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Keynesian Dynamics and the Wage Price Spiral. A Baseline Disequilibrium Approach

by

Toichiro Asada, Pu Chen, Carl Chiarella
and Peter Flaschel

University of Bielefeld
Department of Economics
Center for Empirical Macroeconomics
P.O. Box 100 131

33501 Bielefeld, Germany

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Toichiro Asada
Faculty of Economics
Chuo University, Tokyo
Japan

Pu Chen
Faculty of Economics
Bielefeld University, Bielefeld
Germany

Carl Chiarella
School of Finance and Economics
University of Technology, Sydney
Australia

Peter Flaschel
Faculty of Economics
Bielefeld University
Germany

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Abstract:

We reformulate and extend the standard AS-AD growth model of the Neoclassical Synthesis (Stage I) with its traditional microfoundations. The model still has an LM curve in the place of a Taylor interest rate rule, exhibits sticky wages as well as sticky prices, myopic perfect foresight of current inflation rates and adaptively formed medium run expectations concerning the investment and inflation climate in which the economy is operating. The resulting nonlinear 5D model of labor and goods market disequilibrium dynamics avoids striking anomalies of the standard model of the Neoclassical synthesis (Stage I). It exhibits instead Keynesian feedback dynamics proper with in particular asymptotic stability of its unique interior steady state for low adjustment speeds and with cyclical loss of stability – by way of Hopf bifurcations – when some adjustment speeds are made sufficiently large, even leading to purely explosive dynamics sooner or later. In such cases downward money wage rigidity can be used to make the dynamics bounded and thus viable. In this way we obtain and analyze a baseline DAS-AD model with Keynesian feedback channels whose rich set of stability features is the source of business cycle fluctuations. These outcomes of the model stand in contrast to those of the currently fashionable New Keynesian alternative (the Neoclassical Synthesis, Stage II) that we suggest is more limited in scope.

Keywords: DAS-AD growth, wage and price Phillips curves, real interest effects, real wage effects, (in)stability, persistent business cycles, inflation and deflation.

JEL CLASSIFICATION SYSTEM: **E24**, **E31**, **E32**.

1 Introduction

In this paper we reformulate and extend[1] the standard AS-AD growth dynamics of the Neoclassical Synthesis (Stage I) with its traditional microfoundations, as it is for example treated in detail in Sargent (1987, Ch.5). However our extension has not yet replaced the LM curve with a now standard Taylor rule, as is done in the New Keynesian approaches. The model has sticky wages as well as sticky prices, underutilized labor as well as capital stock, myopic perfect foresight of current wage and price inflation rates and adaptively formed medium run expectations concerning the investment and inflation climate in which the economy is operating. The resulting nonlinear 5D model of labor and goods market disequilibrium dynamics (with a traditional LM treatment of the financial part of the economy) avoids the striking anomalies of the conventional model of the Neoclassical synthesis, stage I. [2] Instead it exhibits Keynesian feedback dynamics proper with in particular asymptotic stability of its unique interior steady state solution for low adjustment speeds of wages, prices, and expectations. The loss of stability occurs cyclically, by way of Hopf bifurcations, when these adjustment speeds are made sufficiently large, even leading eventually to purely explosive dynamics.

Locally we thus obtain and prove the existence in general of damped, persistent or explosive fluctuations in the rates of capacity utilization of both labor and capital, and of wage and price inflation rates accompanied by interest rate fluctuations that (due to the conventional working of the Keynes-effect) move in line with the goods price level. Our modification and extension of traditional AS-AD growth dynamics, as investigated from the orthodox point of view in Sargent (1987), thus provides us with a Keynesian theory of the business cycle. This is so even in the case of myopic perfect foresight, where the structure of the traditional approach dichotomizes into independent supply-side real dynamics – that cannot be influenced by monetary policy – and a subsequently determined inflation dynamics that are purely explosive if the price level is taken as a predetermined variable. In our new type of Keynesian labor and goods market dynamics we therefore can treat myopic perfect foresight of both firms and wage earners without the need for the methodology of the rational expectations approach to unstable saddlepoint dynamics.

If this model loses asymptotic stability it does so in a cyclical fashion, by way of so-called Hopf-bifurcations, which may give rise to persistent fluctuations around the steady state. However, this particular loss of stability (generated if some of the speed of adjustment parameters become sufficiently large) is only of a local nature, since eventually purely explosive behavior is the generally observed outcome, as can be checked by means of numerical simulations. The considered model type therefore cannot be considered as being complete in such circumstances, since some mechanism is required to bound the fluctuations to economically viable regions. Downward money wage rigidity is the mechanism we use for this purpose. Extended in this way we therefore obtain and study a

[1]The essential idea of the model presented here was first proposed in Chiarella, Flaschel, Groh and Semmler (2003) in a short response to the comments of Velupillai (2003)on our earlier work. Due to brevity the model could not be investigated there.

[2]These anomalies include in particular saddle point dynamics that imply instability unless some poorly motivated jumps are imposed on certain variables, here on both the price and the wage level.

baseline model of the DAS-AD variety with a rich set of stability implications for the various types of business cycle fluctuations that it can generate.

The dynamic outcomes of this baseline disequilibrium AS-AD model can be usefully contrasted with those of the currently fashionable microfounded New Keynesian alternative (the Neoclassical synthesis, stage II) that in our view is more limited in scope, at least as far as interacting feedback mechanisms and thereby implied dynamic possibilities are concerned. This comparison reveals in particular that one does not always end up with the typical (in our view strange) dynamics of rational expectation models, due to certain types of forward looking behavior, if such behavior is coupled with plausible backward looking behavior for the medium-run evolution of the economy. Furthermore, our dual Phillips curves approach to the wage price spiral indeed performs quite well [3], when estimated empirically and in particular does not give rise to the situation observed for the New (Keynesian) Phillips curve, found to be completely at odds with the facts in the literature [4]. In our approach standard Keynesian feedback mechanisms are coupled with a wage price spiral having a considerable degree of inertia, with the result that these feedback mechanisms work as is known from partial analysis in their interaction with the added wage and price level dynamics.

In the next section we briefly reconsider the fully integrated Keynesian AS-AD model of the Neoclassical Synthesis, stage I, and show that it gives rise to an implausible real/nominal dichotomy – with an appended nominal dynamics of purely explosive type – when operated under myopic perfect foresight with respect to the price rate of inflation. Furthermore, money wage levels must then be allowed to jump just as the price level, despite the presence of a conventional money wage Phillips curve, in order to overcome the observed nominal instability by means of the rational expectations solution methodology. We conclude from this that this model type is not suitable for a Keynesian approach to economic dynamics. In section 3 we then briefly discuss the New Keynesian approach to economic dynamics and find there too, that it raises more questions than it helps to answer. Section 4 therefore proposes a new and nevertheless traditional approach to Keynesian dynamics proper, by taking note of the empirical facts that both labor and capital can be under- or overutilized, that both wages and prices can be sticky and that there are certain climate expressions surrounding the current state of the economy which add sufficient inertia to the considered dynamics.

The resulting 5D model type is analyzed with respect to its stability features in section 5 and shown to give rise to local asymptotic stability when certain Keynesian feedback chains – to some extent well-known to be destabilizing from a partial perspective – are made sufficiently weak, including a real wage adjustment mechanism that is not so well established in the literature. The presented informal analysis is made rigoros in an appendix to this paper, where the calculation of the Routh-Hurwitz conditions for the involved Jacobians is considered in great detail and where the occurrence of Hopf bifurcations, i.e., in particular cyclical loss of stability is also proved. Section 6 of the

[3] See Flaschel and Krolzig (2004), Flaschel, Kauermann and Semmler (2004) and Chen and Flaschel (2004).

[4] In this connection, see for example Mankiw (2001) and with much more emphasis Eller and Gordon (2003), whereas Gali, Gertler and Lopez-Salido (2003) argue in favor of a hybrid form of the New Phillips Curve.

paper concludes and provides an outlook on numerical and empirical analyses of the model of this paper to be undertaken in two companion papers to the present one.

2 Traditional AS-AD under myopic perfect foresight. The 'rational expectations' supply side solution

In this section we briefly discuss the traditional AS-AD growth dynamics with prices set equal to marginal wage costs, and nominal wage inflation driven by an expectations augmented Phillips curve. Introducing myopic perfect foresight (i.e., the assumption of no errors with respect to the short-run rate of price inflation) into such a Phillips curve will alter the dynamics implied by the model in a radical way, in fact towards a globally stable neoclassical growth dynamics with real wage rigidity and thus fluctuating rates of under- or over-employment. Furthermore, price level dynamics no longer feed back into these real dynamics and are now unstable in the large. The accepted approach in the literature is then to go on from myopic perfect foresight to 'rational expectations' and to construct a purely foreword looking solution (which incorporates the whole future of the economy) by way of the so-called jump-variable technique of Sargent and Wallace (1973). This represents in our view however not a reasonable solution to the dynamic results obtained in this model type under myopic perfect foresight, as we shall show in this paper.

The case of myopic perfect foresight in a dynamic AD-AS model of business fluctuations and growth has been considered in very detailed form in Sargent (1987, Ch.5). The model of Sargent's (1987, Ch.5) so-called Keynesian dynamics is given by a standard combination of AD based on IS-LM, and AS based on the condition that prices always equal marginal wage costs, plus finally an expectations augmented money wage Phillips Curve or WPC. The specific features that characterize this textbook treatment of AS-AD are that investment includes profitability considerations besides the real rate of interest, that there is not immediately a reduced form PC employed in this dynamic analysis, and most importantly that expectations are rational (i.e., myopic perfect foresight in the deterministic context). Consumption is based on current disposable income in the traditional way, the LM curve is of standard type and there is neoclassical smooth factor substitution and the assumption that prices are set according to the marginal productivity principle and thus optimal from the viewpoint of the firm. These more or less standard ingredients give rise to the following set of equations that determine the statically endogenous variables: consumption, investment, government expenditure, output, interest, prices, taxes, the profit rate, employment and the rate of employment $C, I, G, Y, r, p, T, \rho, L^d, V^l$ and on this basis the dynamically endogenous variables: the capital stock, labor supply and the nominal wage level K, L, w, for which laws of motion are provided in the equations shown below.

$$
\begin{aligned}
C &= c(Y + rB/p - \delta K - T) && (1)\\
I/K &= i(\rho - (r-\pi)) + n, \quad \rho = \frac{Y - \delta K - \omega L^d}{K},\ \omega = \frac{w}{p} && (2)\\
G &= gK, \quad g = \text{ const.} && (3)\\
Y &\overset{IS}{=} C + I + \delta K + G && (4)\\
M &\overset{LM}{=} p(h_1 Y + h_2(r_0 - r)W) && (5)\\
Y &= F(K, L^d) && (6)\\
p &\overset{AS}{=} w/F_L(K, L^d) && (7)\\
\hat{w} &\overset{PC}{=} \beta_w(V^l - \bar{V}^l) + \pi, \quad V^l = L^d/L && (8)\\
\pi &\overset{MPF}{=} \hat{p} && (9)\\
\hat{K} &= I/K && (10)\\
\hat{L} &= n \quad (= \hat{M} \quad \text{for analytical simplicity}) && (11)
\end{aligned}
$$

We make the simplifying assumptions that all behavior is based on linear relationships in order to concentrate on the intrinsic nonlinearities of this type of AS-AD growth model. Furthermore, following Sargent (1987, Ch.5), we assume that $t = (T - rB/p)/K$ is a given magnitude and thus, like real government expenditure per unit of capital, g, a parameter of the model. This excludes feedbacks from government bond accumulation and thus from the government budget equation on real economic activity. We thus concentrate on the working of the private sector with minimal interference from the side of fiscal policy, which is not an issue considered in this paper. The model is fully backed-up by budget equations as in Sargent (1987): pure equity financing of firms, money and bond financing of the government budget deficit and money, bond and equity accumulation in the sector of private households. There is flow consistency if the new inflow of money and bonds is always accepted by private households. Finally, Walras' Law of Stocks and the perfect substitute assumption for government bonds and equities ensure that equity price dynamics remain implicit. The LM–curve is thus the main representation of the financial part of the model, which is therefore still of a very simple type at this stage of its development.

The treatment of the resulting dynamics turns out to be not very difficult. In fact, equations (8) and (9) imply a real–wage dynamics of the type:

$$\hat{\omega} = \beta_w(l^d/l - \bar{V}^l), \quad l^d = L^d/K, l = L/K.$$

From $\dot{K} = I = S = Y - \delta K - C - G$ and $\dot{L} = nL$ we furthermore get

$$\hat{l} = n - (y - \delta - c(y - \delta - t) - g) = n - (1-c)y - (1-c)\delta + ct - g,$$

with $y = Y/K = F(1, l^d) = f(l^d)$.

Finally, by eq. (7) we obtain

$$\omega = f'(l^d), \text{ i.e. }, \quad l^d = (f')^{-1}(\omega) = h(\omega), \quad h' < 0$$

Thus the real dynamics of the model may be expressed by the autonomous 2D dynamical system:

$$\begin{aligned}\hat{\omega} &= \beta_w(h(\omega)/l - \bar{V}^l) \\ \hat{l} &= n - (1-c)\delta - g + ct - (1-c)f(h(\omega))\end{aligned}$$

It is easy to show, see e.g. Flaschel (1993), that this system is well–defined in the positive orthant of the phase space, has a unique interior steady–state, which moreover is globally asymptotically stable in the considered domain. In fact, this is just a Solow (1956) growth dynamics with a real–wage Phillips curve (real wage rigidity) and thus classical underemployment (or over-employment dynamics if $\bar{V}^l < 1!$). There may be a full–employment ceiling in this model type, but this is an issue of secondary importance here.

The unique interior steady state of the considered dynamics is given by

$$\begin{aligned}y_o &= \frac{1}{1-c}[(1-c)\delta + n + g - ct] = \frac{1}{1-c}[n + g - t] + \delta + t \\ l_o^d &= f^{-1}(y_o), \quad \omega_o = f'(l_o^d), \quad l_o = l_o^d/\bar{V}^l \\ m_o &= h_1 y_o, \quad \hat{p}_o = 0, \quad r_o = \rho_o = f(l_o^d) - \delta - \omega_o l_o^d\end{aligned}$$

Keynes' (1936) approach is nearly absent in this type of analysis, which seems to be Keynesian in nature (AS–AD), but which – due to the neglect of short–run errors in inflation forecasting – has become in fact of very neoclassical type. The marginal propensity of consume, the stabilizing element in Keynesian theory, is still present, but neither investment nor money demand plays a role in the real dynamics we have obtained from eq.s (1) – (11). Volatile investment decisions and financial markets are thus simply irrelevant for the real dynamics of this AS–AD growth model when *myopic* perfect foresight on the current rate of price inflation is assumed. What, then, remains for the role of Keynesian "troublemakers", investment efficiency and liquidity preference? The answer again is, in technical terms, a very simple one:

We have for given $\omega = \omega(t) = (w/p)(t)$ as implied by the real dynamics (due to the $I = S$ assumption):

$$(1-c)f(h(\omega)) - (1-c)\delta + ct - g = i(f(l) - \delta - \omega h(\omega) - r + \hat{p}) + n, \quad i.e.$$

$$\hat{p} = \frac{1}{i}[(1-c)f(h(\omega) - (1-c)\delta + ct - g - n] - (f(l) - \delta - \omega h(\omega)) + r = g(\omega, l) + r$$

with an added reduced-form LM-equation of the type

$$r = (h_1 f(h(\omega)) - m)/h_2 + r_0, \quad m = \frac{M}{pK}.$$

The foregoing equations imply

$$\hat{m} = \hat{l}(\omega) - g(\omega, l) - r_o + \frac{m - h_1 f(h(\omega))}{h_2}$$

as the non-autonomous[5] differential equation for the evolution of real money balances in general and as the reduced form representation of the nominal dynamics.[6] Due to this feedback chain, $\hat{m}$ depends positively on the level of m and in traditional approaches the jump–variable technique needs to be implemented in order to tame such explosive nominal processes; see Flaschel (1993), Turnovsky (1995) and Flaschel, Franke and Semmler (1997) for details. Advocates of the jump–variable technique, therefore are led to conclude that investment efficiency and liquidity preference only play a role in appended purely nominal processes and this solely in a stabilizing way, though with accelerating components in the case of anticipated monetary and other shocks. A truly neoclassical synthesis.

By contrast, we believe that Keynesian IS-LM growth dynamics proper (demand driven growth and business fluctuations) must remain intact if (generally minor) errors in inflationary expectations are excluded from consideration in order to simplify the analysis of the dynamical system to be considered. A correctly formulated Keynesian approach to economic dynamics and fluctuating growth should not give rise to a dichotomized system with classical real and IS-LM inflation dynamics, here in fact of the most basic jump variable type, namely

$$\hat{m} = \frac{m - h_1 y_o}{h_2} \quad [\hat{p} = -\frac{(M/K)_o \frac{1}{p} - h_1 y_o}{h_2}],$$

if it is assumed for simplicity that the real part is already at its steady state. This dynamic equation is of the same kind as the one for the Cagan monetary model and can be treated with respect to its forward-looking solution in the same way, as it is discussed in detail for example in Turnovsky (1995, 3.3/4), i.e., the nominal dynamics assumed to hold under the jump-variable hypothesis is then of a very well-known type.

However a first hint that the model is not a consistently formulated one and also not consistently solved is given by the fact that nominal wages must here jump with the price level $p \quad (w = \omega p)$, since the real wage ω is now moving continuously in time according to the derived real dynamics. The level of money wages is thus now capable of adjusting instantaneously, which is in contradiction to the assumption of only sluggishly adjusting nominal wages according to the assumed money wage PC.[7] Furthermore, a properly formulated Keynesian growth dynamics should – besides allowing for un- or over-employed labor – also allow for un- or over- employment of the capital stock, at least in certain episodes. Thus the price level, like the wage level, should adjust somewhat sluggishly; see also Barro (1994) in this regard. We will come back to this observation after the next section which is devoted to new developments in the area of Keynesian dynamics, the so-called New Keynesian approach of the macrodynamic literature.

[5]Non-autonomous since the independent section of the (ω, l) block will feed into the RHS as time function.

[6]Note that we have $g(\omega, l) = -\rho_o$ in the steady state.

[7]See Flaschel (1993) and Flaschel, Franke and Semmler (1997) for further investigations along these lines.

3 New Keynesian AS–AD dynamics. A continuous-time comparison

The baseline model of New Keynesian macrodynamics (see e.g. Gali (2000)) is, when transferred to continuous time, basically given by the following set of equations, if myopic perfect foresight is now assumed with respect to 'next period's' rate of inflation in a non–stochastic environment:

$$\begin{aligned} Y &\overset{IS}{=} Y(r-\hat{p}), \quad Y'<0 \\ r-\hat{p} &\overset{TR}{=} (r-\hat{p})_0 + \beta_{r_1}(\pi-\bar{\pi}) + \beta_{r_2}(Y-\bar{Y}), \end{aligned}$$

where

$$\bar{Y} \overset{NAIRU}{=} Y((r-\hat{p})_0),$$

is the natural rate of employment, and

$$\dot{\pi} \overset{PC}{=} (1-\beta)\pi - \beta_p(Y-\bar{Y}), \quad \pi=\hat{p}.$$

The model consists of an IS–curve, a Taylor interest rate policy rule (TR) and a Phillips curve (PC), centered around the natural level dynamics of output $\bar{Y}$ to which the steady state level of the real rate of interest $(r-\hat{p})_0$ is corresponding.

We have simplified the New Keynesian baseline model further by using the short–term real rate of interest in the IS–curve in the place of a long–term one as in Gali (2000). Furthermore we do not explicitly pay attention to marginal costs and substitution in the formulation of the PC of the model. The IS– and the PC–curve are reduced–form equations with a price–level oriented motivation for the PC–curve (indicated by the subscript p of the adjustment speed). The model is far from being as explicit as the model of the previous section, but can nevertheless be usefully compared with it with respect to its dynamical features and its reliance on the jump-variable technique.

In the TR, here we have explicitly employed inflation- as well as output-gaps (as is often done), and have assumed that these gaps drive a wedge between the actual real and the steady state real rate of interest by the nominal interest rate steering of the monetary authority. For the PC, we see that the discrete time formulation of the New PC of New Keynesian Theory reduces in continuous time to the perfectly foreseen change in the rate of inflation, up to a level term $(1-\beta)\pi$, due to the discount factor on future inflation that is employed in this type of approach. It therefore follows that a positive output gap decelerates inflation in contrast to what more traditional Phillips curves and the data suggest.

The employed TR can be reduced to (by insertion of the IS–curve):

$$\begin{aligned} g(r-\hat{p}) &:= r-\hat{p}-\beta_{r_2}(Y(r-\hat{p})-\bar{Y}) = (r-\hat{p})_0 + \beta_{r_1}(\pi-\bar{\pi}), \text{ i.e.} \\ r-\hat{p} &= g^{-1}((r-\hat{p})_0 + \beta_{r_2}(\pi-\bar{\pi})) =: h(\pi-\bar{\pi}) \end{aligned}$$

which is a well–defined representation, since $g'>0$ holds true. We thus get that the actual real rate of interest is a well–defined and strictly increasing function of the inflation gap, due to the interest rate policy adopted and the working of the goods market. Inserting this result into the New Keynesian PC gives

$$\dot{\pi} = -(1-\beta)\pi - \beta_p(Y(g(\pi - \bar{\pi}) - \bar{Y}) =: k(\pi), \quad k' > 0$$

with $Y' < 0, g' > 0$. This latter equation implies a positive relationship between inflation π and its rate of change if β is chosen sufficiently close to 1 (which is a meaningful assumption for this comparison of the employed discount factor with the parameters that characterize the real and the nominal dynamics).

Again, advocates of the jump–variable technique are able to tame the explosive dynamics suggested by the above law of motion as in the preceding section, but now applied to the rate of inflation π in the place of the price–level p (or real balances m). The price–level is thus no longer allowed to jump in this type of approach (which is very reasonable from the empirical perspective), but only its rate of change π, and this in the usual way, here to its new steady state value in the case of unanticipated shocks, and through in time accelerating adjustment to the new steady–state value in the case of anticipated shocks (which have changed the steady state position of the economy). We thus have isolated from the model the law of motion that drives price inflation, which now has to be used to determine the dynamics of output and interest in addition.

In this New Keynesian dynamics we thus have an evolution of the inflation rate that depends indeed again on the characteristics of the real sector, and that feeds back into this sector according to the interdependent evolution of

$$\begin{aligned} Y &= Y(r - \pi) \text{ and} \\ r - \pi &= h(\pi - \bar{\pi}). \end{aligned}$$

This is clearly an advantage in comparison to the dynamics considered in the preceding section. However, the model of the preceding section is much more explicit and coherent in its structural presentation, in particular with respect to long–run dynamics and the role of wage formation in the employment and investment decisions of firms.

We are fairly skeptical as to whether the New PC – in particular due to its slope – is really an improvement over traditional structural approaches which employ separate equations for wage and price dynamics. We also notice that hybrid approaches with respect to forward and backward looking behavior have now started to receive some attention by researchers in this area. Based on these observations we are thus going to present an alternative approach to New Keynesian macrodynamics that may be characterized still as traditional, but of a mature Keynesian type (but in any case not really as 'new'). This approach contains both forward and backward looking elements and allows for persistence of inflation – without assuming adaptive expectations for the prediction of future short run rates of inflation.

Remark: Assuming a dynamic adjustment rule for the nominal interest rate in the place of the static one of this section, such as for example:

$$\dot{r} \stackrel{TR}{=} \beta_{r_1}(r_o - r) + \beta_{r_2}(\pi - \bar{\pi}) + \beta_{r_3}(Y - \bar{Y}),$$

would imply together with

$$\dot{\pi} \stackrel{PC}{=} (1-\beta)\pi - \beta_p(Y-\bar{Y}), \quad \pi = \hat{p}$$

a 2D saddlepoint dynamics to which the jump-variable technique can again be applied in the usual way (if the interest rate smoothing parameter β_{r_1} is not chosen too large).

4 Keynesian AS-AD Disequilibrium Dynamics: An alternative baseline model

We have already observed that a Keynesian model of aggregate demand fluctuations should (independently of whether justification can be found in Keynes' General Theory) allow for under- (or over-)utilized labor as well as capital in order to be general enough from the descriptive point of view. As Barro (1994) for example observes IS-LM is (or should be) based on imperfectly flexible wages and prices and thus on the consideration of wage as well as price Phillips Curves. This is precisely what we will do in the following, augmented by the observation that medium-run aspects count both in wage and price adjustment as well as in investment behavior, here still expressed in simple terms by the introduction of the concept of an inflation as well as an investment climate. These economic climate terms are based on past observation, while we have model-consistent expectations with respect to short-run wage and price inflation. The modification of the traditional AS-AD model of section 2 that we shall introduce now thus treats expectations in a hybrid way, myopic perfect foresight on the current rates of wage and price inflation on the one hand and adaptive updating of economic climate expressions[8], with exponential weighting scheme here especially, on the other hand.

In light of the foregoing discussion, we assume here two Phillips Curves or PC's in the place of only one. In this way we provide wage and price dynamics separately, both based on a measure of demand pressure $V^l - \bar{V}^l, V^c - \bar{V}^c$, in the market for labor and for goods, respectively. We here denote by V^l the rate of employment on the labor market and by $\bar{V}^l$ the NAIRU-level of this rate, and similarly by V^c the rate of capacity utilization of the capital stock and $\bar{V}^c$ the normal rate of capacity utilization of firms. These demand pressure influences on wage and price dynamics, or on the formation of wage and price inflation, $\hat{w}, \hat{p}$, are here both augmented by a weighted average of cost-pressure terms based on forward looking myopic perfect foresight and a backward looking measure of the prevailing inflationary climate, symbolized by π^m. Cost pressure perceived by workers is thus a weighted average of the currently evolving price inflation $\hat{p}$ and some longer-run concept of price inflation, π^m, based on past observations. Similarly, cost pressure perceived by firms is given by a weighted average of the currently evolving (perfectly foreseen) wage inflation $\hat{w}$ and again the measure of the inflationary climate in which the economy is operating. We thus arrive at the following two Phillips Curves for wage and price inflation, which in this core version of the model are formulated in a fairly symmetric way.

[8] Here with exponential weighting schemes.

Structural form of the wage-price dynamics:

$$\begin{aligned}\hat{w} &= \beta_w(V^l - \bar{V}^l) + \kappa_w \hat{p} + (1-\kappa_w)\pi^m,\\ \hat{p} &= \beta_p(V^c - \bar{V}^c) + \kappa_p \hat{w} + (1-\kappa_p)\pi^m.\end{aligned}$$

Inflationary expectations over the medium run, π^m, i.e., the inflationary climate in which current inflation is operating, may be adaptively following the actual rate of inflation (by use of some exponential weighting scheme), may be based on a rolling sample (with hump-shaped weighting schemes), or on other possibilities for updating expectations. For simplicity of exposition we shall make use of the conventional adaptive expectations mechanism in the presentation of the full model below. Besides demand pressure we thus use (as cost pressure expressions) in the two PC's weighted averages of this economic climate and the (foreseen) relevant cost pressure term for wage setting and price setting. In this way we get two PC's with very analogous building blocks, which despite their traditional outlook will have interesting and novel implications. These two Phillips curves have been estimated for the US-economy in various ways in Flaschel and Krolzig (2004), Flaschel, Kauermann and Semmler (2004) and Chen and Flaschel (2004) and found to represent a significant improvement over single reduced-form Phillips curves, with wage flexibility being greater than price flexibility with respect to demand pressure in the market for goods and for labor, respectively. Note that such a finding is not possible in the conventional framework of a single reduced-form Phillips curve.

Note that for our current version, the inflationary climate variable does not matter for the evolution of the real wage $\omega = w/p$, the law of motion of which is given by:

$$\hat{\omega} = \kappa[(1-\kappa_p)\beta_w(V^l - \bar{V}^l) - (1-\kappa_w)\beta_p(V^c - \bar{V}^c)], \quad \kappa = 1/(1-\kappa_w\kappa_p)$$

This follows easily from the obviously equivalent representation of the above two PC's:

$$\begin{aligned}\hat{w} - \pi^m &= \beta_w(V^l - \bar{V}^l) + \kappa_w(\hat{p} - \pi^m),\\ \hat{p} - \pi^m &= \beta_p(V^c - \bar{V}^c) + \kappa_p(\hat{w} - \pi^m),\end{aligned}$$

by solving for the variables $\hat{w} - \pi^m$ and $\hat{p} - \pi^m$. It also implies the two across-markets or reduced form PC's given by:

$$\begin{aligned}\hat{p} = &\quad \kappa[\beta_p(V^c - \bar{V}^c) + \kappa_p\beta_w(V^l - \bar{V}^l)] + \pi^m,\\ \hat{w} = &\quad \kappa[\beta_w(V^l - \bar{V}^l) + \kappa_w\beta_p(V^c - \bar{V}^c)] + \pi^m,\end{aligned}$$

which represent a considerable generalization of the conventional view of a single-market price PC with only one measure of demand pressure, the one in the labor market. This traditional expectations-augmented PC formally resembles the above reduced form $\hat{p}$-equation if Okun's Law holds in the sense of a strict positive correlation between $V^c - \bar{V}^c$, $V^c = Y/Y^p$ and $V^l - \bar{V}^l$, $V^l = L^d/L$, our measures of demand pressures on the market for goods and for labor. Yet, the coefficient in front of the traditional PC would even in this situation be a mixture of all of the $\beta's$ and $\kappa's$ of the two originally given PC's and thus represent a composition of goods and labor market characteristics.

With respect to the investment climate we proceed similarly and assume that this climate is adaptively following the current risk premium $\epsilon = \rho - (r - \hat{p}))$, the excess of the actual profit rate over the actual real rate of interest (which is perfectly foreseen). This gives[9]

$$\dot{\epsilon}^m = \beta_{\epsilon^m}(\epsilon - \epsilon^m), \quad \epsilon = \rho + \hat{p} - r,$$

which is directly comparable to

$$\dot{\pi}^m = \beta_{\pi^m}(\pi - \pi^m), \quad \pi = \hat{p}.$$

We believe that it is very natural to assume that economic climate expressions evolve sluggishly in view of their observed short-run analogs. It is however easily possible to introduce also forward looking components into the updating of the climate expressions, for example based on the p^* concept of central banks or potential output profitability calculations. The investment function of the model of this section is now given simply by $i(\epsilon^m)$ in the place of $i(\epsilon)$.

We have now covered all modifications needed to overcome the extreme conclusions of the traditional AS-AD approach under myopic perfect foresight as they were sketched in section 2. The model thus now simply incorporates sluggish price adjustment besides sluggish wage adjustment and makes use of certain delays in the cost pressure terms of its wage and price PC and in its investment function. In the Sargent (1987) approach to Keynesian dynamics we have that $\beta_{\epsilon^m}, \beta_{\pi^m}, \beta_p$ are all set equal to infinity and $\bar{U}_c$ set equal to one, which implies that only current inflation rate and excess profitabilities matter for the evolution of the economy and that prices are perfectly flexible so that full capacity utilization, not only normal capacity utilization, is always achieved.

This brings us to one point that still needs definition and explanation, namely the concept of the rate of capacity utilization that we will be using in this approach in the presence of neoclassical smooth factor substitution, but Keynesian over- or under-employment of the capital stock. Actual use of productive capacity is of course defined in reference to actual output Y. As measure of potential output Y^p we associate with actual output Y the profit-maximizing output with respect to currently given wages and prices. Capacity utilization V^c is therefore measured relative to the profit maximizing output level and thus given by[10]

$$V^c = Y/Y^p \quad \text{with} \quad Y^p = F(K, L^p),\ \omega = F_L(K, L^p).$$

where Y is determined from the IS-LM equilibrium block in the usual way. We have assumed in the price PC as normal rate of capacity utilization a rate that is less than one and thus assume in general that demand pressure leads to price inflation, before potential output has been reached, in line what is assumed in the wage PC and demand pressure on the labor market. The idea behind this assumption is that there is imperfect competition on the market for goods so that firms raise prices before profits become zero on the margin.

[9] In our response to Velupillai (2003), see Chiarella, Flaschel, Groh and Semmler (2003), we have used a slightly different expression for the updating of the investment climate, in this regard see the introductory observation in section 6 below.

[10] In intensive form expressions the following gives rise to $V^c = y/y^p$ with $y^p = f((f')^{-1}(\omega))$ in terms of the notation we introduced in section 2.

Sargent (1987, Ch.5) therefore not only has myopic behavior throughout, but also always the perfect – but empirically questionable – establishment of the condition that the price is given by marginal wage costs. This 'limit case' of the dynamic AS-AD model of this section is not a meaningful model, in particular since it is not at all closely related in its dynamic properties to situations of very fast adjustment of prices and climate expressions to currently correctly observed inflation rates and excess profitability.

There is still another motivation available for the imperfect price level adjustment we are assuming instead. For reasons of simplicity, we here consider the case of a Cobb-Douglas production function, given by $Y = K^{\alpha}L^{1-\alpha}$, solely. According to the above we have

$$p = w/F_L(K, L^p) = w/[(1-\alpha)K^{\alpha}(L^p)^{-\alpha}]$$

which for given wages and prices defines potential employment. Similarly, we define competitive prices as the level of prices p_c where

$$p_c = w/F_L(K, L^d) = w/[(1-\alpha)K^{\alpha}(L^d)^{-\alpha}]$$

holds true. From these definitions we get the relationship:

$$\frac{p}{p_c} = \frac{(1-\alpha)K^{\alpha}(L^d)^{-\alpha}}{(1-\alpha)K^{\alpha}(L^p)^{-\alpha}} = (L^p/L^d)^{\alpha}$$

Due to this we obtain from the definitions of L^d, L^p and their implication $Y/Y^p = (L^d/L^p)^{1-\alpha}$ an expression that relates the above price ratio to the rate of capacity utilization as defined in this section:

$$\frac{p}{p_c} = (\frac{Y}{Y^p})^{\frac{-\alpha}{1-\alpha}} \quad \text{or} \quad \frac{p_c}{p} = (\frac{Y}{Y^p})^{\frac{\alpha}{1-\alpha}} = (V^c)^{\frac{\alpha}{1-\alpha}}.$$

We thus get that (for $\bar{V}^c = 1$) upward convergence of the rate of capacity utilization to full capacity utilization is positively correlated with downward convergence of actual prices to their competitive value and vice versa. In particular in the special case $\alpha = 0.5$ we would get as reformulated price dynamics the formula:

$$\hat{p} \;\; = \;\; \beta_p(p_c/p - 1) + \kappa_p\hat{w} + (1-\kappa_p)\pi^m.$$

which to some extent resembles the New Phillips curve of the New Keynesian approach as far as the reflection of demand pressure forces are concerned. Price inflation is thus increasing when competitive prices (and thus marginal wage costs) are above the actual ones and decreasing otherwise (neglecting the cost-push terms for the moment). This shows that our understanding of the rate of capacity utilization in the framework of neoclassical smooth factor substitution is related to demand pressure terms as used in New Keynesian approaches[11] and thus further motivated in its adoption. Actual prices will fall if they are above marginal wage costs to a sufficient degree. However, our approach suggests that actual prices start rising before marginal wage costs are in fact

[11] Though we have shown that it is a price gap, the comparison of actual prices and marginal wage costs, that enters the price PC []see also Powell and Murphy (1997) for a closely related approach, there applied to an empirical study of the Australian economy].

established, i.e. in particular, we have that actual prices are always higher than the competitive ones in the steady state.

We note that the steady state of the now considered Keynesian dynamics is the same as the one of the dynamics of section 2 (with $\epsilon_o^m = 0, V_o^c = \bar{V}^c, V_o^l = \bar{V}^l, y_o^p = y_o/V_o^c, l_o^p = f^{-1}(y_o^p)$ in addition). Furthermore, the dynamical equations considered above have of course to be augmented still by the ones that have remained unchanged by the modifications just considered. The intensive form of all resulting static and dynamics equations is presented in the next section where we start the stability analysis of the baseline model of this section.

The modifications of the AS-AD model of section 2 proposed in the present section imply that it no longer dichotomizes and that the jump-variable technique can no longer be sensibly applied. Instead, the steady state of the dynamics is locally asymptotically stable under conditions that are reasonable from a Keynesian perspective, loses its asymptotic stability by way of cycles (by way of so-called Hopf-bifurcations) and becomes sooner or later globally unstable if (generally speaking) adjustment speeds become too high, i.e., in particular approach the limit case considered section 2.

We no longer have state variables in the model that can be considered as being not predetermined, but in fact can reduce the dynamics to an autonomous system in the five state variables real wage, real balances per unit of capital, full employment labor intensity, and the expressions for the inflation and the investment climate. These and all other dynamic state variables of the model move continuously in time. Thus if the model is subject to explosive forces, it requires extrinsic nonlinearities in economic behavior that would come into affect far off the steady state and bound the dynamics to an economically meaningful domain in the 5D state space. Asada, Chiarella, Flaschel and Hung (2004) provide details of such an approach and its numerical investigation.

Summing up we can state that we have arrived at a model type that is much more complex, but also much more convincing, that the labor market dynamics of the traditional AS-AD dynamics of the Neoclassical Synthesis (Stage I). We now have 5 in the place of only three laws of motion, which incorporate myopic perfect foresight without any significant impact on the resulting Keynesian dynamics. We can handle factor utilization problems both for labor and capital without assuming a fixed propositions technology, i.e., in AS-AD growth with neoclassical smooth factor substitution. We have sluggish wage as well as price adjustment processes with cost pressure terms that are both forward and backward looking, and that allow for the distinction between temporary and permanent inflation shocks. We have a unique interior steady state solution of (one must stress) supply side type, generally surrounded by business fluctuations of Keynesian short-run as well las medium-run type. Our DAS-AD growth dynamics therefore exhibits a variety of features that are much more in line with a Keynesian understanding of the features of the trade cycle than is the case for the conventional modelling of AS-AD growth dynamics.

Taken together the model of this section consists of five laws of motion for real wages, real balances, the investment climate, labor intensity and the inflationary climate:

$$\begin{aligned}
\hat{\omega} &= \kappa[(1-\kappa_p)\beta_w(l^d/l - \bar{V}^l) - (1-\kappa_w)\beta_p(y/y^p - \bar{V}^c)] && (12)\\
\hat{m} &= -\hat{p} - i\epsilon^m && (13)\\
\dot{\epsilon}^m &= \beta_{\epsilon^m}(\rho + \hat{p} - r - \epsilon^m) && (14)\\
\hat{l} &= -i\epsilon^m && (15)\\
\dot{\pi}^m &= \beta_{\pi^m}(\hat{p} - \pi^m) && (16)
\end{aligned}$$

with $\hat{p} = \kappa[\beta_p(y/y^p(\omega) - \bar{V}^c) + \kappa_p\beta_w(l^d/l - \bar{V}^l)] + \pi^m$.

We here already employ reduced-form expressions throughout and consider the dynamics of the real wage, ω, real balances per unit of capital, m, the investment climate ϵ^m, labor intensity, l, and the inflationary climate, π^m on the basis of the simplifying assumptions that natural growth n determines also the trend growth term in the investment function as well as money supply growth. The above dynamical system is to be supplemented by the following static relationships for output, potential output and employment (all per unit of capital) and the rate of interest and the rate of profit:

$$\begin{aligned}
y &= \frac{1}{1-c}[i\epsilon^m + n + g - t] + \delta + t && (17)\\
y^p &= f((f')^{-1}(\omega)), \quad F(1, L^p/K) = f(l^p) = y^p, F_L(1, L^p/K)) = f'(l^p) = \omega && (18)\\
l^d &= f^{-1}(y) && (19)\\
r &= r_o + (h_1 y - m)/h_2 && (20)\\
\rho &= y - \delta - \omega l^d && (21)
\end{aligned}$$

which have to be inserted into them in order to obtain an autonomous system of 5 differential equations that is in a natural or intrinsic way nonlinear. We note however that there are many items that reappear in various equations or are similar to each other implying that stability analysis can exploit a variety of linear dependencies in the calculation of the conditions for local asymptotic stability. This dynamical system will be investigated in the next section in somewhat informal terms and, with modifications, in the section thereafter in a rigorous way.

5 Feedback-guided β-Stability Analysis

In this section we illustrate an important method to prove local asymptotic stability of the interior steady state through partial motivations from the feedback chains that characterize this baseline model of Keynesian dynamics. Since the model is an extension of the standard AS-AD growth model we know from the literature that there is a real rate of interest effect typically involved, first analyzed by formal methods in Tobin (1975), see also Groth (1992). There is the stabilizing Keynes-effect based on activity-reducing nominal interest rate increases following price level increases, which provides a check to further price increases. Secondly, if the expected real rate of interest is driving investment and consumption decisions (increases leading to decreased aggregate demand), there is the stimulating (partial) effect of increases in the expected rate of inflation that may lead to further inflation and further increases in expected inflation under appropriate conditions. This is the so-called Mundell-effect that works opposite to the Keynes-effect, but also through the real rate of interest rate channel as just seen.

The Keynes-effect is the stronger the smaller the parameter h_2 characterizing the interest rate sensitivity of money demand becomes, since the reduced-form LM equation of our model simply reads:

$$r = r_o + (h_1 y - m)/h_2, \quad y = Y/K, m = M/(pK).$$

The Mundell-effect is the stronger the faster the inflationary climate adjusts to the present level of price inflation, since we have

$$\dot{\pi}^m = \beta_{\pi^m}(\hat{p} - \pi^m) = \beta_{\pi^m}\kappa[\beta_p(V^c - \bar{V}^c) + \kappa_p\beta_w(V^l - \bar{V}^l)]$$

and since both rates of capacity utilization depend positively on the investment climate ϵ^m which in turn is driven by excess profitability $\epsilon = \rho + \hat{p} - r$. Excess profitability – as shown – in turn depends positively on the inflation rate and thus on the inflationary climate as the reduced-form price Phillips curve shows in particular.

There is a further potentially (at least partially) destabilizing feedback mechanism as the model is formulated. Excess profitability depends positively on the rate of return on capital ρ and thus negatively on the real wage ω. We thus get – since consumption does not depend on the real wage – that real wage increases depress economic activity (though with the delay that is caused by our concept of an investment climate transmitting excess profitability to investment behavior). From our reduced-form real wage dynamics

$$\hat{\omega} \;\; = \;\; \kappa[(1-\kappa_p)\beta_w(V^l - \bar{V}^l) - (1-\kappa_w)\beta_p(V^c - \bar{V}^c)].$$

we thus obtain that price flexibility should be bad for economic stability due to the minus sign in front of the parameter β_p while the opposite should hold true for the parameter that characterizes wage flexibility. This is a situation as it was already investigated in Rose (1967). It gives the reason for our statement that wage flexibility gives rise to normal and price flexibility to adverse Rose effects as far as real wage adjustments are concerned. Besides real rate of interest effect, establishing opposing Keynes- and Mundell-effects, we thus have also another real adjustment process in the considered model where now wage and price flexibility are in opposition to each other, see Chiarella and Flaschel (2000) and Chiarella, Flaschel, Groh and Semmler (2000) for further discussion of these as well as other feedback mechanisms in Keynesian growth dynamics.

There is still another adjustment speed parameter in the model, the one that determines how fast the investment climate is updated in the light of current excess profitability. This parameter will play no decisive role in the stability investigations that follow, but will become important in the alternative stability analysis to be considered in the next section. In the present stability analysis we will however focus on the role played by $h_2, \beta_w, \beta_p, \beta_{\pi^m}$ in order to provide one example of asymptotic stability of the interior steady state position by appropriate choices of these parameter values, basically in line with the above feedback channels of partial Keynesian macrodynamics.

This adds to the description of the dynamical system (12) – (16) whose stability properties are now to be investigated by means of varying adjustment speed parameters. With the feedback scenarios considered above in mind, we first observe that the inflationary climate can be frozen at its steady state value, here $\pi^m_o = \hat{M} - n = 0$, if $\beta_{\pi^m} = 0$ is assumed. The system thereby becomes 4D and it can indeed be further reduced to 3D if in addition $\beta_w = 0$ is assumed, since this decouples the l-dynamics from the remaining system ω, m, ϵ^m.

We intentionally will consider the stability of these 3D subdynamics – and its subsequent extensions – in informal terms here, reserving rigorous calculations for an alternative scenario to be presented and investigated in the next section. In this way we hope to show to the reader how one can proceed from low to high dimensional analysis in such stability investigations. This method has been already applied to various other often much more complicated dynamical systems, see Asada, Chiarella, Flaschel and Franke (2003) for a variety of typical examples.

Proposition 1:

Assume that the parameters h_2, β_p are chosen sufficiently small and that the κ_w, κ_p parameters do not equal 1. Then: The interior steady state of the reduced 3D dynamical system

$$\begin{aligned}
\hat{\omega} &= -\kappa(1-\kappa_w)\beta_p(y/y^p(\omega) - \bar{V}^c) \\
\hat{m} &= -i\epsilon^m - \kappa\beta_p(y/y^p(\omega) - \bar{V}^c) \\
\dot{\epsilon}^m &= \beta_{\epsilon^m}(\rho + \kappa\beta_p(y/y^p(\omega) - \bar{V}^c) - r - \epsilon^m)
\end{aligned}$$

is locally asymptotically stable.

Sketch of proof: Assuming h_2, β_p sufficiently small gives for the Jacobian J at the steady state the sign structure:

$$J = \begin{pmatrix} - & 0 & - \\ - & 0 & - \\ - & + & - \end{pmatrix}.$$

Furthermore, the entries J_{23}, J_{33} can be made as large as desired by choosing h_2, the carrier of the Keynes-effect, sufficiently small. This immediately implies that all principal minors of order 2 are then nonnegative (their sum a_2 is positive), while trace $J < 0$ is directly visible ($= -a_1$). And for $\det J = -a_3$ one easily gets by way of the linear dependencies present in the Jacobian of the considered 3D dynamics:

$$0 > \det J > -J_{11}J_{23}J_{32}$$

which – taken together – implies that all coefficients a_1, a_2, a_3 of the Routh Hurwitz polynomial are positive and in addition fulfill $a_1a_2 - a_3 > 0$. ■

Proposition 2:

Assume in addition that the parameters β_w is now positive and chosen sufficiently small. Then: The interior steady state of the implied 4D dynamical system (where the law of motion for l has now been integrated)

$$\begin{aligned}
\hat{\omega} &= \kappa[(1-\kappa_p)\beta_w(l^d/l - \bar{V}^l) - (1-\kappa_w)\beta_p(y/y^p - \bar{V}^c)] \\
\hat{m} &= -i\epsilon^m - \kappa[\beta_p(y/y^p - \bar{V}^c) + \kappa_p\beta_w(l^d/l - \bar{V}^l)] \\
\dot{\epsilon}^m &= \beta_{\epsilon^m}(\rho + \kappa[\beta_p(y/y^p(\omega) - \bar{V}^c) + \kappa_p\beta_w(l^d/l - \bar{V}^l)] - r - \epsilon^m) \\
\hat{l} &= -i\epsilon^m
\end{aligned}$$

is locally asymptotically stable.

Sketch of proof: Exploiting the many linear dependencies shown in the considered dynamical system one can easily reduce the right hand side of the Jacobian of the dynamics at the steady state to:

$$\begin{aligned}
\hat{\omega} &= (1-\kappa_p)\beta_w(l^d/l - \bar{V}^l) \\
\hat{m} &= -\beta_p(y/y^p(\omega) - \bar{V}^c) \\
\dot{\epsilon}^m &= \beta_{\epsilon^m}(\rho - r - \epsilon^m) \\
\hat{l} &= -i\epsilon^m
\end{aligned}$$

without any change in the sign of its determinant. Continuing in this way one can then even obtain:

$$\begin{aligned}
\hat{\omega} &= (1-\kappa_p)\beta_w(l_o^d/l - \bar{V}^l) \\
\hat{m} &= -\beta_p(y_o/y^p(\omega) - \bar{V}^c) \\
\dot{\epsilon}^m &= -\beta_{\epsilon^m}\frac{h_1 y_o - m}{h_2} \\
\hat{l} &= -i\epsilon^m,
\end{aligned}$$

again without change in the signs of the determinants to be calculated at each step. The sign of the determinant of the now corresponding Jacobian is however easily shown to be positive. The eigenvalue zero of the situation where the 4D system is considered for $\beta_w = 0$ thus must become negative if the change in β_w is sufficiently small, since the other three eigenvalues must then continue to have negative real parts. ■

Proposition 3:

Assume in addition that the parameters β_{π^m} is now positive and chosen sufficiently small. Then: The interior steady state of the full 5D dynamical *system (where the state variable π^m is now moving)*

$$
\begin{aligned}
\hat{\omega} &= \kappa[(1-\kappa_p)\beta_w(l^d/l-\bar{V}^l)-(1-\kappa_w)\beta_p(y/y^p-\bar{V}^c)] \\
\hat{m} &= -\pi^m - i\epsilon^m - \kappa[\beta_p(y/y^p-\bar{V}^c)+\kappa_p\beta_w(l^d/l-\bar{V}^l)] \\
\dot{\epsilon}^m &= \beta_{\epsilon^m}(\rho+\kappa[\beta_p(y/y^p(\omega)-\bar{V}^c)+\kappa_p\beta_w(l^d/l-\bar{V}^l)]+\pi^m-r-\epsilon^m) \\
\hat{l} &= -i\epsilon^m \\
\dot{\pi}^m &= \beta_{\pi^m}(\kappa[\beta_p(y/y^p(\omega)-\bar{V}^c)+\kappa_p\beta_w(l^d/l-\bar{V}^l)])
\end{aligned}
$$

is locally asymptotically stable.

Sketch of proof: As for proposition 2, by now simply making use of the rows corresponding to the laws of motion for l and m in order to reduce the row corresponding to the law of motion for π^m to the form $(0,0,0,0,-)$, again without change in the sign of the determinants of the accompanying Jacobians. The fifth eigenvalue must therefore change from zero to a negative value if the parameter β_π is made positive (but not too large). ■

We note that – due to the unchanged sign of the calculated determinants – loss of stability can only occur by way of so-called Hopf-bifurcations, i.e. in particular, by way of economic fluctuations.

We observe that the parameters β_p and β_{π^m} have been chosen such that adverse Rose and destabilizing Mundell-effects are both week and accompanied by a strongly stabilizing Keynes-effect. Due to our reliance on the continuity of eigenvalues with respect to parameter changes we however had to choose in addition that also β_w should be sufficiently small. This is possibly not really necessary, since wage flexibility is stabilizing from the partial perspective. Note however that the size of the parameter ϵ^m is not at all restricted in the present approach to $\beta-$stability. This will be different in the stability analysis that follows in the next section.

We finally observe that loss of stability can only occur – according to the above – by way of Hopf-bifurcations, i.e., in particular through the generation of cycles in the real-nominal interactions of the model. Such loss of stability is here possible if prices become sufficiently flexible compared to wage flexibility, leading to an adverse type of real wage adjustment, and if the inflationary climate expression is updated sufficiently fast, i.e., if the system looses the inertia we have built into it to a sufficient degree. These are typical feedback structures of a properly formulated Keynesian dynamics that may give rise to local instability and thus the need to add further extrinsic or behavioral nonlinearities to the model in order to bound the generated business fluctuations. Such issues will be explored in companion papers from the numerical and the empirical perspective, see Asada, Chiarella, Flaschel and Hung (2004) and Chen, Chiarella and Flaschel (2004).

6 Outlook

We have considered in this paper an extension and modification of the traditional approach to AS-AD growth dynamics that allows us to avoid the dynamical anomalies of the neoclassical synthesis, stage I, and also a strange feedback structure of New Keynesian approach, the neoclassical synthesis (stage II), that both arise from the pure dominance of the assumption of perfect foresight within these two frameworks. Conventional wisdom in these approaches then avoids the stability problems of these model types by just assuming global asymptotic stability through the adoption of non-predetermined variables and the application of the so-called jump-variable technique.

This approach of the Rational Expectations School is however much more than just the consideration of rational expectations, but in fact the assumption not only of hyper-perfect foresight coupled with a solution method that avoids all potential instabilities of macrodynamic economic systems. In the present context, this approach would impose the condition that prices – and also nominal wages – must be allowed to jump in a particular way in order to establish by assumption the stability of the investigated dynamics.

By contrast, our alternative approach – which allows for sluggish wage as well as price adjustment, in view of unbalanced labor as well as product markets, and also for certain economic climate variables, representing the medium-run evolution of inflation and profitability differentials – completely bypasses such stability assumptions. Instead it shows in a very detailed way local asymptotic stability under certain assumptions (very plausible from the perspective of a Keynesian theory), cyclical loss of stability when these assumptions are violated (if speeds of adjustment become sufficiently high), and even explosive fluctuations in the case of further increases of the crucial speeds of adjustment of the model. In the latter case extrinsic nonlinearities have to be introduced in order to tame the explosive dynamics as in some of the examples in Chiarella and Flaschel (2000, Ch.6,7).

The stability features of our properly formulated Keynesian dynamics are based on specific interactions of traditional Keynes- and Mundell-effects or real rate of interest effects (here present only in the employed investment function) with so-called Rose or real-wage effects, see Chiarella and Flaschel (2000) for their introduction, which in the present framework simply means that increasing wage flexibility is stabilizing and increasing price flexibility destabilizing, due to the fact that aggregate demand here depends negatively on the real wage (due to the assumed investment function) and due to the extended types of Phillips curves we have employed in our new approach to traditional Keynesian growth dynamics. The interaction of these three effects is what explains the obtained stability results under the not very demanding assumption of myopic perfect foresight and thus gives rise to a traditional type of Keynesian business cycle theory, not at all plagued by the anomalies of the textbook AS-AD dynamics.

The model of this paper will be numerically explored and estimated in two companion papers, Asada, Chiarella, Flaschel and Hung (2004) and Chen, Chiarella, Flaschel and Semmler (2004) in order to analyze in greater depth and also with an empirical background the interaction of the various feedback channels present in the considered dynamics. At that point we will then also make use of Taylor interest rate policy rules in the place of the traditional LM curve so far employed. Our work on related models suggests that the interest rate policy rule may not be sufficient to tame the explosive dynamics in all conceivable cases, and, we will then also make use of nonlinearities such as a kinked money wage Phillips curve – representing downward money wage rigidity – and Blanchard and Katz type (2000) error correction mechanisms in order to make the dynamics viable and thus economically meaningful in the cases where the steady state is a repeller. Taking all this together our general conclusion will be that this framework not only overcomes the anomalies of the Neoclassical Synthesis, stage I, but also provides a coherent alternative to the New Keynesian theory of the business cycle, the Neoclassical Synthesis, stage II, as sketched in Gali (2000).

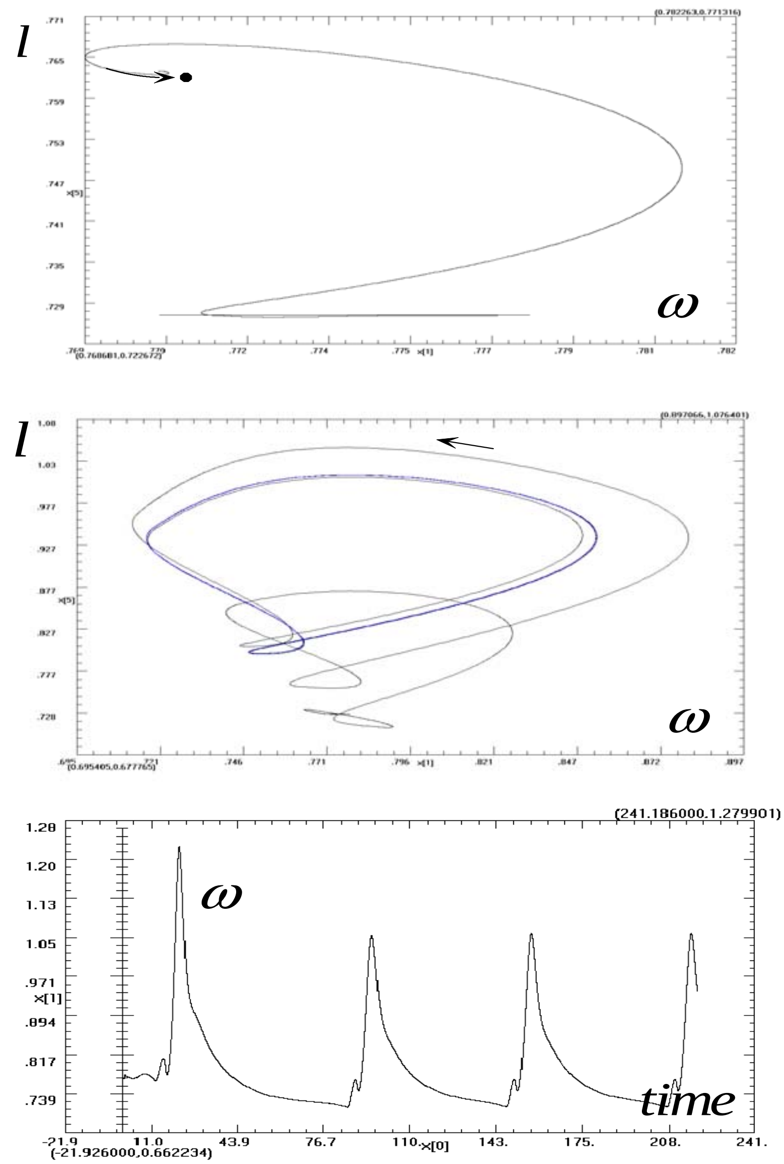

Figure 1: Stable depressions or persistent fluctuations through downwardly rigid money wages.

The figure 1 briefly provides a typical outcome of the dynamics if downwardly rigid money wages are added to an explosive sitution where the economy is not at all a viable one and subject to immediate breakdown without such rigidity. If the money wage Phillips curve is augmented by the assumption that money wages can rise as desribed by it, but cannot fall, we get a situation of a continuum of steady states (for $\hat{M} = n$), due zero root hysteresis, and then the result that the economy converges rapidly to the situation of a stable depression, which depends in its depth on the intitial shock the economy is subject to. If, by contrast, money wages can fall, but will do so at most at the rate of for example 0.01, the steady state remains uniquely determined and – though surrounded by strongly explosive forces – and is not totally unstable, due to the limit cycle situation that is then generated by the operation of the floor to money wage declines. This type of floor makes depressions much longer than recoveries, but avoids that the economy can be trapped in a stable depression. The two situations just discussed are illustrated by the figure 1.

Our alternative Keynesian dynamics is based on disequilibrium in the market for goods and for labor, on sluggish adjustment of prices as well as wages and on myopic perfect foresight interacting with certain economic climate expression – creating the necessary inertia – with a rich array of dynamic outcomes that provide great potential for future generalizations. Some of these generalizations are considered in Chiarella, Flaschel, Groh and Semmler (2000) and Chiarella, Flaschel and Franke (2004). Our overall approach, which may be called a disequilibrium approach to business cycle modelling, provides a theoretical framework within which to consider the contributions of authors such as Zarnowitz (1999), who also stresses the dynamic interaction of many traditional macroeconomic building blocks and the feedback mechanisms they are generating.

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Mathematical Appendix: Rigorous stability analysis

The objective of this section is to consider, on the one hand, a modified version of the DAS-AD dynamics of the preceding section as it was proposed in Chiarella, Flaschel, Groh and Semmler (2003) in a reply to Velupillai (2003) and to demonstrate, on the other hand, in a very detailed and also new way propositions on asymptotic stability and Hopf bifurcations by a different approach to the use of the Routh-Hurwitz conditions for local asymptotic stability. We thereby provide another set of sufficient conditions for such stability which supplement the ones of the preceding section. The result here will be that we now have to choose the parameters $h_1, \beta_w, \beta_{\pi^m}$ sufficiently small and h_2, β_{ϵ^m} in a certain middle range in order to get the stability propositions looked for. Astonishingly enough a condition on the parameter β_p, characterizing price flexibility, can be completely avoided now.

The model of the preceding section is here changed, since we now employ for the formation of the investment climate the law of motion

$$\dot{\epsilon}^m = \beta_{\epsilon^m}(\rho + \hat{p} - r - \epsilon_o^m), \quad \text{in the place of} \quad \dot{\epsilon}^m = \beta_{\epsilon^m}(\rho + \hat{p} - r - \epsilon^m).$$

This however simply means that the weights with which past excess profitability are aggregated are now changed, as can be shown by way of integration of the two laws of motion, since

$$\epsilon^m = \epsilon(t_o)e^{-\beta_{\epsilon^m}(t-t_o)} + \beta_{\epsilon^m}\int_{t_o}^{t} e^{-\beta_{\epsilon^m}(t-s)}\epsilon(s)ds$$

is now simply replaced by:

$$\epsilon^m = \beta_{\epsilon^m}\int_{t_o}^{t}\epsilon(s)ds$$

Instead of an exponential weighting scheme we now use an unweighted aggregate of past observation as measure of the investment climate in which the economy is operating.

6.1 The model

In this section we analyze mathematically the five-dimensional macrodynamic model that is obtained from the structural equations of the preceding section (including the above modification of the model) and that has already been briefly investigated in Chiarella, Flaschel, Groh, and Semmler (2003). This model is represented – in its initial format – by the following sets of algebraic and dynamic equations which appropriately transformed will provide us with an autonomous system of five interdependent differential equations. The local asymptotic stability properties of the model will be investigated in great detail in this section by making use of the fact that the various adjustment speeds of the considered model (in fact $\beta_w, \beta_p, \beta_{\pi^m}, \beta_{\epsilon^m}$) allow to reduce the dynamics to cases (only one is considered here) where the Routh-Hurwitz conditions can be considered and proved explicitly, while the higher dimensional cases are then treated by continuity arguments with respect to the eigenvalues of the full dynamics. This method of proof has been established in Chiarella and Flaschel (2000) and has since then been used in a variety of other cases, see for example Chiarella, Flaschel, Franke and Semmler (2002) for typical examples. We call this approach to the stability investigation of large(r) macrodynamical systems the β-stability method for obvious reasons. Note here also that this proof strategy generally gives rise to Hopf-bifurcations when some $\beta-$adjustment speeds become so large that local stability gets lost, giving rise to persistent fluctuations in such situations.

The static part of the equations is represented as follows.

$$y = [i(\varepsilon^m) + g - t]/s + \delta + t = y(\varepsilon^m) \quad ; \quad y_{\varepsilon^m} = i_{\varepsilon^m}/s > 0 \tag{22}$$

$$r = r_0 + (h_1 y - m)/h_2 = r_0 + (h_1 y(\varepsilon^m) - m)/h_2 = r(\varepsilon^m, m)$$

; $r_{\varepsilon^m} = h_1 y_{\varepsilon^m}/h_2 > 0, \quad r_m = -1/h_2 < 0$ (2)

$$\rho = y - \delta - \omega l^d(y) = y(\varepsilon^m) - \omega l^d(y(\varepsilon^m)) = \rho(\varepsilon^m, \omega) \quad ; \quad l^d = f^{-1}(y) = l^d(y),$$

$$\rho_{\varepsilon^m} = (1 - \omega l^d_y) y_{\varepsilon^m} = \{1 - \omega/f'(l^d)\} y_{\varepsilon^m} > 0, \quad \rho_\omega = -l^d < 0 \tag{23}$$

$$V^l = l^d/l = l^d(y)/l = l^d(y(\varepsilon^m))/l = V^l(\varepsilon^m, l) \quad ;$$

$$V^l_{\varepsilon^m} = l^d_y y_{\varepsilon^m}/l > 0, \quad V^l_l = -l^d/l^2 < 0 \tag{24}$$

$V^c = y/y^p(\omega) = y(\varepsilon^m)/y^p(\omega) = V^c(\varepsilon^m, \omega) \quad ; \quad y^p(\omega)$ given by solving

$$\omega = f'(l^p), \quad y^p = f(l^p) \quad ; \quad V^c_{\varepsilon^m} = y_{\varepsilon^m}/y^p > 0, \quad V^c_\omega = -y y^p_\omega/(y^p)^2 > 0 \tag{25}$$

where $y_{\varepsilon^m} = y'(\varepsilon_m)$, $i_{\varepsilon^m} = i'(\varepsilon^m)$, $r_{\varepsilon^m} = \partial r/\partial \varepsilon^m$, $r_m = \partial r/\partial m$ etc. The meanings of the symbols are as follows:

$y = Y/K$ =actual gross output–capital ratio, $i = I/K = \dot{K}/K$ = rate of net investment (rate of capital accumulation), $g = G/K$ = government expenditure – capital ratio (fixed), $t = T/K$ = tax – capital ratio (fixed), s = marginal propensity to save (fixed, $0 < s < 1$), δ = rate of capital depreciation (fixed, $0 \leqq \delta \leqq 1$), Y = actual real gross output (real gross national income), K = real capital stock, $I = \dot{K}$ = real net investment, G = real government expenditure, T = real tax, ε^m = investment climate, r = nominal rate of interest, $m = M/(pK)$ = real money balance per capital, M = nominal money supply, p = price level, ρ = net rate of profit, $\omega = w/p$ = real wage rate, w = nominal wage rate, $l^d = L^d/K$ = employment – capital ratio, L^d = labor employment, $l = L/K$ = full – employment labor intensity, L = labor supply, V^l = rate of employment, y^p = full capacity gross output-capital ratio, V^c = rate of capacity utilization.

We can derive Eq. (2) as follows. We can express the equilibrium condition for money market as $m = h_1 y + h_2(r_0 - r)$, where the right hand side is the linear real money demand function per capital stock ($h_1 > 0$, $h_2 > 0$, $r_0 > 0$). Solving this equation with respect to r, we have Eq. (2).

It is assumed that output is demand-constrained, i.e., $l^d < l^p$, which means that $f'(l^d) > f'(l^p) = \omega$ because of the assumption of decreasing marginal productivity of labor, $f''(l) < 0$. This is the reason why we have $\rho_{\varepsilon^m} > 0$ in Eq. (23). In this case, we also have $y^p_\omega = f'(l^p)/f''(l^d)$ <0, implying $V^c_\omega > 0$ in Eq. (25). In the short run, ε^m, m, ω and l are given data. Correspondingly, y, r, ρ, V^l, and V^c are determined by the equations (22) – (25).

The <u>dynamic part</u> of the equations is given as follows.

$$\hat{w} = \dot{w}/w = \beta_w(V^l - \bar{V}^l) + \kappa_w \hat{p} + (1-\kappa_w)\pi^m \quad ; \quad \beta_w > 0, \quad 0 < \kappa_w < 1 \tag{26}$$

$$\hat{p} = \dot{p}/p = \beta_p(V^c - \bar{V}^c) + \kappa_p \hat{w} + (1-\kappa_p)\pi^m \quad ; \quad \beta_p > 0, \quad 0 < \kappa_p < 1 \tag{27}$$

$$\dot{\pi}^m = \beta_{\pi^m}(\hat{p} - \pi^m) \quad ; \quad \beta_{\pi^m} > 0 \tag{28}$$

$$\dot{\varepsilon}^m = \beta_{\varepsilon^m}\varepsilon, \quad \varepsilon = \rho - (r - \hat{p}) \quad ; \quad \beta_{\varepsilon^m} > 0 \tag{29}$$

$$\hat{l} = \dot{l}/l = n - i(\varepsilon^m) \tag{30}$$

$$\hat{m} = \dot{m}/m = \hat{M} - \hat{p} - \hat{K} = \mu - \hat{p} - i(\varepsilon^m) \tag{31}$$

where π^m = medium-term inflation climate, ε = current risk premium on investment, $n = \hat{L}$ = rate of growth of labor supply (natural rate of growth) which is assumed to be constant, and $\mu = \hat{M}$ = rate of growth of nominal money supply which is assumed to be constant.

6.2 Five-dimensional dynamical system

The system in the previous section can be reduced to the following nonlinear five-dimensional system of differential equations.

(i) $\dot{\omega} = \frac{\omega}{1-\kappa_p\kappa_w}[(1-\kappa_p)\beta_w\{V^l(\varepsilon^m, l) - \bar{V}^l\} - \beta_p(1-\kappa_w)\{V^c(\varepsilon^m, \omega) - \bar{V}^c\}]$

$$\equiv F_1(\omega, l, \varepsilon^m)$$

(ii) $\dot{l} = l\{n - i(\varepsilon^m)\} \equiv F_2(l, \varepsilon^m)$

(iii) $\dot{m} = m[\mu - \frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w}\{V^l(\varepsilon^m, l) - \bar{V}^l\} - \frac{\beta_p}{1-\kappa_p\kappa_w}\{V^c(\varepsilon^m, \omega) - \bar{V}^c\} - \pi^m - i(\varepsilon^m)]$

$$\equiv F_3(\omega, l, m, \varepsilon^m, \pi^m)$$

(iv) $\dot{\varepsilon}^m = \beta_{\varepsilon^m}[\rho(\varepsilon^m, \omega) - r(\varepsilon^m, m) + \frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w}\{V^l(\varepsilon^m, l) - \bar{V}^l\}$

$$+\frac{\beta_p}{1-\kappa_p\kappa_w}\{V^c(\varepsilon^m, \omega) - \bar{V}^c\} + \pi^m] \equiv \beta_{\varepsilon^m} G_4(\omega, l, m, \varepsilon^m, \pi^m)$$

$$\equiv F_4(\omega, l, m, \varepsilon^m, \pi^m \quad ; \quad \beta_{\varepsilon^m})$$

(v) $\dot{\pi}^m = \beta_{\pi^m}[\frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w}\{V^l(\varepsilon^m, l) - \bar{V}^l\} + \frac{\beta_p}{1-\kappa_p\kappa_w}\{V^c(\varepsilon^m, \omega) - \bar{V}^c\}]$

$$\equiv \beta_{\pi^m} G_5(\omega, l, \varepsilon^m) \equiv F_5(\omega, l, \varepsilon^m \quad ; \quad \beta_{\pi^m}) \quad (S_1)$$

Next, let us consider how to derive these equations. First, we can rewrite equations (26) and (27) in terms of the matrix notation as follows.

$$\begin{bmatrix} 1 & -\kappa_w \\ -\kappa_p & 1 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \beta_w(V^l - \bar{V}^l) + (1-\kappa_w)\pi^m \\ \beta_p(V^c - \bar{V}^c) + (1-\kappa_p)\pi^m \end{bmatrix} \tag{32}$$

Solving this equation, we obtain the following reduced form of $\hat{w}$ and $\hat{p}$.

$$\hat{w} = \begin{vmatrix} \beta_w(V^l - \bar{V}^l) + (1-\kappa_w)\pi^m & -\kappa_w \\ \beta_p(V^c - \bar{V}^c) + (1-\kappa_p)\pi^m & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & -\kappa_w \\ -\kappa_p & 1 \end{vmatrix}$$

$$= \frac{\beta_w}{1-\kappa_p\kappa_w}(V^l - \bar{V}^l) + \frac{\beta_p\kappa_w}{1-\kappa_p\kappa_w}(V^c - \bar{V}^c) + \pi^m \tag{33}$$

$$\hat{p} = \begin{vmatrix} 1 & \beta_w(V^l - \bar{V}^l) + (1-\kappa_w)\pi^w \\ -\kappa_p & \beta_p(V^c - \bar{V}^c) + (1-\kappa_p)\pi^w \end{vmatrix} \Big/ \begin{vmatrix} 1 & -\kappa_w \\ -\kappa_p & 1 \end{vmatrix}$$

$$= \frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w}(V^l - \bar{V}^l) + \frac{\beta_p}{1-\kappa_p\kappa_w}(V^c - \bar{V}^c) + \pi^m \tag{34}$$

Substituting the equations (33) and (34) into the equality $\hat{\omega} = \hat{w} - \hat{p}$, we obtain Eq. (S_1)(i). Eq. (S_1)(ii) follows from Eq. (30). Substituting Eq. (34) into the equations (31), (29) and (28), we have Eq. (S_1)(iii), (iv), and (v) respectively.

6.3 Long run equilibrium solution

Next, let us investigate the properties of the stationary solution (long run equilibrium solution) of the system (S_1) which satisfies $\dot{\omega} = \dot{l} = \dot{m} = \dot{\varepsilon}^m = \dot{\pi}^m = 0$. Substituting $\dot{\omega} = \dot{\pi}^m = 0$ into the equations (S_1)(i) (v), we have the following system of equations.

$$\begin{bmatrix} (1-\kappa_p)\beta_w & -\beta_p(1-\kappa_w) \\ \kappa_p\beta_w & \beta_p \end{bmatrix} \begin{bmatrix} V^l - \bar{V}^l \\ V^c - \bar{V}^c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{35}$$

The solution of this system of equations becomes $V^l - \bar{V}^l = 0$ and

$V^c - \bar{V}^c = 0$ because we have the following inequality.

$$\begin{vmatrix} (1-\kappa_p)\beta_w & -\beta_p(1-\kappa_w) \\ \kappa_p\beta_w & \beta_p \end{vmatrix} = (1-\kappa_p)\beta_p\beta_w + \beta_p\kappa_p\beta_w(1-\kappa_w) > 0 \tag{36}$$

Therefore, we can characterize the long run equilibrium solution as follows.

(i) $V^l(\varepsilon^m, l) = l^d(y(\varepsilon^m))/l = \bar{V}^l$

(ii) $V^c(\varepsilon^m, \omega) = y(\varepsilon^m)/y^p(\omega) = \bar{V}^c$

(iii) $i(\varepsilon^m) = n$

(iv) $\pi^m = \mu - n$

(v) $\rho(\varepsilon^m, \omega) - r(\varepsilon^m, m) + \mu - n = 0$ (17)

We shall write the vector of the equilibrium values as $(\omega*, l*, m*, \varepsilon^m*, \pi^m*)$. π^m* is uniquely determined by Eq. (17)(iv). Since $i_{\varepsilon^m} > 0$, ε^m* is uniquely determined by Eq. (17)(iii) if it exists. We shall assume that $\varepsilon^m* > 0$ in fact exists. In this case, we obtain the unique $l* > 0$ substituting $\varepsilon^m = \varepsilon^m*$ into Eq. (17)(i). We can also determine unique $\omega* > 0$ (if it exists) by substituting $\varepsilon^m = \varepsilon^m*$ into Eq. (17)(ii), since $y^p_\omega < 0$. Finally, we can determine unique $m* > 0$ (if it exists) by substituting $\varepsilon^m = \varepsilon^m*$ and $\omega = \omega*$ into Eq. (17)(v), since $r_m < 0$.

The above analysis reveals that *at most* one long run equilibrium point exists. In other words, there is no possibility of the existence of the multiple equilibria. In the next section, we shall investigate the local stability / instability of the long run equilibrium point of this five-dimensional system by *assuming* that an economically meaningful long run equilibrium point exists.

6.4 A five-dimensional analysis of local stability

We can write the Jacobian matrix of the system (S_1) which is evaluated *at the equilibrium point* as follows.

$$J_1 = \begin{bmatrix} F_{11} & F_{12} & 0 & F_{14} & 0 \\ 0 & 0 & 0 & F_{24} & 0 \\ F_{31} & F_{32} & 0 & F_{34} & F_{35} \\ \beta_{\varepsilon^m} G_{41} & \beta_{\varepsilon^m} G_{42} & \beta_{\varepsilon^m} G_{43} & \beta_{\varepsilon^m} G_{44} & \beta_{\varepsilon^m} \\ \beta_{\pi^m} G_{51} & \beta_{\pi^m} G_{52} & 0 & \beta_{\pi^m} G_{54} & 0 \end{bmatrix} \tag{37}$$

where $F_{11} = \partial F_1/\partial\omega = -\frac{\omega\beta_p(1-\kappa_w)}{1-\kappa_p\kappa_w} \underset{(+)}{V^c_\omega} < 0$, $F_{12} = \partial F_1/\partial l = \frac{\omega(1-\kappa_p)\beta_w}{1-\kappa_p\kappa_w} \underset{(-)}{V^l_l} < 0$, $F_{14} = \partial F_1/\partial\varepsilon^m = \frac{\omega}{1-\kappa_p\kappa_w}\{(1-\kappa_p)\beta_w \underset{(+)}{V^l_{\varepsilon^m}} - \beta_p(1-\kappa_w) \underset{(+)}{V^c_{\varepsilon^m}}\}$, $F_{24} = \partial F_2/\partial\varepsilon^m = -l \underset{(+)}{i_{\varepsilon^m}} < 0$, $F_{31} = \partial F_3/\partial\omega = -\frac{m\beta_p}{1-\kappa_p\kappa_w} \underset{(+)}{V^c_\omega} < 0$, $F_{32} = \partial F_3/\partial l = -\frac{m\kappa_p\beta_w}{1-\kappa_p\kappa_w} \underset{(-)}{V^l_l} > 0$, $F_{34} = \partial F_3/\partial\varepsilon^m = -\frac{m}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}}) < 0$, $F_{35} = \partial F_3/\partial\pi^m = -m < 0$, $G_{41} = \partial G_4/\partial\omega = \underset{(-)}{\rho_\omega} + \frac{\beta_p}{1-\kappa_p\kappa_w} \underset{(+)}{V^c_\omega}$, $G_{42} = \partial G_4/\partial l = \frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w} \underset{(-)}{V^l_l} < 0$, $G_{43} = \partial G_4/\partial m = 1/h_2 > 0$, $G_{44} = \partial G_4/\partial\varepsilon^m = \underset{(+)}{\rho_{\varepsilon^m}} - (h_1 \underset{(+)}{y_{\varepsilon^m}} / h_2) + \frac{1}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}})$, $G_{51} = \partial G_5/\partial\omega = \frac{\beta_p}{1-\kappa_p\kappa_w} \underset{(+)}{V^c_\omega} > 0$, $G_{52} = \partial G_5/\partial l = \frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w} \underset{(-)}{V^l_l} < 0$, and $G_{54} = \partial G_5/\partial\varepsilon^m = \frac{1}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}}) > 0$.

The sign pattern of the matrix J_1 becomes as follows.

$$signJ_1 = \begin{bmatrix} - & - & 0 & ? & 0 \\ 0 & 0 & 0 & - & 0 \\ - & + & 0 & - & - \\ ? & - & + & ? & + \\ + & - & 0 & + & 0 \end{bmatrix} \tag{38}$$

The characteristic equation of this system can be written as

$$\Gamma_1(\lambda) \equiv |\lambda I - J_1| = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \tag{39}$$

where each coefficient is given as follows.

$$a_1 = -traceJ_1 = -\underset{(-)}{F_{11}} - \beta_{\varepsilon^m}\underset{(?)}{G_{44}} \equiv a_1(\beta_{\varepsilon^m}) \tag{40}$$

a_2 =sum of all principal second-order minors of J_1

$$= \begin{vmatrix} F_{11} & F_{12} \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} F_{11} & 0 \\ F_{31} & 0 \end{vmatrix} + \beta_{\varepsilon^m}\begin{vmatrix} F_{11} & F_{14} \\ G_{41} & G_{44} \end{vmatrix} + \beta_{\pi^m}\begin{vmatrix} F_{11} & 0 \\ G_{51} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ F_{32} & 0 \end{vmatrix} + \beta_{\varepsilon^m}\begin{vmatrix} 0 & F_{24} \\ G_{42} & G_{44} \end{vmatrix}$$

$$+\beta_{\pi^m}\begin{vmatrix} 0 & 0 \\ G_{52} & 0 \end{vmatrix} + \beta_{\varepsilon^m}\begin{vmatrix} 0 & F_{34} \\ G_{43} & G_{44} \end{vmatrix} + \begin{vmatrix} 0 & F_{35} \\ 0 & 0 \end{vmatrix} + \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} G_{44} & 1 \\ G_{54} & 0 \end{vmatrix}$$

$$= \beta_{\varepsilon^m}(-\beta_{\pi^m}\underset{(+)}{G_{54}} + \underset{(-)}{F_{11}}\,\underset{(?)}{G_{44}} - \underset{(?)}{F_{14}}\,\underset{(?)}{G_{41}} - \underset{(-)}{F_{24}}\,\underset{(-)}{G_{42}} - \underset{(-)}{F_{34}}\,\underset{(+)}{G_{43}}) \equiv a_2(\beta_{\varepsilon^m}, \beta_{\pi^m}) \tag{41}$$

a_3 =– (sum of all principal third-order minors of J_1)

$$= -\begin{vmatrix} F_{11} & F_{12} & 0 \\ 0 & 0 & 0 \\ F_{31} & F_{32} & 0 \end{vmatrix} - \beta_{\varepsilon^m}\begin{vmatrix} F_{11} & F_{12} & F_{14} \\ 0 & 0 & F_{24} \\ G_{41} & G_{42} & G_{44} \end{vmatrix} - \beta_{\pi^m}\begin{vmatrix} F_{11} & F_{12} & 0 \\ 0 & 0 & 0 \\ G_{51} & G_{52} & 0 \end{vmatrix} - \beta_{\varepsilon^m}\begin{vmatrix} F_{11} & 0 & F_{14} \\ F_{31} & 0 & F_{34} \\ G_{41} & G_{43} & G_{44} \end{vmatrix}$$

$$-\beta_{\pi^m}\begin{vmatrix} F_{11} & 0 & 0 \\ F_{31} & 0 & F_{35} \\ G_{51} & 0 & 0 \end{vmatrix} - \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} F_{11} & F_{14} & 0 \\ G_{41} & G_{44} & 1 \\ G_{51} & G_{54} & 0 \end{vmatrix} - \beta_{\varepsilon^m}\begin{vmatrix} 0 & 0 & F_{24} \\ F_{32} & 0 & F_{34} \\ G_{42} & G_{43} & G_{44} \end{vmatrix}$$

$$-\beta_{\pi^m}\begin{vmatrix} 0 & 0 & 0 \\ F_{32} & 0 & F_{35} \\ G_{52} & 0 & 0 \end{vmatrix} - \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} 0 & F_{24} & 0 \\ G_{42} & G_{44} & 1 \\ G_{52} & G_{54} & 0 \end{vmatrix} - \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} 0 & F_{34} & F_{35} \\ G_{43} & G_{44} & 1 \\ 0 & G_{54} & 0 \end{vmatrix}$$

$$= \beta_{\varepsilon^m}\{\beta_{\pi^m}(-\underset{(?)}{F_{14}}\,\underset{(+)}{G_{51}} + \underset{(-)}{F_{11}}\,\underset{(+)}{G_{54}} - \underset{(-)}{F_{24}}\,\underset{(-)}{G_{52}} - \underset{(-)}{F_{35}}\,\underset{(+)}{G_{54}}\,\underset{(+)}{G_{43}}) - \underset{(-)}{F_{12}}\,\underset{(-)}{F_{24}}\,\underset{(?)}{G_{41}} + \underset{(-)}{F_{11}}\,\underset{(-)}{G_{42}}\,\underset{(-)}{F_{24}}$$

$$-\underset{(?)}{F_{14}}\,\underset{(+)}{G_{43}}\,\underset{(-)}{F_{31}} + \underset{(-)}{F_{11}}\,\underset{(+)}{G_{43}}\,\underset{(-)}{F_{34}} - \underset{(-)}{F_{24}}\,\underset{(+)}{G_{43}}\,\underset{(+)}{F_{32}}\} \equiv a_3(\beta_{\varepsilon^m}, \beta_{\pi^m}) \tag{42}$$

a_4 =sum of all principal fourth-order minors of J_1

$$= \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} 0 & 0 & F_{24} & 0 \\ F_{32} & 0 & F_{34} & F_{35} \\ G_{42} & G_{43} & G_{44} & 1 \\ G_{52} & 0 & G_{54} & 0 \end{vmatrix} + \beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} F_{11} & 0 & F_{14} & 0 \\ F_{31} & 0 & F_{34} & F_{35} \\ G_{41} & G_{43} & G_{44} & 1 \\ G_{51} & 0 & G_{54} & 0 \end{vmatrix}$$

$$+\beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} F_{11} & F_{12} & F_{14} & 0 \\ 0 & 0 & F_{24} & 0 \\ G_{41} & G_{42} & G_{44} & 1 \\ G_{51} & G_{52} & G_{54} & 0 \end{vmatrix} + \beta_{\pi_m}\begin{vmatrix} F_{11} & F_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ F_{31} & F_{32} & 0 & F_{35} \\ G_{51} & G_{52} & 0 & 0 \end{vmatrix}$$

$$+\beta_{\varepsilon^m}\begin{vmatrix} F_{11} & F_{12} & 0 & F_{14} \\ 0 & 0 & 0 & F_{24} \\ F_{31} & F_{32} & 0 & F_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{vmatrix}$$

$$= \beta_{\varepsilon^m}\{\beta_{\pi^m}(F_{24}\begin{vmatrix} F_{32} & 0 & F_{35} \\ G_{42} & G_{43} & 1 \\ G_{52} & 0 & 0 \end{vmatrix} - G_{43}\begin{vmatrix} F_{11} & F_{14} & 0 \\ F_{31} & F_{34} & F_{35} \\ G_{51} & G_{54} & 0 \end{vmatrix} - F_{24}\begin{vmatrix} F_{11} & F_{12} & 0 \\ G_{41} & G_{42} & 1 \\ G_{51} & G_{52} & 0 \end{vmatrix})$$

$$+F_{24}\begin{vmatrix} F_{11} & F_{12} & 0 \\ F_{31} & F_{32} & 0 \\ G_{41} & G_{42} & G_{43} \end{vmatrix}\}$$

$$= \beta_{\varepsilon^m}\{\beta_{\pi^m}(-\underset{(-)}{F_{24}}\,\underset{(-)}{F_{35}}\,\underset{(+)}{G_{43}}\,\underset{(-)}{G_{52}} - \underset{(+)}{G_{43}}\,\underset{(?)}{F_{14}}\,\underset{(-)}{F_{35}}\,\underset{(+)}{G_{51}} + \underset{(+)}{G_{43}}\,\underset{(-)}{F_{11}}\,\underset{(+)}{G_{54}}\,\underset{(-)}{F_{35}} - \underset{(-)}{F_{24}}\,\underset{(-)}{F_{12}}\,\underset{(+)}{G_{51}}$$

$$+\underset{(-)}{F_{24}}\,\underset{(-)}{F_{11}}\,\underset{(-)}{G_{52}}) + \underset{(-)}{F_{24}}\,\underset{(+)}{G_{43}}(\underset{(-)}{F_{11}}\,\underset{(+)}{F_{32}} - \underset{(-)}{F_{12}}\,\underset{(-)}{F_{31}})\} \equiv a_4(\beta_{\varepsilon^m}, \beta_{\pi^m}) \tag{43}$$

$$a_5 = -\det J_1 = -\beta_{\varepsilon^m}\beta_{\pi^m}\begin{vmatrix} F_{11} & F_{12} & 0 & F_{14} & 0 \\ 0 & 0 & 0 & F_{24} & 0 \\ F_{31} & F_{32} & 0 & F_{34} & F_{35} \\ G_{41} & G_{42} & G_{43} & G_{44} & 1 \\ G_{51} & G_{52} & 0 & G_{54} & 0 \end{vmatrix}$$

$$= -\beta_{\varepsilon^m}\beta_{\pi^m}F_{24}\begin{vmatrix} F_{11} & F_{12} & 0 & 0 \\ F_{31} & F_{32} & 0 & F_{35} \\ G_{41} & G_{42} & G_{43} & 1 \\ G_{51} & G_{52} & 0 & 0 \end{vmatrix} = -\beta_{\varepsilon^m}\beta_{\pi^m}F_{24}G_{43}\begin{vmatrix} F_{11} & F_{12} & 0 \\ F_{31} & F_{32} & F_{35} \\ G_{51} & G_{52} & 0 \end{vmatrix}$$

$$= \beta_{\varepsilon^m}\beta_{\pi^m}\,\underset{(-)}{F_{24}}\,\underset{(+)}{G_{43}}\,\underset{(-)}{F_{35}}(-\underset{(-)}{F_{12}}\,\underset{(+)}{G_{51}} + \underset{(-)}{F_{11}}\,\underset{(-)}{G_{52}}) \equiv a_5(\beta_{\varepsilon^m}, \beta_{\pi^m}) > 0 \tag{44}$$

Next, let us consider the conditions for local stability of the equilibrium point in this system. It is well known that the Routh-Hurwitz conditions for stable roots in this five-dimensional system can be expressed as follows (cf. Gandolfo (1996) chap. 16).

(i) $\Delta_1 \equiv a_1 > 0$

(ii) $\Delta_2 \equiv \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 - a_3 > 0$

(iii) $\Delta_3 \equiv \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3\Delta_2 + a_1(a_5 - a_1 a_4) = a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 + a_1 a_5 > 0$

(iv) $\Delta_4 \equiv \begin{vmatrix} a_1 & a_3 & a_5 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} = a_4\Delta_3 - a_5 \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & 1 & a_2 \end{vmatrix}$

$$= a_4\Delta_3 + a_5(-a_1 a_2^2 - a_5 + a_2 a_3 + a_1 a_4)$$

$$= a_4\Delta_3 + a_5(a_1 a_4 - a_5 - a_2\Delta_2)0$$

(v) $\Delta_5 \equiv \begin{vmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ 1 & a_2 & a_4 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & 1 & a_2 & a_4 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 \end{vmatrix} = a_5\Delta_4 > 0$ (26)

It is easy to see that two inequalities $a_1 > 0$ and $a_5 > 0$ are a set of *necessary* conditions for the local stability of this system. The condition $a_5 > 0$ is always satisfied because of Eq. (44). However, a_1 depends on the value of the parameter β_{ε^m} because of Eq. (40). Furthermore, we can see that G_{44} is an increasing function of the sensitivity of the money demand with respect to the nominal rate of interest (h_2), and we have $\lim_{h_2 \to 0} G_{44} = -\infty$, $\lim_{h_2 \to +\infty} G_{44} > 0$. The following proposition follows from this fact.

Proposition 4

Suppose that h_2 is so large that $G_{44} > 0$. Then, the equilibrium point of the system (S_1) is locally unstable if the inequality

$$\beta_{\varepsilon^m} > -\underset{(-)}{F_{11}} \,/\, \underset{(+)}{G_{44}} \tag{45}$$

is satisfied.

Proof. If the inequality (45) is satisfied, we have $a_1 < 0$, which violates one

of the Routh-Hurwitz conditions for stable roots.

This proposition implies that the system becomes dynamically unstable if the values of the parameters h_2 and β_{ε^m} are sufficiently large. This proposition provides us a *sufficient* condition for local *instability*. On the other hand, the following proposition provides us an interesting set of *sufficient* conditions for the local *stability*.

Proposition 5

Suppose that the following set of inequalities is satisfied at the parameter values $\beta_{\varepsilon^m} = \beta^0_{\varepsilon^m} > 0$ and $\beta_{\pi^m} = 0$.

$$a_1(\beta^0_{\varepsilon^m}) > 0, \quad a_3(\beta^0_{\varepsilon^m}, 0) > 0,$$

$$a_1(\beta^0_{\varepsilon^m})a_2(\beta^0_{\varepsilon^m}, 0)a_3(\beta^0_{\varepsilon^m}, 0) - a_1(\beta^0_{\varepsilon^m})^2 a_4(\beta^0_{\varepsilon^m}, 0) - a_3(\beta^0_{\varepsilon^m}, 0)^2 > 0 \tag{46}$$

Then, a set of inequalities (26)(i) – (v) is satisfied at $\beta_{\varepsilon^m} = \beta^0_{\varepsilon^m}$ for all sufficiently small $\beta_{\pi^m} > 0$.

Proof.

We have the following relationships at $[\beta_{\varepsilon^m}, \beta_{\pi^m}] = [\beta^0_{\varepsilon^m}, 0]$ because $a_5(\beta^0_{\varepsilon^m}, 0) = 0$.

(i) $\Delta_1 = a_1(\beta^0_{\varepsilon^m})$

(ii) $\Delta_2 = a_1(\beta^0_{\varepsilon^m})a_2(\beta^0_{\varepsilon^m}, 0) - a_3(\beta^0_{\varepsilon^m}, 0)$

(iii) $\Delta_3 = a_3(\beta^0_{\varepsilon^m}, 0)\Delta_2 - a_1(\beta^0_{\varepsilon^m})^2 a_4(\beta^0_{\varepsilon^m}, 0)$

$$= a_1(\beta^0_{\varepsilon^m})a_2(\beta^0_{\varepsilon^m}, 0)a_3(\beta^0_{\varepsilon^m}, 0) - a_1(\beta^0_{\varepsilon^m})^2 a_4(\beta^0_{\varepsilon^m}, 0) - a_3(\beta^0_{\varepsilon^m}, 0)^2$$

(iv) $\Delta_4 = a_4(\beta^0_{\varepsilon^m}, 0)\Delta_3$ (29)

We can easily see from these relationships that four conditions $\Delta_j > 0$ $(j = 1, 2, 3, 4)$ are satisfied at $[\beta_{\varepsilon^m}, \beta_{\pi^m}] = [\beta^0_{\varepsilon^m}, 0]$ if a set of inequalities (46) are satisfied, because we have $a_4(\beta^0_{\varepsilon^m}, 0) = \beta^0_{\varepsilon^m} \underset{(-)}{F_{24}}\,\underset{(+)}{G_{43}}(\underset{(-)}{F_{11}}\,\underset{(+)}{F_{32}} - \underset{(-)}{F_{12}}\,\underset{(-)}{F_{31}})0$. It is clear that four inequalities $\Delta_j > 0$ ($j = 1, 2, 3, 4$) are also satisfied at $\beta_{\varepsilon^m} = \beta^0_{\varepsilon^m}$ for all sufficiently small $\beta_{\pi^m} > 0$, because each coefficient is the continuous function of the parameter β_{π^m}. The inequality $\Delta_5 > 0$ is also satisfied at $\beta_{\varepsilon^m} = \beta^0_{\varepsilon^m}$ for all sufficiently small $\beta_{\pi^m} > 0$, because we have $a_5(\beta^0_{\varepsilon^m}, \beta_{\pi^m}) > 0$ if $\beta_{\pi^m} > 0$.

Proposition 5 implies that the equilibrium point of the system (S_1) is locally asymptotically stable at $\beta_{\varepsilon^m} = \beta^0_{\pi^m} > 0$ for all sufficiently small $\beta_{\pi^m} > 0$ if a set of inequalities (46) is satisfied. In the next section, we shall show that these inequalities in fact correspond to the exact local stability conditions of a degenerated four-dimensional system.

6.5 Local stability and Hopf Bifurcations in a degenerated four-dimensional system, and implications for the 5D dynamics

It is easy to see that the characteristic equation (39) becomes as follows as $\beta_{\pi^m} \to 0$.

$$\lim_{\beta_{\pi^m} \to 0} \Gamma_1(\lambda) = \lim_{\beta_{\pi^m} \to 0} |\lambda I - J_1| = \lambda\,|\lambda I - J_2| = 0 \tag{47}$$

where J_2 is the following (4×4) submatrix of the (5×5) matrix J_1.

$$J_2 = \begin{bmatrix} F_{11} & F_{12} & 0 & F_{14} \\ 0 & 0 & 0 & F_{24} \\ F_{31} & F_{32} & 0 & F_{34} \\ \beta_{\varepsilon^m} G_{41} & \beta_{\varepsilon^m} G_{42} & \beta_{\varepsilon^m} G_{43} & \beta_{\varepsilon^m} G_{44} \end{bmatrix} \tag{48}$$

Eq. (47) has a root $\lambda = 0$, and other four roots are determined by the following equation.

$$\Gamma_2(\lambda) \equiv |\lambda I - J_2| = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0 \tag{49}$$

where each coefficient becomes as follows.

$$b_1 = a_1(\beta_{\varepsilon^m}) = A - \beta_{\varepsilon^m} B \tag{50}$$

$$b_2 = a_2(\beta_{\varepsilon^m}, 0) = \beta_{\varepsilon^m} C \tag{51}$$

$$b_3 = a_3(\beta_{\varepsilon^m}, 0) = \beta_{\pi^m} D \tag{52}$$

$$b_4 = a_4(\beta_{\varepsilon^m}, 0) = \beta_{\varepsilon^m} E \tag{53}$$

In these expressions, A, B, C, D, and E are constants which are given as follows.

$$A = -\underset{(-)}{F_{11}} = \frac{\omega\beta_p(1-\kappa_w)}{1-\kappa_p\kappa_w}\underset{(+)}{V^c_\omega} > 0,$$

$$B = \underset{(?)}{G_{44}} = \underset{(+)}{\rho_{\varepsilon^m}} - (h_1 \underset{(+)}{y_{\varepsilon_m}} / h_2) + \frac{1}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}}),$$

$$C = \underset{(-)}{F_{11}}\,\underset{(?)}{G_{44}} - \underset{(?)}{F_{14}}\,\underset{(?)}{G_{41}} - \underset{(-)}{F_{24}}\,\underset{(-)}{G_{42}} - \underset{(-)}{F_{34}}\,\underset{(+)}{G_{43}}$$

$$= -\frac{\omega\beta_p(1-\kappa_w)}{1-\kappa_p\kappa_w}\underset{(+)}{V^c_\omega}\{\underset{(+)}{\rho_{\varepsilon^m}} - (h_1 \underset{(+)}{y_{\varepsilon^m}} / h_2) + \frac{1}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}})\}$$

$$-\frac{\omega}{1-\kappa_p\kappa_w}\{(1-\kappa_p)\beta_w \underset{(+)}{V^l_{\varepsilon^m}} - \beta_p(1-\kappa_w)\underset{(+)}{V^c_{\varepsilon^m}}\}(\underset{(-)}{\rho_\omega} + \frac{\beta_p}{1-\kappa_p\kappa_w}\underset{(+)}{V^c_\omega})$$

$$+\frac{\kappa_p\beta_w}{1-\kappa_p\kappa_w}\underset{(-)}{V^l_l}\, l\, \underset{(+)}{i_{\varepsilon^m}} + \frac{m}{1-\kappa_p\kappa_w}(\kappa_p\beta_w \underset{(+)}{V^l_{\varepsilon^m}} + \beta_p \underset{(+)}{V^c_{\varepsilon^m}})(1/h_2),$$

$$D = -\underset{(-)}{F_{12}}\,\underset{(-)}{F_{24}}\,\underset{(?)}{G_{41}} + \underset{(-)}{F_{11}}\,\underset{(-)}{G_{42}}\,\underset{(-)}{F_{24}} + \underset{(+)}{G_{43}}(-\underset{(?)}{F_{14}}\,\underset{(-)}{F_{31}} + \underset{(-)}{F_{11}}\,\underset{(-)}{F_{34}} - \underset{(-)}{F_{24}}\,\underset{(+)}{F_{32}})$$

$$= \frac{\beta_w}{1-\kappa_p\kappa_w}[\omega(1-\kappa_p)\underset{(-)}{V^l_l}\, l\, \underset{(+)}{i_{\varepsilon^m}}(\underset{(-)}{\rho_\omega} + \frac{\beta_p}{1-\kappa_p\kappa_w}\underset{(+)}{V^c_\omega}) + \frac{\omega\beta_p(1-\kappa_w)\kappa_p}{1-\kappa_p\kappa_w}\underset{(+)}{V^c_\omega}\,\underset{(-)}{V^l_l}\, l\, \underset{(+)}{i_{\varepsilon^m}}]$$

$$+\frac{\beta_w}{1-\kappa_p\kappa_w}(1/h_2)m[\frac{\omega\beta_p}{1-\kappa_p\kappa_w}\{(1-\kappa_p)+(1-\kappa_w)\kappa_p\}\underset{(+)}{V^l_{\varepsilon^m}}\,\underset{(+)}{V^c_\omega}-\kappa_p l\,\underset{(+)}{i_{\varepsilon^m}}\,\underset{(-)}{V^l_l}]$$

$$\equiv\frac{\beta_w}{1-\kappa_p\kappa_w}H,$$

$$E=\underset{(-)}{F_{24}}\,\underset{(+)}{G_{43}}(\underset{(-)}{F_{11}}\,\underset{(+)}{F_{32}}-\underset{(-)}{F_{12}}\,\underset{(-)}{F_{31}})$$

$$=\frac{m\omega\beta_p\beta_w}{(1-\kappa_p\kappa_w)^2}(1/h_2)l\,\underset{(+)}{i_{\varepsilon^m}}\{-(1-\kappa_w)\kappa_p\,\underset{(+)}{V^c_{\varepsilon^m}}\,\underset{(-)}{V^l_l}-(1-\kappa_p)\,\underset{(-)}{V^l_l}\,\underset{(+)}{V^c_\omega}\}>0. \tag{54}$$

In fact, Eq. (49) is identical to the characteristic equation of a degenerated four-dimensional system, which we can construct by freezing the inflation climate π^m at the equilibrium level $\mu-n$ in the system (S_1)(i) – (iv). For simplicity, we shall call this degenerated four-dimensional system as the system (S_2).

We can express the Routh-Hurwitz conditions for stable roots in this four-dimensional system as follows (cf. Gandolfo (1996) chap. 16).

(i) $\Phi_1\equiv b_1>0$

(ii) $\Phi_2\equiv\begin{vmatrix} b_1 & b_3\\ 1 & b_2\end{vmatrix}=b_1b_2-b_3>0$

(iii) $\Phi_3\equiv\begin{vmatrix} b_1 & b_3 & 0\\ 1 & b_2 & b_4\\ 0 & b_1 & b_3\end{vmatrix}=b_3\Phi_2-b_1^2b_4=b_1b_2b_3-b_1^2b_4-b_3^2>0$

(iv) $\Phi_4\equiv\begin{vmatrix} b_1 & b_3 & 0 & 0\\ 1 & b_2 & b_4 & 0\\ 0 & b_1 & b_3 & 0\\ 0 & 1 & b_2 & b_4\end{vmatrix}=b_4\Phi_3>0$ (38)

A set of inequalities (38) is equivalent to the following set of conditions.

$$b_1>0,\quad b_3>0,\quad b_4>0,\quad \Phi_3\equiv b_1b_2b_3-b_1^2b_4-b_3^2>0 \tag{55}$$

Remark 1

A set of inequalities (55) automatically implies the inequality $b_2>0$.

By the way, the inequality $b_4>0$ is always satisfied for all $\beta_{\varepsilon^m}>0$. Therefore, the exact local stability conditions of the system (S_2) can be reduced to the following three inequalities as far as $\beta_{\varepsilon^m}>0$.

$$b_1>0,\quad b_3>0,\quad \Phi_3>0 \tag{56}$$

It is important to note that a set of inequalities (56) is exactly the same as a set of conditions (46). Now, we can easily obtain the following proposition.

Proposition 6

The equilibrium point of the system (S_2) is locally unstable for all $\beta_{\varepsilon^m} > 0$ if *either* of the inequalities $C < 0$ *or* $D < 0$ is satisfied.

Proof.

If $C < 0$, we have $b_2 < 0$ for all $\beta_{\varepsilon^m} > 0$, which violates one of the Routh-Hurwitz conditions for stable roots. If $D < 0$, we have $b_3 < 0$ for all $\beta_{\varepsilon^m} > 0$, which also violates one of the conditions for stable roots.

This proposition implies that *both* of the inequalities $C > 0$ and $D > 0$ are *necessary* conditions for the local stability of the system (S_2). We can see from Eq. (54) that these conditions are satisfied if the value of the parameter h_2 (sensitivity of the money demand with respect to the changes of the nominal rate of interest) is sufficiently *small*.

By the way, we have the following relationships from Eq. (54) ($A > 0$ is independent of the changes of the parameter β_w).

$$B(0) \equiv \lim_{\beta_w \to 0} B = \underset{(+)}{\rho_{\varepsilon^m}} - (h_1 \underset{(+)}{y_{\varepsilon^m}} / h_2) + \frac{\beta_p}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_{\varepsilon^m}} \tag{57}$$

$$\begin{aligned} C(0) \equiv \lim_{\beta_w \to 0} C = \frac{\beta_p}{1 - \kappa_p \kappa_w} [\omega(1 - \kappa_w)\{- \underset{(+)}{\rho_{\varepsilon^m}} + (h_1 \underset{(+)}{y_{\varepsilon^m}} / h_2) - \frac{\beta_p}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_{\varepsilon^m}} \\ + \underset{(+)}{V^c_{\varepsilon^m}} (\underset{(-)}{\rho_\omega} + \frac{\beta_p}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_\omega})\} + m \underset{(+)}{V^c_{\varepsilon^m}} (1/h_2)] \end{aligned} \tag{58}$$

We shall study the local stability of the equilibrium point of the four-dimensional system (S_2) under the following assumption.

Assumption 1.

$B(0) > 0, \quad C(0) > 0$,and $H > 0$.

The condition $B(0) > 0$ implies that

$$1/h_2 < (\underset{(+)}{\rho_{\varepsilon^m}} + \frac{\beta_p}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_{\varepsilon^m}}) / (h_1 \underset{(+)}{y_{\varepsilon^m}}) \equiv Q. \tag{59}$$

This means that the value of the parameter h_2 is not too small. The conditions $C(0) > 0$ and $H > 0$ imply the following two inequalities.

$$\begin{aligned} 1/h_2 > \beta_p \{\omega(1 - \kappa_w) (\underset{(+)}{\rho_{\varepsilon^m}} + \frac{\beta_p}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_{\varepsilon^m}}) + \underset{(+)}{V^c_{\varepsilon^m}} (- \underset{(-)}{\rho_\omega} - \frac{\beta_w}{1 - \kappa_p \kappa_w} \underset{(+)}{V^c_\omega})\} / \{\omega(1 - \kappa_w) h_1 \underset{(+)}{y_{\varepsilon^m}} \\ + m \underset{(+)}{V^c_{\varepsilon^m}}\} \equiv T \end{aligned} \tag{60}$$

$$1/h_2 > \{-\omega(1-\kappa_p)\underset{(-)}{V_l^l}\, l\, \underset{(+)}{i_{\varepsilon^m}}(\underset{(-)}{\rho_\omega} + \frac{\beta_p}{1-\kappa_p\kappa_w}\underset{(+)}{V_\omega^c}) - \frac{\omega\beta_p(1-\kappa_w)\kappa_p}{1-\kappa_p\kappa_w}\underset{(+)}{V_\omega^c}\,\underset{(-)}{V_l^l}\, l\, \underset{(+)}{i_{\varepsilon^m}}\}/\frac{m\omega\beta_p}{1-\kappa_p\kappa_w}$$

$$\{(1-\kappa_p)+(1-\kappa_w)\kappa_p\}\underset{(+)}{V_{\varepsilon^m}^l}\,\underset{(+)}{V_\omega^c} - m\kappa_p l\, \underset{(+)}{i_{\varepsilon^m}}\,\underset{(-)}{V_l^l}\} \equiv W \tag{61}$$

These two inequalities mean that the value of the parameter h_2 is not too large. That is to say, **Assumption 1** is equivalent to the following set of inequalities.

$$\max[T, W] < 1/h_2 < Q \tag{62}$$

This set of inequalities is meaningless unless

$$\max[T, W] < Q. \tag{63}$$

The inequality (63) will in fact be satisfied if the value of the parameter h_1 (sensitivity of the money demand with respect to the changes of the real income) is sufficiently small, since we have $\lim_{h_1 \to 0} Q = +\infty$, $\lim_{h_1 \to 0} T < +\infty$, and $W < +\infty$. The small h_1 means the mild slope of the LM curve (see Eq. (2)). To sum up, **Assumption 1** will in fact be satisfied if h_1 is relatively small and h_2 is at the intermediate level.

Under **Assumption 1**, we have

$B > 0$, $C > 0$,and $D > 0$ (47)

for all sufficiently small $\beta_w > 0$. In this case, we can simplify a set of local stability conditions as follows.

$$0 < \beta_{\varepsilon^m} < A/B, \quad \Phi_3 > 0 \tag{64}$$

We can write the function Φ_3 as follows.

$$\Phi_3(\beta_{\varepsilon^m}) = (A - \beta_{\varepsilon^m}B)\beta_{\varepsilon^m}^2 CD - (A - \beta_{\varepsilon^m}B)^2\beta_{\varepsilon^m}E - \beta_{\varepsilon^m}^2 D^2$$

$$= -B(CD + BE)\beta_{\varepsilon^m}^3 + \{(AC - D)D + 2ABE\}\beta_{\varepsilon^m}^2 - A^2E\beta_{\varepsilon^m} \tag{65}$$

Suppose that $\beta_w > 0$ is so small that a set of inequalities (64) is satisfied. Since $\Phi_3(0) = 0$ and $\Phi_3'(0) = -A^2E < 0$, we have $\Phi_3 < 0$ for all sufficient small $\beta_{\varepsilon^m} > 0$. This observation implies that the equilibrium point of this system becomes locally unstable for all sufficient small $\beta_{\varepsilon^m} > 0$. On the other hand, we already know that the system becomes unstable for all $\beta_{\varepsilon^m} > A/B$. Therefore, we have the instability result for very small as well as very large β_{ε^m}. If $\beta_{\varepsilon^m} = 0$, the investment climate does not move, i. e., $\varepsilon^m = \varepsilon^m(0)$ for all time. In this case the movement of l is governed by the equation $\dot{l} = l\{n - i(\varepsilon^m(0))\}$, so that l continues to increase or continues to decrease unless $n = i(\varepsilon^m(0))$. Obviously, this means instability, and this property applies also for sufficiently small $\beta_{\varepsilon^m} > 0$. On the other hand, the instability of the system in case of the large adjustment parameter can be interpreted as an 'overshooting' phenomenon. Next, let us investigate whether the stable region exists at the intermediate range of the adjustment parameter values or not.

The equation $\Phi_3(\beta_{\varepsilon^m}) = 0$ has the following three roots.

(i) $\beta^0_{\varepsilon^m} = 0$

(ii) $\beta^1_{\varepsilon^m} = \frac{\{(AC-D)D+2ABE\}-\sqrt{\{(AC-D)D+2ABE\}^2-4B(CD+BE)A^2E}}{2B(CD+BE)}$

$$= \frac{\{(AC-D)D+2ABE\} - D\sqrt{(AC-D)^2-4ABE}}{2B(CD+BE)}$$

(iii) $\beta^2_{\varepsilon^m} = \frac{\{(AC-D)D+2ABE\}+D\sqrt{(AC-D)^2-4ABE}}{2B(CD+BE)}$ (50)

An interval with $\Phi_3 > 0$ exists in the region $\beta_{\varepsilon^m} \in (0, \quad +\infty)$ *if and only if* $\beta^1_{\varepsilon^m}$ and $\beta^2_{\varepsilon^m}$ are real roots such that $0 < \beta^1_{\varepsilon^m} < \beta^2_{\varepsilon^m}$. We can prove that in fact that is the case if the value of the parameter $\beta_w > 0$ under **Assumption 1.**

We can easily see that the following properties are satisfied.

$$D(0) \equiv \lim_{\beta_w \to 0} D = 0 \tag{66}$$

$$E(0) \equiv \lim_{\beta_w \to 0} E = 0 \tag{67}$$

$$\lim_{\beta_w \to 0} (AC - D) = AC(0) > 0. \tag{68}$$

In this case, $\beta^1_{\varepsilon^m}$ and $\beta^2_{\varepsilon^m}$ in Eq. (50) become to be the real roots such that $0 < \beta^1_{\varepsilon^m} < \beta^2_{\varepsilon^m}$ for sufficient small $\beta_w > 0$, because of the inequality (68) and the fact that $\lim_{\beta_w \to 0} (ABE) = 0$. This situation is illustrated in figure 2.

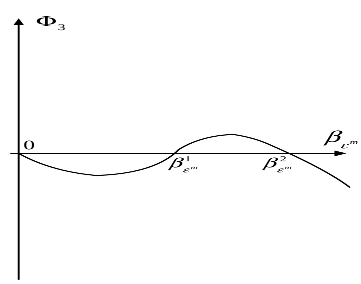

Figure 2: The parameter Φ_3 as a function of β_{ϵ^m}.

Furthermore, we can show that

$$A/B - \beta_{\varepsilon^m}^2 = \frac{D\{(AC - D) - \sqrt{(AC - D)^2 - 4ABE}\}}{2B(CD + BE)} \quad , \tag{69}$$

which becomes to be positive for sufficiently small $\beta_w > 0$.

We can obtain the following important proposition from the above observations.

Proposition 7

Suppose that $\beta_w > 0$ is sufficiently small. Then, under **Assumption 1**, there exist the parameter values $\beta_{\varepsilon^m}^1$ and $\beta_{\varepsilon^m}^2$ such that $0 < \beta_{\varepsilon^m}^1 < \beta_{\varepsilon^m}^2$ which satisfy the following properties.

(i) The equilibrium point of the degenerated four-dimensional system (S_2) is locally asymptotically stable for all $\beta_{\varepsilon^m} \in (\beta_{\varepsilon^m}^1, \beta_{\varepsilon^m}^2)$, and it is locally unstable for all $\beta_{\varepsilon^m} \in (0, \beta_{\varepsilon^m}^1) \cup (\beta_{\varepsilon^m}^2, +\infty)$.

(ii) The equilibrium point of the original five-dimensional system (S_1) is locally asymptotically stable for all sufficiently small $\beta_{\pi^m} > 0$ if $\beta_{\varepsilon^m} \in (\beta_{\varepsilon^m}^1, \beta_{\varepsilon^m}^2)$.

(iii) The equilibrium point of the original five-dimensional system (S_1) is locally unstable for all sufficiently small $\beta_{\pi^m} > 0$ if $\beta_{\varepsilon^m} \in (0, \beta_{\varepsilon^m}^1) \cup (\beta_{\varepsilon^m}^2, +\infty)$.

Proof.

(i) We already know from the above observations that there exist the parameter values $\beta_{\varepsilon^m}^1$ and $\beta_{\varepsilon^m}^2$ such that $0 < \beta_{\varepsilon^m}^1 < \beta_{\varepsilon^m}^2$ with the following properties if the relevant assumptions are satisfied. (22) For all $\beta_{\varepsilon^m} \in (\beta_{\varepsilon^m}^1, \beta_{\varepsilon^m}^2)$, we have both of $\Phi_3 > 0$ and $0 < \beta_{\varepsilon^m} < A/B$, so that all of the Routh-Hurwitz conditions for stable roots of the system (S_2) are satisfied. (2) For all $\beta_{\varepsilon^m} \in (0, \beta_{\varepsilon^m}^1) \cup (\beta_{\varepsilon^m}^2, +\infty)$, we have $\Phi_3 < 0$, so that at least one of the Routh-Hurwitz conditions of the system (S_2) is violated.

(ii) If $\beta_{\varepsilon^m} \in (\beta_{\varepsilon^m}^1, \beta_{\varepsilon^m}^2)$, all of the inequalities (46) are satisfied, so that we can apply the result of **Proposition 5.**

(iii)If $\beta_{\varepsilon^m} \in (0, \beta_{\varepsilon^m}^1) \cup (\beta_{\varepsilon^m}^2, +\infty)$, the characteristic equation (47) has at least one root with positive real part. In this case, the characteristic equation (39) also has at least one root with positive real part for all sufficiently small $\beta_{\pi^m} > 0$ by continuity.

By the way, at the points $\beta_{\varepsilon^m} = \beta_{\varepsilon^m}^1$ and $\beta_{\varepsilon^m} = \beta_{\varepsilon^m}^2$, we have the following properties.

$$b_1 > 0, \quad b_3 > 0, \quad b_4 > 0, \quad \Phi_3 = 0, \quad \Phi'(\beta_{\varepsilon^m}) \neq 0 \tag{70}$$

This means that at these points the 'simple' Hopf Bifurcations occur in the four-dimensional system (S_2) (as for the mathematical proof, see Liu (1994), Yoshida and Asada (2001), or Asada and Yoshida (2002)). The 'simple' Hopf Bifurcation is the particular type of the Hopf Bifurcation at which all the characteristic roots *except* a pair of purely imaginary ones have negative real parts. In other words, at these points the characteristic equation (49) has a pair of purely imaginary roots and two roots with negative real parts.

Furthermore, we can observe that there is no other Hopf Bifurcation point in this system because of the following reason. Asada and Yoshida (2002) proved that both of the conditions $b_4 \neq 0$ and $\Phi_3 = 0$ are *necessary* conditions for the occurrence of the Hopf Bifurcation, whether it is simple or non-simple, in the four-dimensional system. The point $\beta_{\varepsilon^m} = 0$ is the only other point which satisfies $\Phi = 0$, but at that point we have $b_4 = 0$. Therefore, the point $\beta_{\varepsilon^m} = 0$ is *not* the Hopf Bifurcation point. These analysis leads us to the following final important proposition, which establishes the existence of the cyclical fluctuation in both of the degenerated system and the original system.

Proposition 8.

(i) There exist some non-constant periodic solutions of the degenerated four-dimensional system (S_2) at some parameter values $\beta_{\varepsilon^m} > 0$ which are sufficiently close to $\beta^i_{\varepsilon^m}$ ($i = 1, 2$) which are defined in **Proposition 7.**

(ii) At the parameter values $\beta_{\varepsilon^m} 0$ which are sufficiently close to

$\beta^i_{\varepsilon^m}$($i = 1, 2$) which are defined in **Proposition 7**, the characteristic equation (39) of the original five-dimensional system (S_1) has a pair of complex roots for all sufficiently small $\beta_{\pi^m} > 0$.

Proposition 8 (ii) follows from the continuity of the characteristic roots with respect to the changes of the coefficients of the characteristic equation. This proposition establishes the existence of the cyclical fluctuation in the original five-dimensional nonlinear dynamical system (S_1).

Remark 2.

If we can find a parameter value $\beta_{\varepsilon^m} = \beta_{\varepsilon^m}* > 0$ at which all of the conditions $\Delta_j > 0$ ($j = 1, 2, 3$), $\Delta_4 = 0$, $\Delta'_4(\beta_{\varepsilon^m}) \neq 0$ are satisfied, we can establish the existence of a (simple) Hopf Bifurcation in the original five-dimensional system (cf. Liu (1994)). In this case, we can establish the existence of the closed orbit in the original five-dimensional system. (In fact, we also need another condition $a_5 > 0$, but this condition is always satisfied in this model.) However, the existence of the closed orbit (existence of a pair of purely imaginary roots) is not necessary for the existence of the cyclical fluctuation. Rather, the existence of a pair of complex roots is enough for the existence of the cyclical movement.