beta r - \theta\mu]^2 + \lambda\mu^2 - \gamma_1\kappa[\beta r + \theta\mu] + \gamma_2\alpha r, \tag{19}$$

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[2]We follow here Turnovsky (1981) and use an adaptive expectation dynamics. This is in line with Rudebusch and Svensson (1999) in the sense that only past information is used in the inflation equation.

where γ_1 and γ_2 are costate variables. The optimal conditions turn out to be

$$\frac{\partial H_c}{\partial r} = 0, \tag{20}$$

$$\dot{\gamma}_1 = -\frac{\partial H_c}{\partial \pi_e} + \rho\gamma_1, \tag{21}$$

$$\dot{\gamma}_2 = -\frac{\partial H_c}{\partial \mu} + \rho\gamma_2, \tag{22}$$

$$\lim_{t\to\infty} \gamma_1 e^{-\rho t} = 0, \tag{23}$$

$$\lim_{t\to\infty} \gamma_2 e^{-\rho t} = 0. \tag{24}$$

From (20) we know

$$r = \frac{1}{\beta}[\pi_e - \theta\mu + \frac{\kappa\gamma_1}{2} - \frac{\alpha\gamma_2}{2\beta}]. \tag{25}$$

After substituting (25) into (17), (18), (21) and (22), we obtain the following dynamic system

$$\dot{\pi}_e = -\kappa(\pi_e + \frac{\kappa}{2}\gamma_1 - \frac{\alpha}{2\beta}\gamma_2), \tag{26}$$

$$\dot{\mu} = \frac{\alpha}{\beta}[\pi_e - \theta\mu + \frac{\kappa}{2}\gamma_1 - \frac{\alpha}{2\beta}\gamma_2], \tag{27}$$

$$\dot{\gamma}_1 = (\rho + \kappa)\gamma_1 - \frac{\alpha}{\beta}\gamma_2, \tag{28}$$

$$\dot{\gamma}_2 = (\rho + \frac{\alpha\theta}{\beta})\gamma_2 - 2\lambda^2\mu. \tag{29}$$

Let $\dot{\pi}_e = \dot{\mu} = \dot{\gamma}_1 = \dot{\gamma}_2 = 0$, we obtain the unique equilibrium:

$$\pi_e^* = 0, \mu^* = 0 \ (u^* = u_n), \gamma_1^* = 0, \gamma_2^* = 0,$$

and therefore $r^* = 0$ and $\pi^* = 0$. In order to consider the out-of-steady-state dynamics around the equilibrium, we compute the Jacobian matrix of the system (26)-(29) evaluated at the equilibrium:

$$J = \begin{pmatrix} -\kappa & 0 & -\frac{\kappa^2}{2} & \frac{\alpha\kappa}{2\beta} \\ \frac{\alpha}{\beta} & -\frac{\alpha\theta}{\beta} & \frac{\alpha\kappa}{2\beta} & -\frac{\alpha^2}{2\beta^2} \\ 0 & 0 & \rho+\kappa & -\frac{\alpha}{\beta} \\ 0 & -2\lambda^2 & 0 & \rho+\frac{\alpha\theta}{\beta} \end{pmatrix}.$$

The characteristic equation of J can be written as

$$\triangle(x) \equiv |xI - J| = x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4 = 0. \tag{30}$$

The Routh-Hurwitz sufficient and necessary condition for local stability in this case is[3]

$$b_i > 0 \ (i = 1, 2, 3, 4), \quad b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 > 0.$$

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[3] The reader is referred to some advanced textbooks or papers on dynamic systems, Yoshida and Asada (2001) for example, for the computation of b_i.

But it is obvious that
$$b_1 = -2\rho < 0,$$
therefore we conclude that the equilibrium is unstable. This is consistent with the results from the case without expectation. For example, if we take $\alpha = 0.6$, $\kappa = 0.9$, $\beta = 0.6$, $\theta = 0.5$, $\rho = 0.1$ and $\lambda = 0.5$, the eigenvalues of J turn out to be
$$x_1 = -1.060, x_2 = -0.421, x_3 = 1.160, x_4 = 0.521.$$
This indicates that the unique equilibrium is unstable, since two characteristic roots are positive.

3 Monetary Policy with Nonlinear Phillips Curve

In the previous section we have explored monetary policy in a traditional model with a linear Phillips curve. There exists a unique saddle point equilibrium, no matter whether the expectation is taken into account or not.

Recently most of the literature, however, casts doubt on the linearity of the Phillips curve. Dupasquier and Ricketts (1998a), for example, survey several models of the nonlinearity in the Phillips curve. The five models surveyed are the *capacity constraint model*, the *mis-perception* or *signal extraction model*, the *costly adjustment model*, the *downward nominal wage rigidity model* and the *monopolistically competitive model*. These models explain the nonlinearity of the Phillips curve from various perspectives. As also mentioned by Akerlof (2002), the nonlinearity of the Phillips curve has been an important issue of macroeconomics, since it has crucial implications for the monetary authorities.

Many empirical studies have been undertaken to explore the nonlinearity of the Phillips curve. Dupasquier and Ricketts (1998a), for example, explore the nonlinearity in the Phillips curve for Canada and the U.S. and conclude that there is stronger evidence of a nonlinearity for the U.S. than for Canada. Aguiar and Martins (2002), however, test three kinds of nonlinearities (quadratic, hyperbole and exponential) in the Phillips curve and Okun's law with the aggregate Euro-area macroeconomic data and find that the Phillips curve turns out to be linear, but Okun's law is nonlinear.

Other studies on the nonlinearity of the Phillips curve include Gómez and Julio (2000), Clements and Sensier (2002), Dupasquier and Ricketts (1998b), Chadha, Masson and Meredith (1992), Laxton, Meredith and Rose (1995) and Bean (2000). Monetary policy with a nonlinear Phillips curve has also been explored, see Schaling (1999), Tambakis (1998) Flaschel, Gong and Semmler (2001), and Nobay and Peel (2000) for example.

The problem is, if the Phillips curve is nonlinear, what does it look like? Some researchers, Filardo (1998) for instance, have explored this problem both theoretically and empirically. One possible shape of the nonlinear Phillips curve is convex. This possibility has been explored by Clark, Laxton and Rose (1996), Schaling (1999), Laxton, Rose and Tambakis (1998), Tambakis (1998), Bean (2000), Zhang and Semmler (2003) and so on. A convex Phillips curve is shown
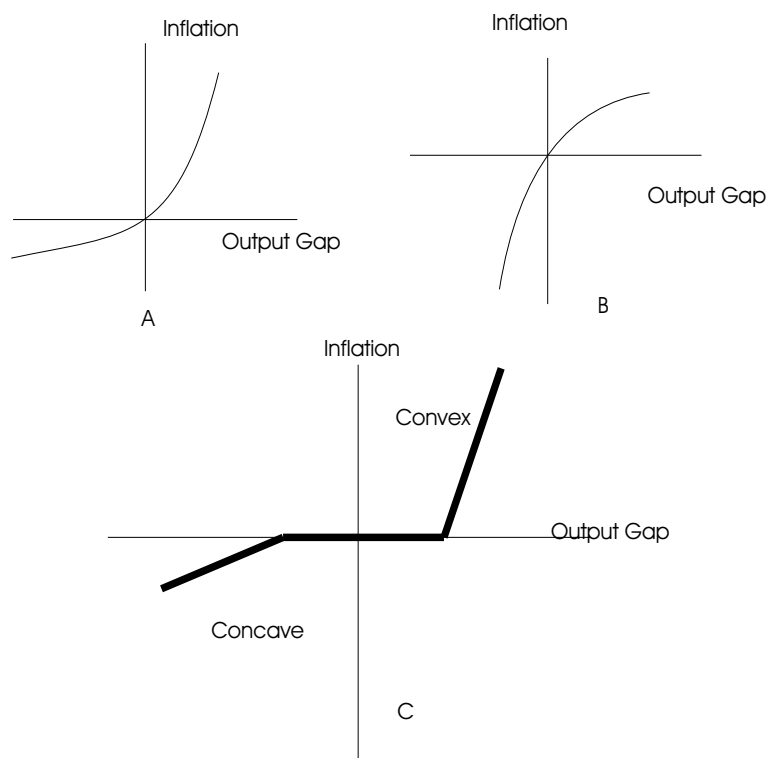
Figure 1: Possible Shapes of Nonlinear Phillips Curve

in Figure 1A. Filardo (1998) describes the property of a convex Phillips curve as follows:

> "...which (the convex Phillips curve) graphically translates into an upward sloping curve that steepens as output rises relative to trend. Intuitively, the steepening slope indicates increased sensitivity of inflation to the economy's strength. As the slope of the convex curve steepens, inflation becomes more sensitive because a given change in inflation requires a progressively smaller output adjustment. ... The convex Phillips curve is consistent with an economy subject to capacity constraints. ... As the economy becomes stronger and capacity constraints increasingly restrict firms' ability to expand output, an increase in demand is more likely to show up as higher inflation than as higher output." (Filardo , 1998)

Another possible shape of the nonlinear Phillips curve is concave. This possibility has been proposed, for example, by Stiglitz (1997). A concave Phillips curve is shown in Figure 1B. The property of a concave Phillips curve can be described as follows

> "... which (the concave Phillips curve) graphically translates into an upward sloping curve that flattens as output rises relative to trend. Intuitively, the concave curve's flattening slope reflects the declining sensitivity of inflation to the strength of the economy. ...

Theoretically, a concave Phillips curve is consistent with an economy where firms are not purely competitive. If firms have some pricing power and thus the ability and desire to influence their market share, they will be more reluctant to raise prices than to lower them." (Filardo, 1998)

Eisner (1997) also reports results that are consistent with a concave Phillips curve. Filardo (1998) further claims that the implications for the output cost of fighting inflation may be different for different shapes of the Phillips curve.

"The concave Phillips curve implies that the cost of fighting inflation rises with the strength of the economy because as the economy strengthens its slope flattens. In contrast, the convex Phillips curve implies that the costs of fighting inflation falls with the strength of the economy because its slope steepens." (Filardo, 1998)

On the basis of these literature, Filardo (1998) explores evidence of a third shape of the nonlinear Phillips curve and proposes that the nonlinear Phillips curve is not purely convex or concave, but a combination of both, namely, convex-concave. He sets up a model as follows

$$\pi_t = \pi_t^e + \beta_w y_{w,t-1} + \beta_b y_{b,t-1} + \beta_s y_{s,t-1} + \varepsilon_t, \tag{31}$$

where π_t and π_t^e are actual and expected inflation rates respectively. y_t is the output gap and ε_t is a supply shock. w, b and s stand for "weak", "balanced" and "strong" respectively. The slope coefficients on the output gap (β_w, β_b and β_s) measure the sensitivity of inflation to economic activity in the *weak, balanced* and *overheated* times. In the case of a linear Phillips curve, $\beta_w = \beta_b = \beta_s$. With the U.S. data from 1959-97, Filardo (1998) finds that $\beta_w = 0.2$, $\beta_b = -0.02$ and $\beta_s = 0.49$. Therefore, he obtains a convex-concave Phillips curve as shown in Figure 1C. This nonlinear Phillips curve implies that when the output gap is positive, the Phillips curve is convex and when the output gap is negative, the Phillips curve is concave. Filardo (1998) further states that the researchers who have proposed a convex *or* concave Phillips curve have considered only one case and overlooked the other. The policy implication of a convex-concave Phillips curve is of course different from that of a convex or concave one.

Based on the literature of nonlinear Phillips curves, this section is devoted to the optimal monetary policy with nonlinear Phillips curves. We will explore monetary policy in two cases: A convex Phillips curve and a convex-concave one.[4] The expectation in the Phillips curve is not taken into account in this section, since the results in the previous section are similar no matter whether the expectation is considered or not. Moreover, the model without expectation is easier to study, since the dimension of the problem is then lower.

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[4]Note that we use the unemployment rate instead of the output in the Philips curve. According to Okun's law, there exists a negative correlation between the unemployment and output, therefore, the convexity or concavity mentioned above will not be changed.

3.1 A Convex Phillips Curve

Some researchers, Schaling (1999) for example, assume that a convex Phillips curve can be defined through the following function

$$f(y) = \frac{\phi y}{1 - \phi \varphi y}, \quad f' > 0, f'' > 0, \phi > 0, 1 > \varphi \geq 0, \tag{32}$$

where y is the output gap and the parameter φ indexes the curvature of $f(\cdot)$. In case $\varphi = 0$, $f(\cdot)$ becomes linear. In line with Okun's law, we just assume $y = -\mu$ and then obtain the following convex Phillips curve[5]

$$\pi = -\beta r - \theta \Pi(\mu), \tag{33}$$

with $\Pi(\cdot)$ defined as

$$\Pi(\mu) = \frac{\phi \mu}{1 + \phi \varphi \mu}. \tag{34}$$

In the research below we assume $\theta = 1$ in the Phillips curve and $\alpha = 1$ in (1) just to simplify the analysis. With the above Phillips curve the current-value Hamiltonian of the optimal control now looks like

$$H_c = [\beta r + \frac{\phi \mu}{1 + \phi \varphi \mu}]^2 + \lambda \mu^2 + \gamma r, \tag{35}$$

from which we obtain the optimal monetary policy

$$r = -\frac{\phi \mu}{\beta(1 + \phi \varphi \mu)} - \frac{\gamma}{2\beta^2} \tag{36}$$

and then by employing (8) and (36), we have the following dynamic system

$$\dot{r} = \frac{1}{\beta^2}[\frac{\phi^2}{\Omega^3}\mu + \lambda \mu + \frac{\beta \phi \rho}{\Omega}\mu] + \rho r, \tag{37}$$

$$\dot{\mu} = r, \tag{38}$$

with $\Omega = 1 + \phi \varphi \mu$. Setting $\dot{r} = \dot{\mu} = 0$, we know the equilibrium of r is

$$r^* = 0$$

and the equilibria of μ can then be solved from the equation

$$\Omega^3 \lambda \mu + \phi \beta \rho \Omega^2 \mu + \phi^2 \mu = 0. \tag{39}$$

It is obvious that $\mu = 0$, therefore $u^* = u_n$ is a solution of (39). As a result, $u^* = u_n$ and $r^* = 0$ is an equilibrium of the system (37)-(38). This is consistent

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[5]In Okun's law ($\mu = -my$), the m is usually smaller than one. We take $m = 1$ just for simplicity since the result below will not be changed if we assume $m < 1$.

with the unique equilibrium with the traditional linear Phillips curve analyzed in the previous section. But in case $\mu \neq 0$, (39) becomes

$$\Omega^3 \lambda + \phi \beta \rho \Omega^2 + \phi^2 = 0, \tag{40}$$

which has three solutions. Note that in order for the condition $f''(\cdot) > 0$ in (32) to be satisfied, μ must be larger than $-\frac{1}{\phi\varphi}$. We call this the *Convex Condition*. The *Convex Condition* implies $\Omega > 0$. But it is obvious that (40) has no solutions satisfying this condition and therefore $(0,0)$ is the only equilibrium of the above dynamic system. This is consistent with the results of the model analyzed in the previous section.

Next we explore the out-of-steady-state equilibrium dynamics around the equilibrium. The Jacobian matrix of the system (37)-(38) evaluated at the equilibrium is

$$J = \begin{pmatrix} \rho & \frac{1}{\beta^2}(\phi^2 + \lambda + \beta\phi\rho) \\ 1 & 0 \end{pmatrix}.$$

It is obvious that the characteristic roots x_1 and x_2 of J satisfy

$$x_1 + x_2 = \rho > 0,$$
$$x_1 x_2 = -\frac{1}{\beta^2}(\phi^2 + \lambda + \beta\phi\rho) < 0.$$

We find that x_1 and x_2 are both real with opposite signs, this implies that the equilibrium is a saddle point.

3.2 A Convex-Concave Phillips Curve

We have explored monetary policy with a convex Phillips curve and find that there exists a unique equilibrium. As stated before, Filardo (1998) finds that the Phillips curve is convex-concave, depending on whether the output gap is positive or negative. A graph of such a curve is shown in Figure 1C. In this section we will explore monetary policy with such a convex-concave Phillips curve. The Phillips curve in 1C has breakpoints and moreover, it is assumed that the derivatives of the curve in the concave and convex zones are constant. This is in fact not consistent with the cases of the two graphs in 1A and B, where the derivatives of the curves are dependent on the sizes of the output gap. Filardo (1998) obtains such a non-smooth graph from the estimation of a simple model, in which he assumes the parameters β_w, β_b and β_s in (31) to be constant. But in reality the three parameters can be continuously state-dependent. If the parameters are continuously state-dependent, the Phillips curve is then smooth and convex-concave. In the research below we assume that the Phillips curve is smooth and convex-concave. In order to consider this possibility we have to design a function which is convex when the output gap is positive and concave when the output gap is negative, and moreover, the absolute values of the derivatives of the function with respect to the output gap are asymmetric around zero. That is, the graph is steeper when the output gap
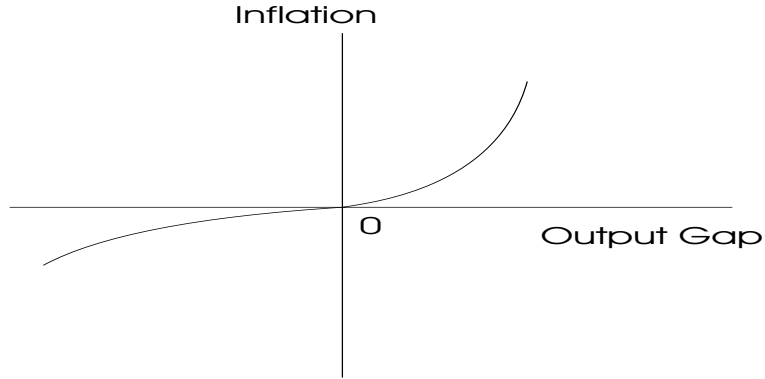
Figure 2: A Convex-Concave Phillips Curve

is positive than when the output gap is negative. Fortunately we can design a function with such properties.

Let y denote the output gap, we define the following function $f(y)$

$$f(y) = \delta(e^{ay} - ay - 1)sgn(y), \quad \delta > 0, a > 0, \tag{41}$$

with

$$sgn(y) = \begin{cases} 1, & \text{if } y \geq 0; \\ -1, & \text{if } y < 0. \end{cases}$$

$\delta(e^{ay} - ay - 1)$ is the so-called LINEX function proposed by Varian (1975) and applied, for example, by Nobay and Peel (1998) and Semmler and Zhang (2002). The LINEX function is nonnegative and asymmetric around zero and, moreover, the parameter a determines the extent of asymmetry.[6] The parameter δ scales the function.

The graph of $f(y)$ is shown in Figure 2. From this figure we see $f(y)$ has the properties discussed above. Let $y = -\mu$, we obtain the following convex-concave Phillips curve

$$\pi = -\beta r - \theta[e^{-a\mu} + a\mu - 1]sgn(\mu), \quad a > 0, \tag{42}$$

where $sgn(\mu)$ equals 1 when $\mu \geq 0$ and -1 when $\mu < 0$. With the Phillips curve defined above the current-value Hamiltonian of the optimal control now looks like

$$H_c = \{\beta r + \theta[e^{-a\mu} + a\mu - 1]sgn(\mu)\}^2 + \lambda\mu^2 + \gamma r, \tag{43}$$

from which we obtain the following optimal monetary policy

$$r = -\frac{\theta}{\beta}[e^{-a\mu} + a\mu - 1]sgn(\mu) - \frac{\gamma}{2\beta^2}. \tag{44}$$

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[6]a can be either positive or negative, but in our model it is positive.

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We have above derived optimal monetary policies from models with different Phillips curves. The optimal monetary policies without expectation are shown in (10), (36) and (44) respectively. The differences between these policies are obvious. Exactly speaking, when the Phillips curve is linear, the optimal monetary policy is also a linear function of the unemployment gap. When the Phillips curve is convex, the optimal monetary policy is a convex function of the unemployment gap. And finally, when the Phillips curve is convex-concave, the optimal monetary policy is a convex-concave function of the unemployment gap. Therefore, the optimal monetary reaction function of the central bank is dependent upon the shape of the Phillips curve.

Dynamics

Next we explore the dynamics of the model. By employing (8) and (44) we obtain the following dynamic system

$$\dot{r} = \frac{1}{\beta^2}\{a\theta^2(1 - e^{-a\mu})(a\mu + e^{-a\mu} - 1)$$

$$+ \beta\theta\rho(e^{-a\mu} + a\mu - 1)sgn(\mu) + \lambda\mu\} + \rho r; \qquad (45)$$

$$\dot{\mu} = r. \qquad (46)$$

Setting $\dot{\mu} = 0$ we get the equilibrium of r, namely $r^* = 0$. With $r^* = 0$ and setting $\dot{r} = 0$, the equilibrium of μ can be solved from the following equation

$$a\theta^2(1 - e^{-a\mu})(a\mu + e^{-a\mu} - 1) + \beta\theta\rho(e^{-a\mu} + a\mu - 1)sgn(\mu) + \lambda\mu = 0. \quad (47)$$

$\mu = 0$, namely $u = u_n$ is the only solution of the above equation. This implies that the dynamic system has a unique equilibrium $r^* = 0$ and $\mu^* = 0$. This is consistent with the results in the previous sections. The Jacobian matrix of the system (45)-(46) evaluated at the equilibrium is

$$J = \begin{pmatrix} \rho & \frac{\lambda}{\beta^2} \\ 1 & 0 \end{pmatrix}.$$

It is obvious that the equilibrium is a saddle point since the characteristic roots of J are both real with oppositive signs.

In this section we have explored monetary policy with two nonlinear Phillips curves. There exists a unique equilibrium (a saddle point), no matter whether the Phillips curve is convex or convex-concave. This is consistent with the results from a model with a linear Phillips curve. But the optimal monetary policy reaction function changes with the shapes of the Phillips curve.

4   Monetary Policy with Endogenous NAIRU

In the previous sections we have explored monetary policy with linear and nonlinear Phillips curves with a constant NAIRU (like an exogenous variable). The

difference of the results lies in the fact that the monetary policy functions change with the shapes of the Phillips curve. The similarity, however, is that there exists a unique equilibrium (a saddle point) despite of the shapes of the Phillips curve.

According to Friedman (1968) and Phelps (1968), monetary policy has no effects on the unemployment rate in the long run because of the so-called money neutrality. This implies a vertical long-run Phillips curve. The recent literature, however, has questioned this assumption. Stiglitz (1997), for example, surveys some factors that may lead to the movement of the NAIRU. Graham and Snower (2002) derive a microfounded long-run downward-sloping Phillips curve and show that a permanent increase in money growth incurs a permanent increase in the inflation rate and a permanent decrease in the unemployment level. Empirically he shows that a 1% increase in the money growth rate can induce a long-run reduction in the unemployment from 15% to 0.5% below its steady-state level. They further claim that the effects can be large and have a long half-life in the short and medium run.

Karanassou, Sala and Snower (2002) show that the changes in money growth can have long-run effects on the unemployment as well as inflation even if there are no money illusion and money neutrality. Akerlof, Dickens and Perry (2000) argue that the long-run Phillips curve is not vertical but instead bowed-inward and then forward-bending. Following Akerlof, Dickens and Perry (2000), Lundborg and Sacklén (2001), based on the Swedish data, show that there exists a negatively sloped long-run Phillips curve. Some researchers further propose that the NAIRU can be affected by the inflation rate and monetary policy.

Different from the traditional view of Friedman, Blanchard (2003) points out that monetary policy can and does affect the natural rate of unemployment. Blanchard and Summers (1988) argue that anything (e.g. a sustained increase in real interest rates) that increases the actual rate of unemployment for sufficiently long is likely to raise the natural rate. Blanchard (2003), moreover, explores several mechanisms in which the real interest rate may affect the natural rate of unemployment. He points out, for instance, that the capital accumulation mechanism plays an important role in accounting for the history of unemployment in Europe over thirty years:

> "Low real interest rates in the 1970s probably partly mitigated the increase in labor costs on profit, limiting the decline in capital accumulation, and thus limiting the increase in the natural rate of unemployment in the 1970s. High real interest rates in the 1980s (an then again, as the result of the German monetary policy response to German reunification, in the early 1990s) had the reverse effect of leading to a larger increase in the natural rate of unemployment during that period. an the decrease in real interest rates since the mid-1990s is probably contributing to the slow decline in unemployment in Europe." (Blanchard, 2003)

Based on a microfounded model, Lengwiler (1998) finds that the NAIRU may be nonconstant and can be influenced by the expected inflation: It can

be downward-sloping or upward-sloping with a vertical long-run Phillips curve (a constant NAIRU) being only an exception. Perez (2000) explores why the NAIRU changes over time and finds that the NAIRU may change with the changes of productivity, minimum wage and so on. Tobin (1998) also points out that the NAIRU may vary over time because of the change of the relationships between unemployment, vacancies, and wage changes.

Although it is still questioned whether the NAIRU can be precisely estimated, some economists have tried with different approaches. An earlier survey on the estimation of the NAIRU is given by Staiger, Stock and Watson (1996). Gordon (1997), for example, estimates the time-varying NAIRU with the U.S. data with and without supply shocks. Apel and Jansson (1999) estimate the potential output and NAIRU of Sweden with a system-based strategy.

Following Blanchard (2003), in this section we will explore optimal monetary policy with an endogenous NAIRU which can be affected by the monetary policy. Before exploring the optimal monetary policy with an endogenous NAIRU, we estimate the time-varying NAIRU for several countries with the model of Gordon (1997).

4.1 Estimates of Time-Varying NAIRU

Gordon (1997) estimates the time-varying NAIRU with the following state-space model:

$$\pi_t = a(L)\pi_{t-1} + b(L)(U_t - U_t^N) + c(L)z_t + e_t, \qquad (48)$$

$$U_t^N = U_{t-1}^N + \eta_t, \qquad (49)$$

where π_t is the inflation rate, U_t the actual unemployment rate and U_t^N the NAIRU which follows a random walk path indicated by equation (49). z_t is a vector of supply shock variables, L is a polynomial in the lag operator, e_t is a serially uncorrelated error term and η_t satisfies the Gaussian distribution with mean zero and variance σ_η^2. Obviously, the variance of η_t plays an important role in the estimation. If it is zero, then the NAIRU is constant and if it is positive, the NAIRU experiences changes. If no constraints are imposed on σ_η^2 the NAIRU will jump up and down and soak up all the residual variation in the inflation variation. This is a standard "stochastic time-varying parameter regression model" that can be estimated using maximum likelihood methods with the help of the Kalman filter.[7]

Gordon (1997) includes a z_t to proxy supply shocks such as changes of relative prices of imports and the change in the relative price of food and energy. If no supply shocks are taken into account, the NAIRU is referred to as "estimated NAIRU without supply shocks". Though there are no fixed rules on what variables should be included as supply shocks, it seems more reasonable to take supply shocks into account than not, since there are undoubtedly other variables than the unemployment rate that affect the inflation rate. In this section

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[7]The reader is referred to Hamilton (1994, Ch. 13) for the Kalman filter.
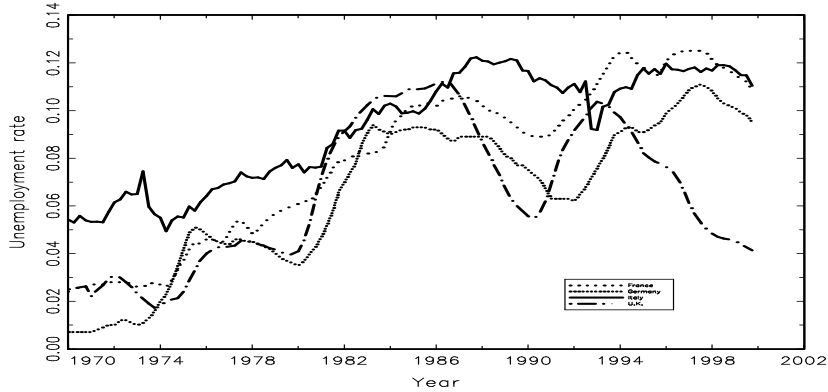
Figure 3: Unemployment Rates of Germany, France, Italy and the U.K.

the supply shocks considered include mainly price changes of imports (im_t), food $(food_t)$, and fuel, electricity and water $(fuel_t)$. As for which variables should be adopted as supply shocks for the individual countries, we undertake an OLS regression for equation (48) before we start the time-varying estimation, assuming that the NAIRU is constant. In most cases we exclude the variables whose t-statistics are insignificant. The data source is the International Statistics Yearbook 2000. As mentioned above, the standard deviation of η_t plays a crucial role. Gordon (1997) assumes it to be 0.2 percent for the U.S. for the period 1955-96. There is little theoretical background on how large η_t should be, but since the NAIRU is usually supposed to be relatively smooth, we constrain the change of the NAIRU within 4 percent, which is also consistent with Gordon (1997). Therefore we assume different values of η_t for different countries, depending on how large we expect the change of the NAIRU to be. The unemployment rates of the four EU countries are presented in Figure 3. The data used in this section are taken from International Statistics Yearbook.

As for Germany, the variance of η_t is assumed to be 7.5×10^{-6} and the price changes of foods, imports, and fuel, electricity and water are taken as supply shocks. The estimates are shown below with t-statistics in parentheses,

$$\pi_t = \underset{(0.056)}{0.004} + \underset{(10.982)}{1.052}\,\pi_{t-1} - \underset{(3.009)}{0.256}\pi_{t-2} + \underset{(0.153)}{0.008}\pi_{t-3} + \underset{(1.105)}{0.013}\,fuel_{t-1}$$
$$+ \underset{(1.994)}{0.061}\,food_{t-1} + \underset{(0.627)}{0.006}\,im_{t-1} - \underset{(0.993)}{0.042}(U_t - U_t^N) + e_t,$$

where $fuel_t$ indicates the price change of fuel, electricity and water, $food_t$ the price change of food and im_t the price change of imports. The estimate of the standard deviation of e_t is 0.006 with t-statistic being 8.506. The time-varying NAIRU of Germany is shown in Figure 4A.

As for France, only one lag of the inflation rate is included in the regression, since the coefficient of the unemployment rate gap tends to zero when more lags
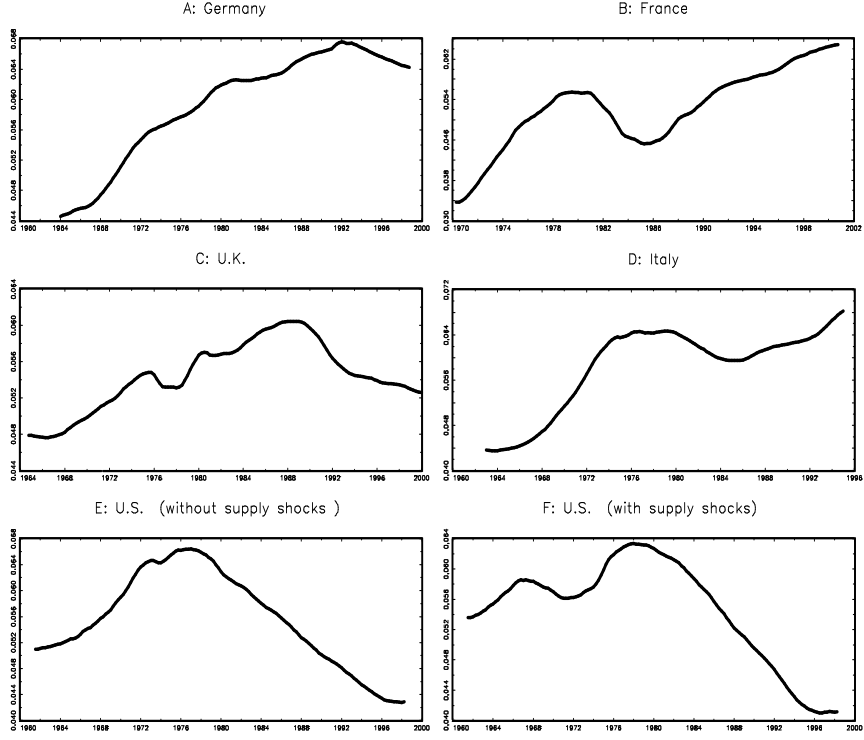
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Figure 4: Time-Varying NAIRU of Germany, France, the U.K., Italy and the U.S.

of the inflation are included. The price changes of food and intermediate goods are taken as supply shocks. Three lags of the price changes of intermediate goods are included to smooth the NAIRU. The result reads as

$$\pi_t = \underset{(0.312)}{0.004} + \underset{(27.967)}{0.989}\,\pi_{t-1} - \underset{(2.058)}{0.085}\,food_{t-1} + \underset{(2.876)}{0.132}\,in_t$$
$$- \underset{(1.313)}{0.090}\,in_{t-1} + \underset{(0.772)}{0.029}\,in_{t-2} - \underset{(1.889)}{0.054}\,(U_t - U_t^N) + e_t,$$

where in_t denotes the price change of intermediate goods. The estimate of the standard deviation of e_t is 0.005 with t-statistic being 8.808, and the variance of η_t is predetermined as 1.3×10^{-5}. The estimate of the NAIRU of France is presented in Figure 4B.

For the same reason as for France, one lag of the inflation rate is included in the regression for the U.K.. The estimation reads

$$\pi_t = \underset{(0.066)}{0.005} + \underset{(11.063)}{0.818}\,\pi_{t-1} + \underset{(2.768)}{0.130}\,food_{t-1}$$
$$+ \underset{(0.691)}{0.017}\,fuel_{t-1} - \underset{(0.300)}{0.072}\,(U_t - U_t^N) + e_t.$$

The estimate of the standard deviation of e_t is 0.013 with t-statistic being 8.764 and the variance of η_t is predetermined as 1.4×10^{-5}. The estimate of the NAIRU of the U.K. is shown in Figure 4C.

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For Italy it seems difficult to get a smooth estimate for the NAIRU if we include only price changes of food, fuel, electricity and water and imports as supply shocks. The main reason seems to be that the inflation rate experienced drastic changes and therefore exerts much influence on the estimate of the NAIRU. Therefore, we try to smooth the estimate of the NAIRU by including the short-term nominal interest rate (nr_t) into the regression, which makes the NAIRU more consistent with the actual unemployment rate. The result is

$$
\pi_t = \underset{(0.001)}{0.0035} + \underset{(6.623)}{1.594}\,\pi_{t-1} - \underset{(4.961)}{0.832}\,\pi_{t-2}
$$
$$
- \underset{(1.722)}{0.247}\,food_{t-1} + \underset{(2.255)}{0.322}\,food_{t-2} - \underset{(1.398)}{0.017}\,fuel_{t-1}
$$
$$
+ \underset{(1.599)}{0.030}\,fuel_{t-2} + \underset{(1.532)}{0.181}\,nr_t - \underset{(0.902)}{0.304}\,(U_t - U_t^N) + e_t.
$$

The estimate of the standard deviation of e_t is 0.010 with t-statistic being 8.982, and the variance of η_t is assumed to be 2.6×10^{-6}. The time-varying NAIRU of Italy is shown in Figure 4D.

We also undertake the estimation of the NAIRU for the U.S. with and without "supply shocks" for 1962.3-1999.4. In the estimation without supply shocks, only four lags of the inflation rate and unemployment gap are included in the regression and the result is

$$
\pi_t = \underset{(0.002)}{0.002} + \underset{(16.045)}{1.321}\,\pi_{t-1} - \underset{(2.199)}{0.243}\,\pi_{t-2}
$$
$$
- \underset{(61.971)}{0.121}\,\pi_{t-3} + \underset{(0.350)}{0.015}\,\pi_{t-4} - \underset{(8.864)}{0.065}\,(U_t - U_t^N) + e_t.
$$

The estimate of the standard deviation of e_t is 0.004 with t-statistic being 15.651 and the variance of η_t is predetermined as 4.5×10^{-6}. The NAIRU of the U.S. without supply shocks is presented in Figure 4E, very similar to the result of Gordon (1997). Considering supply shocks which include price changes in food, energy and imports, we have the following result for the U.S.:

$$
\pi_t = \underset{(0.091)}{0.002} + \underset{(10.592)}{0.957}\,\pi_{t-1} - \underset{(1.350)}{0.151}\,\pi_{t-2}
$$
$$
- \underset{(0.647)}{0.070}\,\pi_{t-3} + \underset{(1.822)}{0.120}\,\pi_{t-4} + \underset{(4.547)}{0.062}\,food_t
$$
$$
+ \underset{(0.468)}{0.007}\,fuel_{t-1} + \underset{(3.808)}{0.025}\,im_{t-1} - \underset{(3.036)}{0.060}\,(U_t - U_t^N) + e_t.
$$

The estimate of the standard deviation of e_t is 0.003 with t-statistic being 16.217 and the variance of η_t is predetermined as 4×10^{-6}. The time-varying NAIRU with supply shocks is shown in Figure 4F.

4.2 Monetary Policy with Endogenous NAIRU

Above we have estimated the time-varying NAIRU with a model proposed by Gordon (1997). As stated before, Blanchard (2003) argues that monetary policy

Parameter	Country			
	Germany	France	U.K.	U.S.
τ_0	0.063 (130.339)	0.051 (24.263)	0.051 (57.513)	0.040 (29.076)
τ_1	0.064 (4.801)	0.085 (1.986)	0.128 (6.222)	0.322 (8.809)
R^2	0.250	0.056	0.356	0.526
Correlation	0.437	0.225	0.337	0.387

Table 1: Regression Results of eq. (50) and Correlation Coefficients of $\bar{r}$ (Computed with the Ex Post Real Rate) and the NAIRU

can and does affect the natural rate of unemployment. Therefore, the problem to tackle next is how monetary policy affects the NAIRU. Taking Europe as example, Blanchard (2003) argues that a tight monetary policy can raise the NAIRU and an expansionary policy may reduce the NAIRU. There seems to exist a positive correlation between the NAIRU and the real interest rate. Therefore below we will analyze the relationship between the NAIRU and the real interest rate. In Table 1 we show the estimation of the following equation from 1982 to the end of the 1990s (T-statistics in parentheses):

$$u_{nt} = \tau_0 + \tau_1 \bar{r}, \tag{50}$$

where $\bar{r}$ denotes the 8-quarter (backward) average of the real interest rate.[8] The real interest rate is defined as the short-term nominal rate minus the actual inflation rate. The reason that we use the 8-quarter backward average of the real rate for estimation is that some researchers argue that the NAIRU is usually affected by the lags of the real rate. The reason that the regression is undertaken only for the period after 1982 is that in the 1970s and at the beginning of the 1980s these countries experienced large fluctuations in the inflation and therefore the real rate also experienced large changes. In Table 1 we find that τ_1 is significant enough. We also show the correlation coefficients of the NAIRU and $\bar{r}$ for the same period in Table 1.

The real rate above is defined as the gap between the nominal rate and the actual inflation. The real rate defined in this way is usually referred to as the ex post real rate. According to the Fisher equation, however, the real rate should be defined as the nominal rate (nr_t) minus the expected inflation, that is

$$r_t = nr_t - \pi_{t-1|t}, \tag{51}$$

where $\pi_{t-1|t}$ denotes the inflation rate from $t-1$ to t expected by the market at time $t-1$. The real rate defined above is usually called the ex ante real rate. How to measure $\pi_{t-1|t}$ is a problem. Blanchard and Summers (1984),

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[8] The interest rates of Germany, France, the U.K. and the U.S. are the German call money rate, 3-month interbank rate, 3-month treasury bill rate and the Federal funds rate respectively. Data source: International Statistics Yearbook.

	Country			
Parameter	Germany	France	U.K.	U.S.
τ_0	0.063 (162.748)	0.050 (22.663)	0.051 (59.251)	0.040 (31.278)
τ_1	0.073 (6.817)	0.097 (2.117)	0.131 (6.337)	0.329 (9.579)
R^2	0.410	0.065	0.371	0.574
Correlation	0.640	0.256	0.609	0.758

Table 2: Regression Results of eq. (50) and Correlation Coefficients of $\bar{r}$ (Computed with the Ex Ante Real Rate) and the NAIRU

for example, measure the expected inflation by an autoregressive process of the inflation rate. Below we will measure the expected inflation by assuming that the economic agents forecast the inflation by learning through the recursive least squares. Namely, we assume

$$\pi_{t-1|t} = c_{0t-1} + c_{1t-1}\pi_{t-1} + c_{2t-1}y_{t-1}, \tag{52}$$

where y_t denotes the output gap. This equation implies that the agents predict the inflation next period by adjusting the coefficients c_0, c_1 and c_2 period by period. Following Sargent (1999), Orphanides and Williams (2002), Evans and Honkapohja (2001) and Zhang and Semmler (2003) we assume that the coefficients evolve in the manner of the recursive least squares

$$C_t = C_{t-1} + t^{-1}V_t^{-1}X_t(\pi_t - X_t'C_{t-1})$$
$$V_t = V_{t-1} + t^{-1}(X_tX_t' - V_{t-1})$$

with $C_t = (c_{0t}\ c_{1t}\ c_{2t})'$ and $X_t = (1\ \pi_{t-1}\ y_{t-1})'$. V_t is the moment matrix of X_t.[9] With the output gap measured by the percentage deviation of the industrial production index (IPI) from its HP filtered trend and the $\pi_{t-1|t}$ computed by the equations above, we show the estimation results for eq. (50) with the ex ante real rate and the correlation coefficients between the NAIRU and $\bar{r}$ computed with the ex ante real rate in Table 2.[10] The results in Table 2 are not essentially different from those in Table 1 except that the correlation coefficients between the NAIRU and $\bar{r}$ in Table 2 are larger than those in Table 1.

The ex ante- and ex post real rates of the U.S. from 1981:1 to 1999:4 are shown in Figure 5. It is obvious that the two rates are not significantly different.

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[9]The reader can refer to Harvey (1981, ch. 7) and Sargent (1999) for the recursive least squares.

[10]The IPI has also been used by Clarida, Gali and Gertler (1998) to measure the output for Germany, France, the U.S., the U.K., Japan and Italy. As surveyed by Orphanides and van Norden (2002), there are many methods to measure the output gap. We find that filtering the IPI using the Band-Pass filter developed by Baxter and King (1995) leaves the measure of the output gap essentially unchanged from the measure with the HP-filter. The Band-Pass filter has also been used by Sargent (1999).
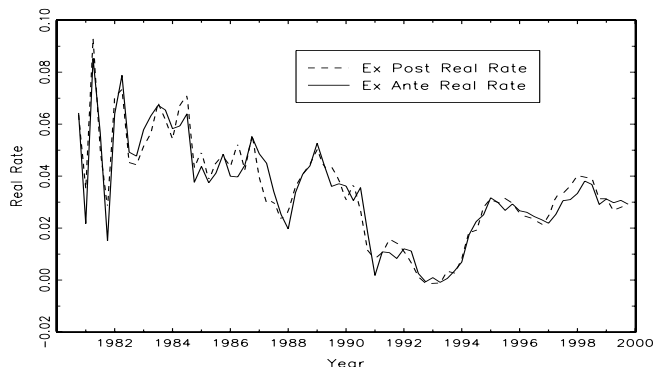
Figure 5: The Ex Ante- and Ex Post Real Rates of the U.S.

The empirical evidence above indicates that there exists a positive relationship between the real rate and the NAIRU. The question then is whether the path of the NAIRU is affected by r in a linear manner similar to that of the actual unemployment given by (2) or in a nonlinear way. In the research below we assume that the the path of the NAIRU can be affected by the monetary policy in a nonlinear manner because of adjustment costs in the investment. Exactly speaking, we assume

$$\dot{u}_n = g(r) = a \tanh(br). \tag{53}$$

We know that $\tanh(x)$ is a function with upper and lower bounds 1 and -1 respectively. It equals zero when $x = 0$. The parameter a scales the function and b determines the slope of $g(r)$ around 0, the larger b is, the steeper the function is around 0. In case b is relatively small, $g(r)$ tends to be linear. Two simulations of $g(r)$ with $a = 0.6$ and different b (1.2 and 0.2) are shown in Figure 6. This model implies that the change of the NAIRU increases with the increase of r and vice versa, but not proportionally. It increases or decreases faster when r is close to zero than when r is far from zero. The change of the NAIRU stops to increase or decrease in case r tends to positive or negative infinity. The reason for the nonproportional change of the NAIRU to the monetary policy is that there exist adjustment costs. Since the 1960s economists have been studying the implication of adjustment costs on the dynamic investment. An earlier study can be found in Jorgenson (1963). He derives the optimal path of the capital stock by assuming an exogenously given output. Lucas (1967) suggests that the adjustment costs can be thought of as a sum of purchase costs and installation costs which are internal to the firm. Gould (1968) designs a quadratic function of adjustment costs and explores their effects on investment of the firm. Some other research on adjustment costs and their effects on the investment can be found in Eisner and Stroz (1963), Treadway (1969), and Feichtinger, Hartl, Kort and Wirl (2001). Adjustment costs prevent firms from increasing or decreasing the investment proportionally with the decrease or increase of the interest rate.
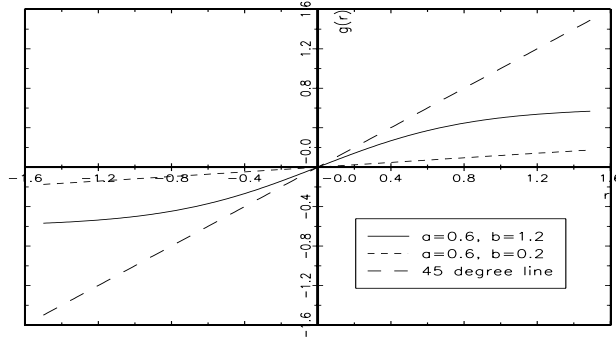
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Figure 6: g(r) with Different b

As a result, the aggregate demand does not decrease or increase proportionally with the increase or decrease of the interest rate and therefore the change of the NAIRU does not increase or decrease proportionally with the interest rate either.[11]

We do not consider the effects of the adjustment costs on the actual unemployment, because there may exist idle capacity and the actual unemployment is supposed to be more affected by the capacity utilization than the NAIRU, which can be considered as the trend of the actual unemployment and may be mainly affected by the investment behavior.

Assuming that the path of the NAIRU can be affected by r as shown in (53), what is then the optimal monetary policy? Next we explore this problem with the model of a linear Phillips curve as given in Section 2, since it is more complicated to analyze the model with the nonlinear Phillips curve. With $\dot{u}_n = g(r)$ eq.(2) changes to

$$\dot{\mu} = \dot{u} - \dot{u}_n = \alpha r - g(r), \tag{54}$$

and the problem of the central bank with expectation taken into account in the Phillips curve reads

$$\underset{r}{Min} \int_0^\infty e^{-\rho t} L dt, \tag{55}$$

subject to

$$\dot{\pi}_e = -\kappa(\beta r + \theta\mu), \tag{56}$$

$$\dot{\mu} = \alpha r - g(r), \tag{57}$$

with L given by (16). The current-value Hamiltonian of the problem above

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[11]Of course $g(r)$ does not have to be symmetric around the origin (0, 0) and we can modify $g(r)$ by using some parameters so that it becomes asymmetric around (0, 0). We will not do so below because the results are not essentially changed no matter whether $g(r)$ is symmetric or not around the origin.

reads

$$H_c = (\pi_e - \beta r - \theta\mu)^2 + \lambda\mu^2 - \gamma_1\kappa(\beta r + \theta\mu) + \gamma_2[\alpha r - g(r)]. \qquad (58)$$

From the first-order condition (20) we know the optimal monetary policy r is the solution of the following equation

$$-2\beta(\pi_e - \beta r - \theta\mu) - \gamma_1\kappa\beta + \gamma_2[\alpha - g'(r)] = 0. \qquad (59)$$

Let $\check{r}(\Omega)$ denote the solution of r from the equation above, with Ω denoting the set of parameters and the variables μ, π_e, γ_1 and γ_2. Following (21)–(24) and (59) we obtain the following dynamic system

$$\dot{\pi}_e = -\kappa[\beta\check{r} + \theta\mu], \qquad (60)$$
$$\dot{\mu} = \alpha\check{r} - g(\check{r}), \qquad (61)$$
$$\dot{\gamma}_1 = -2(\pi_e - \beta\check{r} - \theta\mu) + \rho\gamma_1, \qquad (62)$$
$$\dot{\gamma}_2 = 2\theta(\pi_e - \beta\check{r} - \theta\mu) - 2\lambda\mu + \gamma_1\kappa\theta + \rho\gamma_2, \qquad (63)$$
$$0 = -2\beta(\pi_e - \beta\check{r} - \theta\mu) - \gamma_1\kappa\beta + \gamma_2[\alpha - g'(\check{r})]. \qquad (64)$$

Setting $\dot{\pi}_e = \dot{\mu} = \dot{\gamma}_1 = \dot{\gamma}_2 = 0$, we can compute the equilibria of the economy as follows

$$0 = -\kappa[\beta\check{r} + \theta\mu], \qquad (65)$$
$$0 = \alpha\check{r} - g(\check{r}), \qquad (66)$$
$$0 = -2(\pi_e - \beta\check{r} - \theta\mu) + \rho\gamma_1, \qquad (67)$$
$$0 = 2\theta(\pi_e - \beta\check{r} - \theta\mu) - 2\lambda\mu + \gamma_1\kappa\theta + \rho\gamma_2, \qquad (68)$$
$$0 = -2\beta(\pi_e - \beta\check{r} - \theta\mu) - \gamma_1\kappa\beta + \gamma_2[\alpha - g'(\check{r})]. \qquad (69)$$

We will try some numerical solutions since it is difficult to get analytical solutions of this system. Before trying numerical solutions we explore whether there exist solutions of this system, starting with (66), since this equation contains only one variable, $\check{r}$.

Denote $\Theta(\check{r}) = \alpha\check{r} - g(\check{r})$ and let

$$\Theta(\check{r}) = 0. \qquad (70)$$

It is clear that $\check{r} = 0$ is a solution of the above equation, but with some proper values assigned to the parameters, eq. (70) can have other solutions, therefore the dynamic system (60)–(63) may have multiple equilibria.[12]

In Figure 7 we show two simulations of $\Theta(\check{r})$ with different parameters. It is clear that with $a = 0.6$, $b = 0.2$ and $\alpha = 0.5$ $\Theta(\check{r})$ cuts the horizontal axis once, but with $a = 0.6$, $b = 1.2$ and $\alpha = 0.5$ $\Theta(\check{r})$ cuts the horizontal axis three times, therefore there may exist two other equilibria in the dynamic system (60)–(63) besides the one with $r = 0$.

---

[12]As stated before, we do not consider the effects of adjustment costs on the path of the actual unemployment. But even if r can affect the path of the actual unemployment in a way similar to (53) instead of (2), the results of multiple equilibria should not be essentially changed. If we assume, for example, $\dot{u} = \varrho(r)$ with $\varrho(r)$ defined similarly to $g(r)$ with different parameters. The two functions $g(r)$ and $\varrho(r)$ can then cut each other once or more times, corresponding to one or multiple equilibria of the optimal control problem.
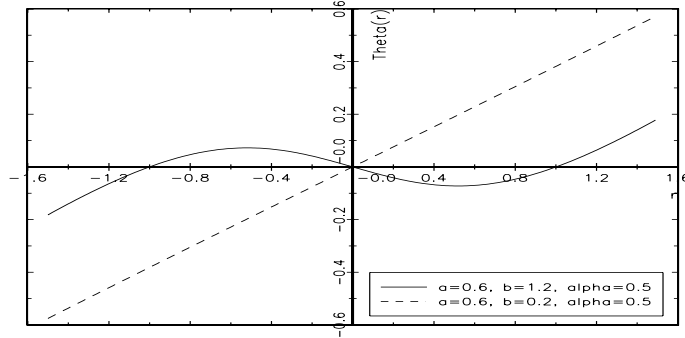
Figure 7: $\Theta(\check{r})$ with Different Sets of Parameters

In order to explore whether the multiple equilibria are robust to parameters chosen, we will try numerical solutions of the system (65)–(69) with a different set of parameters from those employed in Figure 7. Let $a = 0.6$, $b = 6$, $\alpha = 0.75$, $\kappa = 0.9$, $\beta = 0.6$, $\theta = 0.5$, $\rho = 0.1$ and $\lambda = 0.0075$, the three sets of equilibria, a, b and c are given as follows:[13]

$$r_a^* = 0.800, \mu_a^* = -0.960, \pi_{ea}^* = -0.0012, \gamma_{1a}^* = -0.025, \gamma_{2a}^* = -0.0199,$$

$$r_b^* = -0.800, \mu_b^* = 0.960, \pi_{eb}^* = 0.0012, \gamma_{1b}^* = 0.025, \gamma_{2b}^* = 0.0199,$$

and

$$r_c^* = 0, \mu_c^* = 0, \pi_{ec}^* = 0, \gamma_{1c}^* = 0, \gamma_{2c}^* = 0.$$

Substituting the π_e^*, μ^* and r^* into eq. (14) we can observe at least three equilibria values of π, $\pi_a^* = -0.0012$, $\pi_b^* = 0.0012$ and $\pi_c^* = 0$. Therefore we have at least three steady states for inflation and unemployment, $(-0.0012, -0.960)$, $(0.0012, 0.960)$ and $(0, 0)$.[14]

Yet, we can hardly explore the stability of the equilibria because we can not compute the Jacobian matrix without an explicit expression of $\check{r}$. Therefore we will numerically compute the optimal control problem (55)–(57) with the algorithm developed by Grüne (1997) with the parameters given above. Grüne (1997) uses the Bellman equation and dynamic programming to compute numerically the optimal control problem with adaptive grids.[15]

The numerically obtained value function (VF) using the Grüne algorithm is shown in Figure 8, in which we observe that the value function is not smooth

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[13]Because the highly nonlinear system (65)-(69) is solved numerically, there might exist other equilibria which are not detected. Therefore the model may have more than three equilibria.

[14]As above mentioned, given the strong nonlinearities in our functions there might exist other steady state equilibria as well.

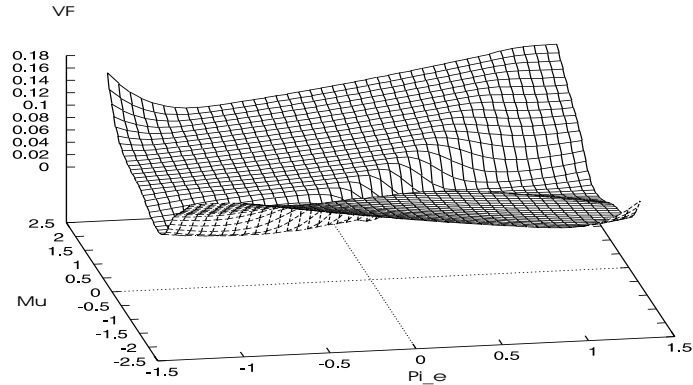[15]For further details of this algorithm see Grüne and Semmler (2003).

Figure 8: Numerical Computation of the Value Function

at the bottom. The vector field of the state variables with π_e on the horizontal axis and μ on the vertical axis is shown in Figure 9. In Figure 9 we indeed can observe three equilibria. As mentioned before, there may exist other equilibria which are difficult to observe in the vector field. In Figure 10 we show the vector field for the state variables with $\lambda = 0.5$, $a = 0.3$ and $b = 2$ and other parameters unchanged and find that there exists a unique equilibrium $(0, 0)$. In Figure 11 we show the optimal trajectories of π_e and μ with different initial values. The equilibria $(-0.0012, -0.960)$ and $(0.0012\ 0.960)$ are stable. Note, however, that in the middle there might exist other fixed points besides $(0, 0)$. Numerically we can observe that the point $(0, 0)$ is unstable, but there seem to be two limit cycles close to $(0, 0)$ which can be detected with some initial values of π_e and μ close to $(0, 0)$. In Figure 12 we show the optimal trajectories of π_e and μ with initial values $(0.0001, 0.0001)$ and $(-0.0002, -0.0011)$. We observe indeed two limit cycles.

Based on some empirical evidence and assuming that the monetary policy affects the path of the NAIRU, in this section we have explored monetary policy with an endogenous NAIRU and find that there may exist multiple equilibria for such type of models. As stated before, Blanchard (2003) remarks that the monetary policy does and can affect the NAIRU. On the other hand, some researchers, Gordon (1997) and Stiglitz (1997), for example, maintain that the NAIRU has some feedback effects on the macroeconomic policy (monetary policy for instance) in the sense that it faces a moving target. Although we know that there are other forces affecting the NAIRU, a model of monetary policy with an exogenous NAIRU can probably not be considered a proper device to study monetary policy effects. Future research is likely to explore more extensively the way how monetary policy affects the NAIRU.
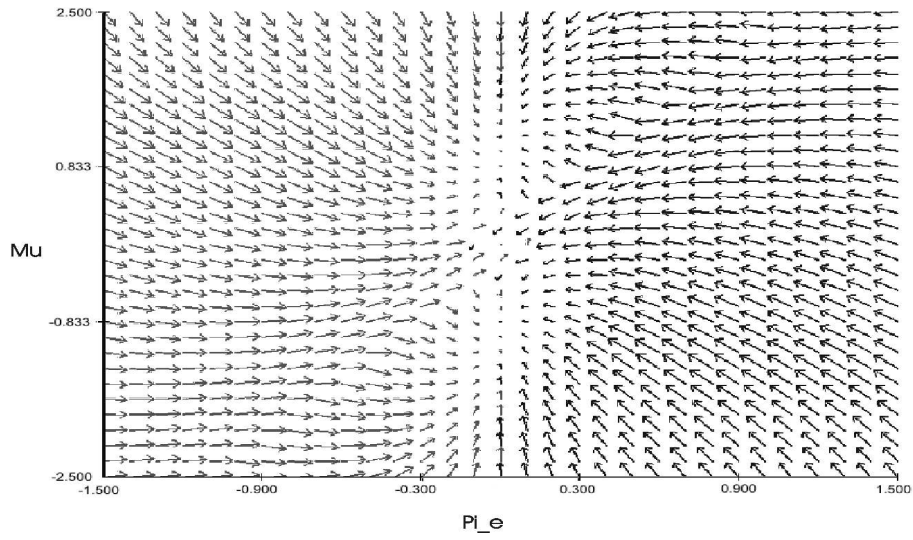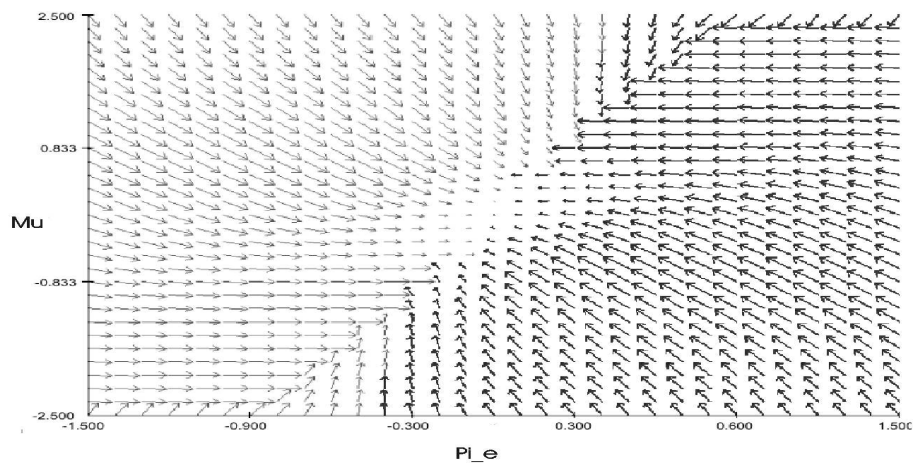
Figure 9: Vector Field with Multiple Equilibria

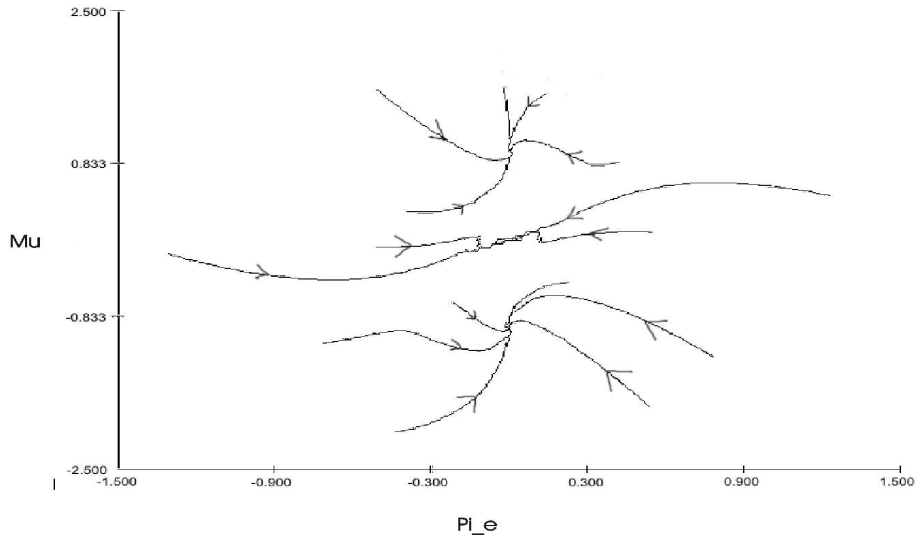

Figure 10: Vector Field with Unique Equilibrium

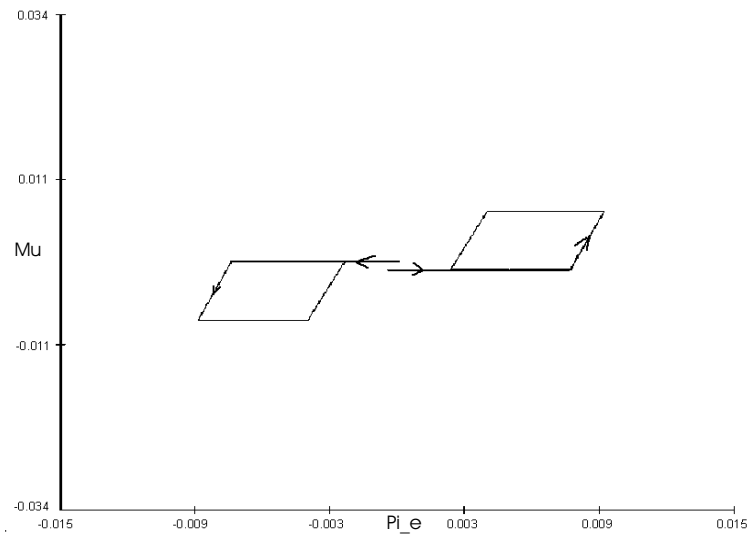Figure 11: Optimal Trajectories of π_e and μ

Figure 12: Limit Cycles with Initial Values Close to $(0,0)$

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5   Conclusion

This paper is concerned with the optimal monetary policy with nonlinear Phillips curves and an endogenous NAIRU. Before exploring the optimal monetary policy with a nonlinear Phillips curve we have studied the optimal monetary policy in a simple model with a linear Phillips curve and find that the optimal monetary policy is a linear function of the unemployment gap and that there exists a unique equilibrium, namely a saddle point. We have then explored the optimal monetary policy with two different shapes of the nonlinear Phillips curve: Convex and convex-concave. We find that the optimal monetary policy changes with the shape of the Phillips curve, but there exists a unique equilibrium (a saddle point) despite of the different nonlinearities of the Phillips curve. Based on this result, we have then relaxed the traditional assumption of a constant and exogenous NAIRU. Recent literature has also proposed that the NAIRU can be influenced by the monetary policy. With some empirical evidence and assuming that the NAIRU can besides other forces be influenced by the monetary policy in a certain manner, we find that there may exist multiple steady state equilibria in the economy, different from the results of models with a constant NAIRU.

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