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Sustainable Living Standards and Adaptive Economizing in Economic Growth

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Sustainable Standards of Living and Adaptive Economizing in Economic Growth[‡§]

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Abstract

We consider economic growth in the presence of a minimal survival consumption level when preferences are lexicographic giving highest priority to the survival of the current and then all future generations. Knowledge of production is incomplete so future output made possible by current saving must be estimated. The resulting capital accumulation path is compared with the inter–temporally optimal one with full knowledge of the production function. Although previous results with the standard utility function show that for sufficiently strong discounting the adaptive strategy converges to the optimal path, in the present case in several scenarios the effect of adaptive economizing under incomplete information is drastic leading to demise of the economy although long-run survival would be possible.

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1 Introduction

A *minimal* sustainable standard of living is a realistic and important feature to take account of in the theory of growth, for if a given generation does not have enough product, it will not survive. Moreover, the only way future generations can survive is to make sure that the current generation survives and can bring its children up to form the next generation of adults. The issue is relevant for—in contrast to the high level of living achieved in the developed countries—substantial numbers of people in developing countries are close to or even below the subsistence threshold. Several authors have studied the implications of the existence of a minimal consumption level for optimal growth by incorporating Stone-Geary preferences into neo-classical growth models (see e.g. Rebelo (1992), Easterly (1994), Chatterjee and Ravikumar). Christiano (1989) uses this approach to demonstrate that consideration of a subsistence level allows to capture the essential features of the evolution of postwar Japanese saving rates much better than growth models assuming standard preferences. Additional empirical motivation for the consideration of subsistence consumption is provided in Steger (2000) where it is shown that a linear growth model with subsistence consumption is able to reproduce several stylized of economic growth.

In this paper we investigate the existence of a subsistence threshold in the context of macroeconomic growth theory by representing household preferences with a lexicographic ordering. The first priority in the lexicographic order is to maximize consumption up to the subsistence threshold. The second priority, given satisfaction of the first, is to insure a subsistence level for all future generations. The third priority, given satisfaction of the first two, is to optimize consumption for all generations. From the formal point of view we think of a Swiss Family Robinson type private ownership economy. The adults are manager-worker-owners to whom all proceeds of production accrue. They determine current consumption and savings based on the trade-off between their own consumption and the standard of living which their descendants could enjoy in the future. The savings are invested, and the augmented capital stock that results constitutes their children's endowment. The new generation of adults repeats the same economizing decision but on the basis of the capital stock inherited from their parents. Such an economy was investigated in some detail in the context of the one–sector growth model in Day and Lin (1992) and in Day (1999), both for the standard inter–temporal equilibrium formulation and for a boundedly rational, adaptive economizing formulation, where agents have incomplete information about future production technology and evaluate the capital stock left to the next generation based on a simple heuristic rather than on infinite horizon optimization.[1] In this paper we consider both cases in the same one–sector growth setting but when each generation faces a minimal consumption level necessary for survival.

Infinite horizon inter–temporal optimization models provide the appropriate analytical framework when designing controls for relatively simple mechanical systems whose physical characteristics are fully understood and when the objective is simple and stationary. Such is assuredly not the case with human endeavor in general. Consequently, individuals do not form detailed economic plans over the very long–run. To the extent tradeoffs are considered, we generally give explicit attention to horizons much shorter than the potential duration of the process as a whole and each generation considers the future on its own terms. Considerable support for this view has been provided by Rust (1994) who found in a variety of settings that the optimal strategies derived from dynamic optimization had weak explanatory power. Further support, especially relevant in the present setting of the one–sector growth model has been derived from laboratory experiments by Noussair and Matheny (2000) who found that over–investment and non–monotonic capital accumulation paths typically occurred. Such phenomena are ruled out for inter-temporally optimal paths but the adaptive economizing model considered in this paper can generate such behavior. A theoretical argument for the use of short horizon planning under incomplete information about the

[1]The relationship between adaptive and optimal or Nash equilibrium strategies has been studied in various contexts: a very early one is Kirman (1977) who showed that linear least squares learning by duopolists who face nonlinear demand curves need not converge to Cournot–Nash equilibria. Situations with various types of strategic interaction in game theoretic settings have been explored by Fudenberg and Levine (1998). The representation of learning by agents who use econometric methods has been explored in depth by Evans and Honkapohja (2001) but for linear models. They emphasize conditions that guarantee convergence to competitive equilibria. Here our concern is to explore what happens when people do *not* use econometric or other sophisticated strategies, not in the belief that people are irrational, but because such strategies depend on information and knowledge that are seldom available or which can only be acquired at great cost in time and resource. After all, it is possible that only trained mathematical economists use such rules and then only in their research.

economic environment has been put forward in Dawid (2005), where it has been shown in a standard one-sector growth model that in cases where individuals have incomplete information about the production function the use of adaptive economizing might lead to higher total discounted utility than infinite horizon optimization.

Given these anecdotal, experimental and theoretical arguments, considerations about the impact of incomplete information and short horizon planning should be included in the economic theory of growth. However, the infinite horizon dynamic optimization formulation under complete information is still an appropriate way to formalize the idea of inter–temporal equilibrium in a one–sector economy, and provides a benchmark against which to evaluate more realistic formulations. For that reason we consider that formulation first. Before taking it up, we describe in §2 the basic ingredients of the analysis: the production function, the lexicographic preferences, and the relationship of the survival consumption threshold to viable capital accumulation. In §3 inter–temporal optimal trajectories are derived and characterized. In §4 the behavior of adaptive economizing trajectories is described when knowledge of production is incomplete and future income must be projected. Then the two approaches are compared. For very small initial capital stocks (not surprisingly) both approaches give identical results: demise. For larger capital stocks, however, their behaviors are different. In addition to the convergence to or fluctuation about optimal growth paths, our analysis shows that for some conditions adaptive economizing may lead to extinction when in fact long–run survival is possible. In particular, extinction always occurs for small discount factors.

Under the inter–temporally optimal policy an increase in the minimal level of consumption has only marginal long–run effects if the initial capital stock is sufficiently large. With adaptive economizing, however such an increase might trigger the long–run extinction of capital even if the initial capital stock is way above the minimal capital stock which can sustain this level of consumption. Hence the implications of a minimal standard of living can indeed be quite dramatic. Nonetheless, if our economy is sufficiently wealthy and productive, a steady state exists which is optimal in the usual sense. Moreover, the adaptive economy can converge to it, given appropriate conditions on its time preference and productivity parameters.

2 Production, Preferences and Viability

2.1 Production and Capital Accumulation

We assume the standard neoclassical production function with capital and labor inputs which in per labor terms is expressed by a function, $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ which is twice continuously differentiable on $(0, \infty)$ with the properties

$$\begin{array}{c} f'(k) > 0, \quad f''(k) < 0 \quad \forall \quad k > 0 \\ f(0) = 0, \quad \lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0. \end{array} \tag{1}$$

Every period the adults decide what amount of output to consume, c_t, where

$$0 \leq c_t < f(k_t). \tag{2}$$

The rest of output, $f(k_t) - c_t$, is invested so the equation of capital accumulation is given by

$$k_{t+1} = \frac{1}{1+n}\Big[(1-\delta)k_t + f(k_t) - c_t\Big], \qquad k_t \geq 0, \tag{3}$$

where δ is the real depreciation rate and n the population growth rate. We denote by k^m the maximal sustainable capital stock, i.e. k^m is the unique positive capital stock satisfying

$$k^m = \frac{1}{1+n}\Big[(1-\delta)k + f(k)\Big].$$

2.2 Lexicographic Preferences

Children cannot survive without parents and if parents want descendants, their first effort must be directed to their own survival. Correspondingly, we assume that the current generation wants, first, to attain at least a level of consumption, $\bar{c}$; second, given that its own survival is assured, it wants its descendants for as many generations as possible to achieve at least the same level; third, and given the assumed survival of all future generations, it wants the most satisfying tradeoff between current consumption and that which future generations could potentially enjoy. Given these preferences, parents maximize their own consumption if income is below subsistence. If consumption has reached this level, they give future generations' consumption priority over more consumption for themselves. When survival at this level is *apparently* attainable for themselves *and* their heirs, the current generation considers the trade–off between its own consumption and that of its descendants, discounting the utility of future generations in the usual way. We suppose that a generation's satisfaction from consumption, given that survival is assured, can be represented by a twice continuously differentiable utility function $u : \mathbb{R}_0^+ \mapsto \mathbb{R}_0^+$ that satisfies

$$\begin{aligned} u'(c) > 0, \quad u''(c) < 0, \quad \forall c > 0 \\ \text{and} \quad \lim_{c \to 0} u'(c) = \infty. \end{aligned} \tag{4}$$

We will refer to feasible trajectories that are best in terms of the lexicographic ordering as L^{**} optimal.

2.3 Minimal Consumption and Viability

When there is no minimal consumption level, population survival is not a problem. The situation is quite different when such a survival threshold exists. Let $\bar{c}$ be this threshold. And let $\bar{k}$ be the unique capital labor ratio such that $f(\bar{k}) = \bar{c}$. Given the monotonicity of the production function defined in (1), it is clear that for all $k_t < \bar{k}$

$$y_t = f(k_t) < \bar{c}. \tag{5}$$

In this case wealth is so low the population cannot survive. Let us refer to $\bar{k}$ as the *disaster wealth level.*

Next, suppose current wealth is above the disaster level. The current generation *can* consume $\bar{c}$. If enough output is invested, children could survive to form the next generation. Is there a wealth level that will enable the economy to survive forever? If that were possible, then certainly it would require $c_t \geq \bar{c}$ for all t. If consumption were always at subsistence, the equation of *maximal capital accumulation* would be

$$k_{t+1} = \phi(k_t) := \max\left\{\frac{1}{1+n}\Big[(1-\delta)k_t + f(k_t) - \bar{c}\Big], \left(\frac{1-\delta}{1+n}\right)k_t\right\}. \tag{6}$$

Every capital accumulation trajectory is under our lexicographic preferences bounded above by this equation.

We note that, $\phi(k)$ is strictly concave and monotonically increasing for all $k > \bar{k}$. If $\bar{c}$ is not too big[2], there will exist two stationary states of (6), say $\hat{k}$ and $\hat{\hat{k}}$, such that $\bar{k} < \hat{k} < \hat{\hat{k}}$, as illustrated in figure 1. The first one is unstable, the second one asymptotically stable if consumption is constant at $c_t = \bar{c}$ for all t.[3] If initial wealth is above the disaster level but insufficient to allow survival of all future generations ($\bar{k} \leq k_0 < \hat{k}$), consumption must follow subsistence until the finite time, t^*, is reached when wealth falls below the disaster level and population must die out. The maximal survival period is t^*. Consumption greater than $\bar{c}$ for any generation below t^* would lower wealth and cause an earlier demise. When $\hat{k} < k_t$, income is large enough to allow existence forever and for some generations to exceed the subsistence level. If $k_t > \hat{\hat{k}}$, the subsistence level can be maintained indefinitely, but even under minimal consumption capital must decumulate and converge from above to $\hat{\hat{k}}$. Accordingly, any steady state of the capital accumulation process with lexicographic preferences must lie in the interval $[\hat{k}, \hat{\hat{k}}]$.

Insert figure 1 here!

[2]If $\bar{c}$ is so large that no fixed point of $\phi(k)$ exists, it is impossible to sustain minimal consumption $\bar{c}$ for all future periods regardless of the initial capital endowment. Hence, long-run viability is impossible. We will not deal with such a scenario but only situations where long run viability is possible for a sufficiently large initial capital stock.

[3]From figure 1 we see that both $\bar{k}$ and $\hat{k}$ increase while $\hat{\hat{k}}$ decreases if $\bar{c}$ increases. These preferences are a special case of the general class of L^{**} utility functions discussed for example in Day (1996).

3 Optimal Growth

3.1 Inviable Societies

Imagine a society that experiences a natural catastrophe or devastating war that reduces its capital stock proportionally more than its population, or one that arises from a new colony that started with inadequate capital for its members, or for which its capital was quite unproductive. If initial wealth is below the disaster level saving for the future is pointless. The best our impoverished agents can do is simply consume their entire income. The population dies out and aggregate capital stock simply decays exponentially at the depreciation rate. Or, if wealth is above the disaster level, the initial survival is assured. According to the second priority, consumption is maintained at the survival threshold until the maximal survival period is reached, when the population dies out. From that point on aggregate capital stock decays exponentially.

Proposition 1. Inviable Societies

(i) *Immediate Demise:* $0 < k_0 < \bar{k}$. *The initial population,* N_0*, consumes the entire production (*$c_0 = f(k_0)$*) and immediately dies out; the aggregate capital stock,* K_t*, decays according to*

$$K_{t+1} = (1-\delta)K_t, \tag{7}$$

where the initial aggregate capital stock $K_0 = N_0 k_0$*.*

(ii) *Finite Survival:* $\bar{k} < k_t < \hat{k}$*. Population subsists for* $t^* - 1$ *periods and dies out in period* t^**, where* t^* *is the smallest* t *such that* $\phi^t(k_0) < \bar{k}$*.*

$$\begin{array}{lll} c_t = \bar{c} & , & t = 0, \dots, t^* - 1, \\ c_{t^*} = f(k_{t^*}) & , & t = t^*, \\ k_{t+1} = \phi(k_t) & , & t = 0, 1, \dots, t^* - 1. \end{array}$$

For all $t \geq t^*$ *aggregate capital stock decays exponentially according to (7).*

3.2 Viable Societies

When initial wealth is sufficient to assure the survival of future generations, i.e. $k_0 \geq \hat{k}$, the optimal consumption strategy takes into account the entire future in the usual way and is characterized by the optimal growth problem[4]

$$\begin{aligned} V(k_0) &:= \max \sum_{t=0}^{\infty} \alpha^t u(c), \\ \text{s.t.} \qquad & \bar{c} \leq c_t \leq f(k_t), \\ k_{t+1} &= \tfrac{1}{1+n}\Big[(1-\delta)k_t + f(k_t) - c_t\Big], \end{aligned} \tag{8}$$

where α is the time preference or discount parameter. With these definitions the equation of L^{**} optimal capital accumulation can be denoted by

$$k_{t+1} = \tau^{\ell}(k_t) = \frac{1}{1+n}\Big[(1-\delta)k_t + f(k_t) - h^*(k_t)\Big]. \tag{9}$$

Here $h^*(k_t)$ denotes the corresponding optimal consumption function. Existence and uniqueness of the optimal accumulation path follows from standard arguments (see Stokey and Lucas (1989)). When $\bar{c} = 0$, the optimal growth problem reduces to the conventional one which is defined by (8) for all $k_0 > 0$. In this case, denote the capital accumulation map by $\tau(k)$. The equation of optimal capital accumulation where $\bar{c} = 0$ has the following properties (see Stokey and Lucas (1989)).

(i) For all $k_0 \in (0, k^m)$ there exists a unique steady state, $\tilde{k}(\alpha) \in (0, k^m)$, that depends on the discount factor, α.

(ii) At the steady state, $\tilde{k}$ associated with α, the net rate of return on capital satisfies

[4]Note that in principle such a formulation is compatible with growth models with Stone-Geary preferences $u(c_t) = \frac{(c_t-\bar{c})^{\sigma}-1}{1-\sigma}$ which have been used in the literature to model subsistence consumption. However, in our setting we implicitly assume that marginal utility of consumption stays finite for $c \to \bar{c}$ whereas under Stone-Geary preferences it goes to infinity for $c \to \bar{c}$. In spite of this, the characterization of optimal growth paths for different parameter ranges that we obtain in this subsection corresponds exactly to the one derived under Stone-Geary preferences with a linear production function (see Rebelo (1992)). Considering preferences of the form $u(c - \bar{c})$ would make our analysis more cumbersome without altering the qualitative insights concerning the impact of adaptive economizing on the growth paths.

$$\rho(\tilde{k}) := \frac{f'(\tilde{k}) - (n+\delta)}{1+n} = \frac{1-\alpha}{\alpha}. \tag{10}$$

(iii) All optimal trajectories converge monotonically to the steady state.

What happens when $\bar{c} > 0$? To answer this, we must find out when optimal trajectories generated by $\tau(k)$ satisfy the minimal consumption constraints. That is, if $\{c_t\}_0^\infty$ is an optimal trajectory when $\bar{c} = 0$, when does c_t exceed $\bar{c}$?

For inviable societies, we know from Proposition 1 that the capital accumulation map is given by

$$\tau^\ell(k) = \phi(k) \quad \forall\, 0 \leq k < \hat{k}$$

for all periods where the population can survive. Therefore $\tau^\ell(k)$ in general differs from the optimal accumulation map $\tau(k)$. In particular, for $k < \min[\tilde{k}, \hat{k}]$ we have $\tau^\ell(k) = \phi(k) < k < \tau(k)$. Consequently, optimal trajectories would be increasing if $\bar{c} = 0$ whereas for $\bar{c} > 0$ they are decreasing as long as the population survives. Consequently $\tau(k)$ does not characterize the L^{**} optimal trajectories for $\bar{c} > 0$ if the initial capital stock is small.

Now consider scenarios where $k_0 > \hat{k}$. A first observation is that the optimal capital accumulation map $\tau^\ell(\cdot)$ has to satisfy

$$\hat{k} \leq \tau^\ell(k) \leq \phi(k)\ \forall k \in [\hat{k}, k^m]. \tag{11}$$

Accordingly, $\tau^\ell(k) = \tau(k)\ \forall k \in [\hat{k}, k^m]$ holds if and only if $\hat{k} \leq \tau(k) \leq \phi(k)\ \forall k \in [\hat{k}, k^m]$. We will see that such a scenario never occurs for $\bar{c} > 0$. Furthermore, it is easy to see that a capital stock $k \in (\hat{k}, \hat{\hat{k}})$ is a steady state of $\tau^\ell(\cdot)$ if and only if it is a steady state of $\tau(\cdot)$.[5] We denote by $\hat{\alpha}$ the unique value of α that satisfies equation (10) when the capital stock $\hat{k}$ coincides with the positive steady state $\tilde{k}$, i.e. for $\alpha = \hat{\alpha}$ we have $\tilde{k} = \hat{k}$. Solving (10) for α we find that

[5]One way to check this observation is to compare the conditions for an interior steady state derived from the Euler equation. If $k \in (\hat{k}, \hat{\hat{k}})$ none of the constraints imposed by the minimal consumption threshold $\bar{c}$ are binding in the neighborhood of k and therefore the condition derived for $\bar{c} > 0$ coincides with the one for $\bar{c} = 0$.

Insert figure 2 here!

$$\hat{\alpha} = \frac{1}{1+\rho(\hat{k})}. \tag{12}$$

Note that $\rho(\hat{\hat{k}}) < 0$ and therefore $\tilde{k} < \hat{\hat{k}}$ holds for all $\alpha \in [0,1]$.

This implies the following relationships concerning $\tilde{k}(\alpha)$.

$$\begin{array}{llll} \text{Weak Discounting:} & \hat{k} \le \tilde{k}(\alpha) \le \hat{\hat{k}} & \text{for all} \quad \alpha \ge \hat{\alpha} & (a) \\ \text{Strong Discounting} & \tilde{k}(\alpha) < \hat{k} & \text{for all} \quad \alpha < \hat{\alpha} & (b) \end{array} \tag{13}$$

Figure 2 shows the situations that correspond to the two inequalities (13).

When $\tilde{k} > \hat{k}$, as in figure 2(a), the situation is relatively straight forward. Basically, L** optimal trajectories are generated by the mapping

$$k_{t+1} = \min\Big\{\phi(k_t), \tau(k_t)\Big\}$$

for all $k_t \in [\hat{k}, k^m]$. Therefore, for $k_0 \in [\hat{k}, k^m]$ trajectories converge to the steady state $\tilde{k}$. When $\tilde{k} < \hat{k}$, however, this steady state is in the inviable region. As shown in figure 2(b), there will exist a point $k^A \in [\tilde{k}, \hat{k}]$ such that $\tau(k)$ is feasible in period t as long as $k_t > k^A$. This means that the current generation could follow the capital accumulation path maximizing the discounted utility stream for $\bar{c} = 0$ without jeopardizing its survival but only at the cost of reducing the number of future surviving generations. Consequently, the L^{**} optimal trajectory in such cases converges towards the minimal capital stock guaranteeing infinite survival. To summarize, we have

Proposition 2. L^{**} **Optimal Growth**

(i) Weak Discounting: $\alpha > \hat{\alpha}$*. There exists a unique wealth* $k^A \in [\hat{k}, \tilde{k}]$ *such that* $\tau(k^A) = \phi(k^A)$*. The equation of* L^{**} *optimal capital accumulation is*

$$k_{t+1} = \tau^{\ell}(k_t) := \begin{cases} \phi(k_t) & , \; k_t \in (\hat{k}, k^A] \\ \tau(k_t) & , \; k_t \in (k^A, k^m] \end{cases}$$

and k_t converges monotonically to $\tilde{k}$ for all $k_0 \in (\hat{k}, k^m]$.

(ii) Strong Discounting: $\alpha < \hat{\alpha}$. For all $k_0 \in (\hat{k}, k^m]$, the equation of L^{**} capital accumulation is

$$k_{t+1} = \tau^{\ell}(k_t) \in [\hat{k}, k_t),$$

with a continuous mapping $\tau^{\ell}(k)$ and k_t converges monotonically to $\hat{k}$ for all $k \in (\hat{k}, k^m]$.

(iii) For $k_0 = \hat{k}$ the economy survives at the steady state, $\hat{k}$ regardless of the discount factor α.

A formal proof is contained in the Appendix.

4 Growth with Adaptive Economizing

Our purpose now is to investigate growth when knowledge of production is incomplete and preferences take into account both the subsistence threshold and the tradeoff between consumption of the current and future generations. However, when predicting consumption decisions of future generations the current generation relies on a simple stationary forecast. We take the adaptive economizing model as a stylized representation of a heuristic planning procedure where individuals reduce a complex infinite horizon planning problem to a simpler problem with short planning horizon. A detailed analysis of the implications of the use of such a planning procedure in the case of standard preferences without a minimal consumption level has been carried out in Day and Lin (1992), Day (1999) and Dawid (2005). Here, we are interested in the implications of adaptive economizing if positive minimal consumption is needed for survival.

4.1 Adaptive Preferences

As before, our Swiss Family Robinson adults have an absolute preference for survival; first for themselves and next, if their own survival is assured, for future generations. They do not compare the discounted utility of all possible consumption trajectories. Instead, we assume they compare the trade–off between their family's current consumption, c, with a level of consumption, c^1, that they believe could be enjoyed forever, given the capital stock they leave to their descendants. This means they compare present consumption with a constant sequence, $c^1, c^1, c^1, \ldots$, that could be sustained if future generations maintain their wealth endowment. At the same time, that endowment can be exploited as the next generation pleases. The lexicographic preference ordering thus described is represented by the utility function,

$$u^{\ell}(c, c^1) = \begin{cases} u(c) & , \text{ if } \ 0 \leq c < \bar{c} \\ u(\bar{c}) + \frac{\alpha}{1-\alpha} u(c^1) & , \text{ if } \ c \geq \bar{c},\ 0 \leq c^1 \leq \bar{c} \\ u(c) + \frac{\alpha}{1-\alpha} u(c^1) & , \text{ if } \ c \geq \bar{c},\ c^1 > \bar{c} \end{cases} , \tag{14}$$

Because the preference ordering does not explicitly take into account all future generations, to distinguish this preference ordering from the L^{**} optimal case, we refer to feasible trajectories that sequentially maximize (14) as ℓ^{**} optimal.[6]

4.2 Investment and Capital Accumulation

Assume that the current generation has only incomplete information about the production function, in particular consider a scenario where they can only observe current production input and output as well as the current capital rate of return $r = f'(k)$. The current generation uses these observations to predict production output for capital stocks different from the current one,

[6] Modern growth theory as applied in the "real business cycle" literature (see e.g. Cooley (1995)) assumes a complete preference ordering over all possible future consumption sequences, even though those consequences will not involve them but all future generations instead. Notice that the term $\frac{\alpha}{1-\alpha} u(c^1)$ in (14) is equivalent to $u(c) + \alpha u(c^1) + \alpha^2 u(c^1) + \cdots + \alpha^t u(c^1) \ldots$, so it takes account of the entire future in terms of the potential constant future utility stream $u(c^1), \ldots$. However, it does not take account of any non–constant future streams. Nonetheless, behavior based on it can converge to the optimal inter–temporal path.

which is equivalent to a first order approximation of the production function. Projected output with the planned capital stock k^1 becomes

$$y^1 = f(k) + f'(k)(k^1 - k) \tag{15}$$

when the current capital stock is k.[7]

Let c stand for current consumption. Let c^1 be the next generation's consumption that, according to the current generation's estimates, could be sustained by future generations, if those generations subsequently choose to maintain the endowment, k^1. The sustainable consumption level projected by the current generation would be

$$c^1 = \Big[y^1 - (n + \delta)k^1\Big], \tag{16}$$

where capital stock, k^1, of course, will depend on current consumption,

$$k^1(k, c) = \frac{1}{1+n}\Big[(1 - \delta)k + f(k) - c\Big]. \tag{17}$$

Inserting expression (15) and (17) into (16), and doing some re–arranging, one finds that the projected sustainable consumption for future generations can be expressed as a function of the current capital stock, current consumption, and the current net rate of return on investment,

$$c^1(k, c) = \Big[1 + \rho(k)\Big]\Big[f(k) - (n + \delta)k\Big] - \rho(k)c, \tag{18}$$

where again

$$\rho(k) = \frac{f'(k) - (n + \delta)}{1 + n}.$$

[7]While it is usually assumed that producers know their production functions, in reality they typically understand their current operating conditions but can only estimate output at points removed from current practice. Large companies have engineering departments that estimate production relationships while farmers have to guess, sometimes with the help of experimental studies and advisors. The use of linear approximations of unknown non–linear relationships is common practice in many planning heuristics used by business firms, see e.g. Nahmias (1993) for examples in the field of Operations Management. Well trained economists can substitute econometric methods for the "naive" projection assumed in (15). Our concern is not with sophisticated methods used by economists but with the behavior that approximates ordinary individuals and firms. Obviously, to do that *we* must know their environment exactly.

We denote the capital stock where $\rho(k) = 0$ by k^r. Existence and uniqueness of this point follows directly from the assumptions about production.

The most preferred combination of present consumption and future sustainable standard of living (c_t, c_t^1) maximizes the ℓ^{**} utility function (14) subject to the relationships between present and future consumption possibilities given by (18) and subject to the constraint that capital cannot be consumed, i.e.,

$$0 \leq c \leq f(k). \tag{19}$$

The solution of this problem is *the adaptive ℓ^{**} consumption strategy*,

$$c = h^\ell(k). \tag{20}$$

Given the imperfect information about production and limited foresight, the agents act rationally but boundedly so. The implied ℓ^{**} adaptive economizing equation of capital accumulation is

$$k_{t+1} = \theta^\ell(k_t) := \frac{1}{1+n}\Big[(1-\delta)k_t + f(k_t) - h^\ell(k_t)\Big]. \tag{21}$$

For convenience we define $\theta(k_t) \equiv \theta^\ell(k_t)$ when $\bar{c} = 0$.

Guaranteeing sustenance for all future generations can be achieved if the initial stock is large enough and the subsistence level, $\bar{c}$, small enough. But, if the capital endowment is not big enough, all generations may not be able to reach $\bar{c}$. Moreover, given that our agents are boundedly rational, they may not be able to provide consumption $\bar{c}$ forever *even if that level is technically feasible.*

4.3 Adaptive Economizing Trajectories

4.3.1 No Subsistence Threshold ($\bar{c} = 0$)

Day and Lin (1992) established the following properties for the Adaptive Economizing model when $\bar{c} = 0$.

(i) There exists a unique, positive steady state, $\tilde{k}$, which coincides with the optimal steady state and satisfies (10).

(ii) When discounting is strong enough, capital converges to the optimal steady state, $\tilde{k}$.

(iii) When discounting is weak enough, consumption, investment, and capital exhibit persistent periodic or chaotic fluctuations about the optimal steady state.

A formal proof is found in Day and Lin (1992) or Day (1999, Chapter 17).

Ironically, these results imply that the more the current generation values the standard of living its heirs can enjoy, the greater the chance of fluctuations and suboptimal savings; whereas, the *less* the current generation cares about the future, the more likely consumption will converge to the inter–temporally optimal path. The reason for this seemingly counter intuitive result is that the first–order approximation of the production function leads to an overestimation of the marginal benefits of savings along increasing capital accumulation paths (but to an underestimation along decreasing capital accumulation paths). Consequently, the present generation saves more than is inter–temporally optimal. if the resulting capital stock overshoots the steady state, the rate of return falls enough so that the next generation saves less than is optimal; capital declines, the rate of return and investment recover, and the fluctuations persist.

So what happens when $\bar{c} > 0$? To answer this question, the discrepancy between the estimated production and the realized production for capital stock $k^1(k_t, c_t)$ must be taken into account. Given (17), the estimated output over–estimates the realized output at $k^1(k_t, c_t)$ if $k^1(k_t, c_t) \neq k_t$. The future standard of living will, therefore, be over–estimated as well. *This may lead to a level of savings that the current generation believes will enable future generations to survive when in fact that is not possible.*

4.3.2 Can Adaptive Economizing Guarantee Survival When it is Possible?

What is the minimum endowment the current generation believes it must leave future generations so they can survive? Because of their incomplete information, this endowment must be estimated. From (18) the maximal level of current consumption compatible with a *belief* that future generations can survive is

$$\tilde{c}(k) := \max\Big\{c|c^1(k,c) \geq \bar{c}\Big\} = \frac{(1+\rho)(f(k)-(n+\delta)k)-\bar{c}}{\rho}.$$

Accordingly, the *projected or subjective future survival endowment* is

$$\ell(k_t) := \frac{1}{1+n}((1-\delta)k + f(k) - \tilde{c}(k)), \quad k > \bar{k}. \tag{22}$$

Whenever $k^1(k_t, c_t) \geq \ell(k_t)$, the current generation believes that viability for all future generations is assured. On the other hand, the current generation can only survive if $k^1(k_t, c_t) \leq \phi(k_t)$. So, all-together the current generation believes that whenever

$$\ell(k_t) \leq k_{t+1} \leq \phi(k_t)$$

survival of the current and all future generations is guaranteed. Under which circumstances is it possible to choose such a k_{t+1}? To answer this question we use the following characterization of the function $\ell(\cdot)$ which is illustrated in figure 3.

Insert figure 3 here!

Lemma 1 . Characteristics of the Minimal Projected Sustainable Endowment

(i) $\ell(\hat{k}) = \hat{k}$*;*

(ii) $\ell(k)$ *is a continuous function such that for all* $k \neq \hat{k}$, $\ell(k) < \hat{k}$.

(iii) There exists a capital stock $k^B \in (\hat{k}, k^r)$ *such that for all* $k \geq k^B$, $\ell(k) = 0$.

(iv) For all $\bar{k} < k < \hat{k}$, $\ell(k) > \phi(k)$.

(v) For all $\hat{k} < k < k^m$, $\ell(k) < \phi(k)$.

A proof is contained in the Appendix.

Points (iv) and (v) of the lemma show that the range of current capital stocks, k_t, where the current generation believes in the survival of the current and all future generations is $[\hat{k}, k^m]$ and therefore exactly coincides with the range of current capital stocks where long term survival is indeed possible. However, under preferences (14) the restrictions on k_{t+1} imposed by (subjective) viability considerations for $k_t \geq \hat{k}$ (i.e. $k_{t+1} \geq \ell(k_t)$) are less stringent than those needed to guarantee actual survival (i.e. $k_{t+1} \geq \hat{k}$). Conditions (i) and (ii) imply that only for a single initial capital stock, namely $k_0 = \hat{k}$ is actual long–run viability guaranteed by subjective viability considerations. For initial capital stocks above the threshold $\hat{k}$ the optimal consumption strategy under complete information would guarantee continuing survival. Given the limited information of the adaptive economizer, however, it is not clear *a priori* whether such an outcome will always be achieved. In particular, if $\ell(k_t) \leq k_{t+1} < \hat{k}_t$ the current generation believes that long-run viability is assured by their actions although the choice of consumption level of this generation actually buries the prospects of long-run survival.

4.3.3 ℓ^{**} Adaptive Economizing Trajectories

Given the properties of the ($\bar{c} = 0$) adaptive economizing trajectories outlined in §4.3.1, and given the nature of the $\ell(k)$ function as shown in figure 4, it is possible to derive trajectories for the ℓ^{**} adaptive economizing strategy. First, in the case of inviable societies capital accumulation paths under adaptive economizing coincide with the optimal ones. We have

Proposition 3. Adaptive Economizing in Inviable Societies
If $0 < k_0 < \hat{k}$ consumption under ℓ^{**} adaptive economizing is identical to the optimal consumption strategy characterized in Proposition 1. In particular, the economy immediately dies out if $k_0 < \bar{k}$, and for $\bar{k} \leq k_0 < \hat{k}$ lives for a finite number of periods t^* with

$$\begin{array}{ll} c_t = \bar{c} & , \ t = 0, \ldots, t^* - 1, \\ c_{t^*} = f(k_{t^*}) & , \ t = t^*, \\ k_{t+1} = \phi(k_t) & , \ t = 0, 1, \ldots, t^* - 1. \end{array}$$

A formal proof is given in the Appendix.

If the initial capital stock is sufficient to allow for a viable economy various possibilities may occur which are illustrated in figure 4.

Insert figure 4 here!

With weak discounting and when adaptive economizing is stable, the situation is essentially the same as the optimal case shown in figure 2(a). In this case the positive steady state $\tilde{k}$ of the adaptive economizing strategy (and hence also of the optimal policy) without minimal consumption lies in the range $[\hat{k}, k^m]$ of long–run viability, which implies that consumption at the steady state is above the minimal level $\bar{c}$. Accordingly, $\tilde{k}$ is also a steady state of the ℓ^{**} adaptive strategy in the presence of a minimal consumption threshold.

However, if the positive steady state is unstable for the adaptive economizing strategy as shown in figures 4(b, c), periodic or chaotic trajectories will emerge around it almost surely. Depending on the size of the fluctuations, the fluctuating path might stay above $\hat{k}$ guaranteeing long-run survival (figure 4(b)) or fall below $\hat{k}$, which implies that the economy can only survive for a finite number of periods (figure 4(c)).

In the case of strong discounting, however the effect of the incomplete information about future productivity becomes very significant. Incentives for current consumption are so strong that under adaptive economizing the capital accumulation paths for almost all initial capital stocks decrease over time. Due to the agents' underestimation of the minimal capital stock that has to be left to the coming generation in order to guarantee long-run survival, the decrease of capital does not stop at the crucial level $\hat{k}$ and therefore demise after a finite number of periods occurs even in cases where long-run survival would be feasible (see figure 4(d)).

These considerations are summarized in Proposition 4.

Proposition 4. ℓ^{**} **Adaptive Economizing Growth** *Let k^D be the smallest capital stock $k \geq \tilde{k}$ such that $\ell(k) = \theta(k)$. If $\theta(k) > \ell(k)$ for all $k \in [\tilde{k}, k^m]$ we set $k^D = k^m$.*

(i) Weak discounting $\hat{\alpha} < \alpha$.

(a) *Let* k^C *be the largest capital stock* $k \leq \tilde{k}$ *such that* $\phi(k) = \theta(k)$. *Then* $\hat{k} < k^C < \tilde{k} < k^D \leq k^m$ *and* $\ell(k) < \theta(k) < \phi(k)$ *for all* $k \in (k^C, k^D)$.

(b) *For* $k_t \in [\hat{k}, k^m]$ *the capital accumulation equation under adaptive economizing is given by*

$$k_{t+1} = \theta^\ell(k_t) := \begin{cases} \min[\phi(k_t), \theta(k_t)] & , \; k_t \in [\hat{k}, k^C] \\ \theta(k_t) & , \; k_t \in (k^C, \min[k^D, k^r]) \\ \max[\ell(k_t), \theta(k_t)] & , \; k_t \in [\min[k^D, k^r], k^r) \\ \theta(k_t) = \frac{1-\delta}{1+n} k_t & , \; k_t \in [k^r, k^m]. \end{cases}$$

(c) *If* $\tilde{k}$ *is asymptotically stable with respect to* θ *the economy is viable for initial capital stocks in the neighborhood of* $\tilde{k}$.

(d) *If* $\tilde{k}$ *is unstable, then depending on the form of the preferences, the discount factor, and the size of the initial capital stock, paths with* $k_0 \in [\hat{k}, k^m]$ *converge to* $\hat{k}$, *fluctuate in* $[\hat{k}, k^m]$, *or fall below* $\hat{k}$ *in which case the economy survives only for a finite number of periods.*

(ii) *Strong discounting* $0 < \alpha < \hat{\alpha}$.

(a) *We have* $k^D \in (\hat{k}, k^r)$ *and for* $k_t \in [\hat{k}, k^m]$ *the capital accumulation equation under adaptive economizing reads*

$$k_{t+1} = \theta^\ell(k_t) = \begin{cases} \ell(k_t) & , \; k_t \in [\hat{k}, k^D) \\ \max\Big(\theta(k_t), \; l(k_t)\Big) & , \; k_t \in [k^D, k^r) \\ \theta(k_t) = \frac{1-\delta}{1+n} k_t & , \; k_t \in [k^r, k^m]. \end{cases}$$

(b) *For almost all initial capital stocks* $k_0 \in [\hat{k}, k^m]$ *the population dies after a finite number of periods and paths of capital accumulation generated by adaptive economizing agents converge towards zero. The only initial capital stocks where long–run viability is achieved are those where the path hits* $\hat{k}$ *after a finite number of iterations.*

A formal proof is given in the Appendix.

5 Discussion

Comparing proposition 4 with proposition 2 we realize that the answer to our question 'Can adaptive economizing guarantee survival when it is possible?' is often negative. In particular, for strong discounting adaptive economizing generically leads to eventual demise although long run survival would be feasible. Even if discounting is sufficiently weak to allow for a steady state above $\hat{k}$, long run survival is not guaranteed. Discount factors close to one imply instability of the steady state and the resulting fluctuations might push the capital stock below $\hat{k}$ inducing demise after a finite number of periods. For initial capital stocks below $\hat{k}$ this fate is unavoidable but for $k_0 > \hat{k}$ the early depletion of the capital stock is due to the limited information and the simplified model of the world used by the decision maker.

Since $\tilde{\alpha}$ increases for increasing $\bar{c}$ the set of discount factors leading to unnecessary extinction becomes larger the higher the desired consumption level $\bar{c}$ is. Strictly speaking these observations apply only to cases where initial capital endowment is not too large. A particularly striking implication of proposition 4 is that in cases of strong capital depreciation, $\frac{(1-\delta)}{1+n}k^m < \hat{k}$, any sufficiently high initial capital endowment of $k_0 \geq k^r$ leads to demise under adaptive economizing regardless of the discount factor and the stability of the steady state $\tilde{k}$. The high initial endowment leads to strong overestimation of the productivity of capital for lower capital stocks and strong overconsumption by the current generation. It should be noted that the drastic implications of adaptive economizing discussed here occur although in our setting the adaptive economizers primary concern is survival of all generations and they are always able to tell correctly whether the current capital stock is in principle sufficient for long run survival or not.

Our results are quite surprising for in the standard growth model adaptive economizing generates paths that are qualitatively similar to the optimal ones as long as the discount factor is small. Also, intuitively one could expect that heavy discounting minimizes the effects of prediction errors of future outputs and imperfect foresight. However, the *smaller* the discount factor the larger is current consumption, the more severe is over–prediction of future production possibilities and the stronger is the resulting over-consumption. Although all generations are interested in allowing future generations the minimal consumption level, this can lead to the demise of the population and gradual decay of the capital stock.

The question now arises whether this unnecessary extinction is due to the simplistic way that the future is considered or to the crude manner of estimating future output. The answer seems to be the latter. Obviously, agents with perfect knowledge about the production function who are endowed with a capital stock above the crucial level $\hat{k}$ would never allow the capital stock to sink below this level, even if they follow the adaptive economizing strategy assumed here. On the other hand, Dawid (2005) has shown in a one-sector growth model framework with standard preferences that, as long as decision makers re–estimate the production function in a linear way every period, oscillations about the positive steady state can occur even if the planning horizon is infinite. Hence, in the scenario with lexicographic preferences, it is possible that unnecessary extinction could occur even if agents consider prospective inter–temporal trade–offs over the entire future.

Under the inter–temporally optimal policy an increase in the minimal level of consumption $\bar{c}$ has only marginal long run effects if the initial capital stock is sufficiently large. Our analysis shows that with adaptive economizing policies such an increase might trigger the long run extinction of capital even if the initial capital stock is way above the minimal capital stock $\hat{k}$ which can sustain this level of consumption. Hence the implications of a minimal standard of living can indeed be quite dramatic.

Throughout the analysis the population growth rate has been assumed to be constant. It influences the critical wealth level $\hat{k}$ as well as the steady state $\tilde{k}$ and the stability of the adaptive economizing process. If this rate were endogenized, the results would be modified. Likewise, as duly noted above, a change in $\bar{c}$ would influence the possible outcomes if it were determined culturally rather than biologically. Even in that event, however, there is surely a lower bound to any reduction in the subsistence level.

As it is, the analysis is suggestive of various extreme situations in which national or man–made disasters reduce capital stocks enough to precipitate decline, or in which a society combined low regard for future generations with an overestimation of labor productivity under reduced per capita capital endowment such that it decumulated its per capita wealth so much that future viability is endangered. To make this point we have relied on a standard representative agent framework. Important issues like wealth distribution call for a heterogenous agents model and have been ignored here. For example societies could attempt to redistribute wealth so that some members survive and be pushed above the critical wealth level $\hat{k}$. Other related considerations such as heterogeneous initial wealth and/or preferences would be worth exploring.

Appendix

Proof of Proposition 2:
(i) First we show that a unique stock $k^A \in [\hat{k}, \tilde{k}]$ exists. It follows from $\tilde{k} \geq \hat{k}$ that $\tau(\hat{k}) \geq \hat{k} = \phi(\hat{k})$. On the other hand, we have $\tau(\tilde{k}) = \tilde{k} \leq \phi(\tilde{k})$ since $\tilde{k} \in [\hat{k}, \hat{\hat{k}}]$. Denote by $h_0^*(k)$ the optimal consumption function for $\bar{c} = 0$. Because $\phi(k) - \tau(k) = h_0^*(k) - \bar{c} \;\forall\; k > \bar{k}$ the continuity and monotonicity of, $h_0^*(k)$, induces monotonicity and continuity of $\phi(k) - \tau(k)$ on $[\hat{k}, k^m]$ and this establishes the existence of a unique stock k^A with $\tau(k) = \phi(k)$.

The fact that both $h_0^*(k)$ and $\tau(k)$ increase monotonically further implies that for any path with initial stock larger than k^A, which is generated by $\tau(k)$, the inequalities $c_t \geq \bar{c}$ and $k_t \geq k^A$ hold for all t. Thus for $k_0 > k^A$ the optimal path of the problem with and without consumption threshold coincide and we have $\tau^\ell(k) = \tau(k)$ on $[k^A, k^m]$.

This leaves us with showing that on the interval $[\hat{k}, k^A]$ the optimal consumption is given by the minimal possible value $\bar{c}$. It follows from the concavity of u and the concavity of the value function of the problem that optimal consumption cannot decrease with an increasing capital stock. Thus, the fact that optimal consumption at k^A equals $\bar{c}$ implies that the same has to hold true on $[\hat{k}, k^A]$.

Finally, it is a standard result in growth theory that $\tau(k)$ is monotonically increasing. Accordingly, $\tau^\ell(k)$ is monotonically increasing on the interval $(\hat{k}, k^m]$ and any path starting in this interval has to converge monotonically to the unique steady state $\tilde{k} \in (\hat{k}, k^m)$.

(ii) With strong discounting we have $\tilde{k} < \hat{k}$ and accordingly $\tau^\ell(k)$ has no fixed point in the interval $(\hat{k}, \hat{\hat{k}})$. Furthermore, it is easy to see that $\hat{\hat{k}}$ cannot be a fixed point of τ^ℓ. Staying at $\hat{\hat{k}}$ yields consumption of $\bar{c}$ every period. Higher consumption can for example be generated by consuming $f(\hat{\hat{k}}) + (1-\delta)\hat{\hat{k}} - \hat{k} > \bar{c}$ in the current period, moving to capital stock to $\hat{k}$, and consuming $\bar{c}$ in all following periods. Hence, $\tau^\ell(\hat{\hat{k}}) < \hat{\hat{k}}$ and together we get $\tau^\ell(k) < k \;\forall\; k \in (\hat{k}, k^m)$, which implies that any L^{**} optimal path with $k_0 > \hat{k}$ is monotonically decreasing. To ensure long run viability τ^l further has to satisfy $\tau^\ell(k) \geq \hat{k}$ for all $k \in (\hat{k}, k^m]$ and we immediately get the claims of (ii).

(iii) Follows directly from $\phi(\hat{k}) = \hat{k}$ and the condition $\hat{k} \leq \tau^{\ell}(k) \leq \phi(k)$. $\square$

Proof of Lemma 1:

(i): For $k = \hat{k}$ we have $c^1(\hat{k}, f(\hat{k}) - (n+\delta)\hat{k}) = \bar{c}$, thus $l(\hat{k}) = \hat{k}$.

(ii): Continuity of ℓ follows from the continuity of $c^1(k, c)$ with respect to k and its monotonicity with respect to c. Note further that $c^1(k, (1-\delta)k + f(k) - (1+n)k^1) = f(k) - f'(k)k + (1+n)\rho(k)k^1$ is increasing in k^1 for $k < k^0$. Concavity of f implies that agents always overestimate future growth and hence $c^1(k, (1-\delta)k + f(k) - (1+n)k^1) > f(k^1) - (n+\delta)k^1$ for all $k^1 \neq k$. Since $f(\hat{k}) - (n+\delta)\hat{k} = \bar{c}$ we have $c^1(k, (1-\delta)k + f(k) - (1+n)\hat{k}) > \bar{c}$ and thus $\ell(k) < \hat{k}$ $\forall k \neq \hat{k}$.

(iii): For current capital stock $k = k^r$ and planned capital stock $k^1 = 0$ we get for the projected future output $y^1 = f(k^r) + f'(k^r)(0 - k^r) = f(k^r) - (n + \delta)k^r > \bar{c}$, which implies that $\ell(k^r) = 0$ and by continuity and monotonicity of $f(k) - f'(k)k$ the existence and uniqueness of $k^B < k^r$ with the given properties follows. Since $\ell(\hat{k}) = \hat{k} > 0$ we must have $k^B > \hat{k}$.

(iv): We first show that $\ell(k) > k$ for $k < \hat{k}$. To see this note that $c^1(k, f(k) - (n+\delta)k) = f(k) - (n+\delta)k$ increases with k on $[0, k^r]$. This implies that $c^1(k, f(k) - (n+\delta)k) < \bar{c}$ for $k < \hat{k}$ and using the monotonicity of $c^1(k, (1-\delta)k + f(k) - (1+n)k^1)$ with respect to k^1 establishes $\ell(k) > k$. Furthermore, it follows from $\hat{k} < k^r$ and $\phi(\hat{k}) = \hat{k}$ that $\phi(\hat{k}) < k$ for $k < \hat{k}$. Hence our result.

(v): Follows directly from $\phi(k) > \hat{k}$ and $\ell(k) < \hat{k}$ for all $k \in (\hat{k}, k^m]$. $\square$

Proof of Proposition 3:

For $k_0 < \bar{k}$ the claim of the proposition follows directly from the lexicographic structure of the preferences u^{ℓ}. For $\bar{k} \leq k_0 < \hat{k}$ we know from lemma 1 that $\ell(k_0) > k_0 > \phi(k_0)$. Accordingly, $c_1(k_0, \bar{c}) < \bar{c}$ and due to the lexicographic structure of u^{ℓ} current consumption is reduced to the minimal level $\bar{c}$. $\square$

Proof of Proposition 4:

(i):

(a): For $\alpha > \hat{\alpha}$ we have $\tilde{k} > \hat{k}$ and therefore $\phi(\tilde{k}) > \theta(\tilde{k}) = \tilde{k} > \hat{k} > \ell(\tilde{k})$. Furthermore, $\theta(\hat{k}) > \hat{k} = \phi(\hat{k})$. Continuity of $\ell(k), \phi(k), \theta(k)$ establishes that $\hat{k} < k^C < \tilde{k} < k^D$, where $\theta(k) > \ell(k)$ might hold on the entire interval $[\tilde{k}, k^m]$ in which case we set $k^D = k^m$.

(b): The optimization problem of the adaptive economizing agent for $k > \hat{k}$ can be written as

$$\max_{k^1} u((1-\delta)k + f(k) - (1+n)k^1) + \tfrac{\alpha}{1-\alpha} u((1-\delta)k + f(k) - (1+n)k^1)$$
$$\text{s.t.} \max\left[\tfrac{1-\delta}{1+n}k, \ell(k)\right] \leq k^1 \leq \phi(k). \tag{23}$$

The optimal solution to this problem under the weaker constraint $\frac{1-\delta}{1+n}k \leq k^1 \leq \frac{1}{1+n}((1-\delta)k + f(k))$ is given by $k^1 = \theta(k)$. Hence, whenever $\theta(k)$ satisfies (23) we have $\theta^\ell = \theta$. Concavity of the objective function implies further that whenever $\theta(k)$ lies outside the range given by (23) the optimal solution lies on the corresponding boundary of the interval. For all $k \in [\hat{k}, \tilde{k})$ we have $\theta(k) > k \geq \hat{k} \geq \ell(k)$ and by definition we have $\theta(k) > \ell(k)$ for all $k \in [\tilde{k}, k^D)$. Therefore, $\theta(k) \geq \ell(k)\ \forall k \in [\hat{k}, k^D]$. Comparing $\phi(k)$ and $\theta(k)$ we first observe that by definition $\phi(k) \geq \theta(k)$ for all $k \in [k^C, \tilde{k}]$. To show that $\phi(k) \geq \theta(k)$ holds also for $k \in [\tilde{k}, k^m]$ we show that the adaptive ℓ^{**} consumption strategy $h^\ell(k)$ for $\bar{c} = 0$ is non-decreasing in k on $[\tilde{k}, k^m]$. We denote this consumption strategy by $h^a(k)$. First note that $\theta(k) < k$ and therefore $h^a(k) > f(k) - (n+\delta)k$ for all $k > \tilde{k}$. Furthermore, if $h^a(k) < f(k)$ the first order condition

$$\begin{aligned} & u'(h^a(k)) \\ & -\frac{\alpha\rho(k))}{1-\alpha}\, u'\left(f(k) - (n+\delta)k + \rho(k)(f(k) - (n+\delta)k - h^a(k))\right) \\ & = \ 0 \end{aligned} \tag{24}$$

has to hold. This implies that for all capital stocks where $h^a(k) < f(k)$ we must have $\rho(k) > 0$. For any k where $\rho(k) \leq 0$ we have $h^a(k) = f(k)$ and therefore $\theta(k) = \frac{1-\delta}{1+n}k < \phi(k)$. Since $\ell(k) = 0\ \forall k \geq k^B$ and $k^B < k^r$ we get $\theta^\ell(k) = \theta(k) = \frac{1-\delta}{1+n}k < \phi(k)$ for all $k \in [k^r, k^m]$.

If $\rho(k) > 0$ we get by implicit differentiation of (24)

$$h^{a\prime}(k) = \frac{\alpha\rho u''((1+\rho)\rho(1+n) + \rho'(f-(n+\delta)k - h^a)) + \alpha u'\rho'}{(1-\alpha)u'' + \alpha\rho^2 u''}. \tag{25}$$

To simplify notation we have omitted the arguments of all functions in this expression. Both the numerator and the denominator are negative, where the negativity of the numerator follows from $(f - (n+\delta)k - h^a) < 0$ and $\rho(k) > 0$. Therefore, we have $h^{a\prime}(k) > 0$ for all values of k where $\theta(k) > \frac{1-\delta}{1+n}k$. Accordingly, $\phi(k) - \theta(k)$ increases with k as long as $\theta(k) > \frac{1-\delta}{1+n}k$ and since $\phi(k^m) > \frac{1-\delta}{1+n}k^m$ we have shown that $\phi(k) > \theta(k)$ for all $k \in [k^B, k^m]$. The claims of (b) now follow directly.
(c) Follows directly from (b) since $\theta^\ell = \theta$ in the neighborhood of $\tilde{k}$.
(d) Due to $l(\hat{k}) = \phi(\hat{k})$ the adaptive economizing problem (23) has $k^1 = \hat{k}$ as the only admissible solution for $k = \hat{k}$ and therefore $\theta^\ell(\hat{k}) = \hat{k}$ and $\hat{k}$ is a fixed point of θ^ℓ. It is however easy to see that $\hat{k}$ is an unstable fixed point and therefore only accumulation paths hitting $\hat{k}$ after a finite number of periods converge to $\hat{k}$. Since $\theta^\ell(k) = \theta(k)$ in the neighborhood of $\tilde{k}$, instability of $\tilde{k}$ with respect to θ yields instability with respect to θ^ℓ. Since $\ell(k) < \hat{k}$ for all $k > \hat{k}$ the resulting fluctuating capital accumulation paths might eventually hit a capital stock below $\hat{k}$. In this case there will be demise with finite survival time as described in point (iv) of proposition 2.
(ii):
(a): Since $\tilde{k} < \hat{k}$ we have $\theta(\hat{k}) < \hat{k} = l(\hat{k})$. On the other hand we have $\theta(k^B) > 0 = \ell(k^B)$. Therefore k^D has to lie in the interval $\hat{k}, k^B$ and due to continuity $\ell(k) > \theta(k)$ also holds for all capital stocks in the interval $[\hat{k}, k^D)$. Thus, $\theta^\ell(k) = \ell(k)$. For $\theta(k) > \ell(k)$ the consumption threshold is not binding and we have $\theta^\ell(k) = \theta(k)$. Note that $h^a(\hat{k}) > \bar{c}$ and the arguments given in the proof of point (b) in part (i) show that current consumption under $\theta(k)$ is therefore larger than $\bar{c}$ for $k \in [\hat{k}, k^m]$. This establishes that $\theta^\ell = \max(\theta(k), \ell(k))$ on $[k^D, k^m]$. Finally, analogous arguments as in the case of weak discounting show that for $k \in [k^r, k^m]$ we have $\ell(k) = 0$ and $\theta(k) = \frac{1-\delta}{1+n}k$.
(b): Because $\theta^\ell(k) < k$ holds for all capital stocks except $k = \hat{k}$ we conclude that any path with $k_0 > \hat{k}$ which does not hit $\hat{k}$ after a finite number of periods either has to converge to $\hat{k}$ from above or enter the region $[0, \hat{k}]$. We know from part (ii)(a) of this proposition and part (ii) of lemma 1 that $\theta^\ell(k) = \ell(k) < \hat{k}$ for $k > \hat{k}$ in the neighborhood of $\hat{k}$ which rules out convergence of a path towards $\hat{k}$ from above. Hence, every path that does not hit $\hat{k}$ after a finite number of periods eventually has to enter $[0, \hat{k})$ which according to proposition 3 induces within a finite number of periods consumption below

the minimal level and therefore the collapse of the economy. □

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Figure Captions

Figure 1: Maximal capital accumulation trajectories given minimum consumption, $\bar{c} > 0$.

Figure 2: Inter–temporally optimal trajectories: (a) weak discounting; (b) strong discounting.

Figure 3: Characteristics of subjective future survival endowments. For $k_{t+1} \in [\ell(k_t), \hat{k}_t]$ the capital endowment left to the next generation appears to insure long-run viability but does not.

Figure 4: Adaptive ℓ^{**} economizing trajectories: a) weak discounting, $\tilde{k}$ is stable with respect to θ; b) weak discounting, $\tilde{k}$ is unstable, fluctuations stay above $\hat{k}$; c) weak discounting, $\tilde{k}$ is unstable, fluctuations lead below $\hat{k}$; d) strong discounting.

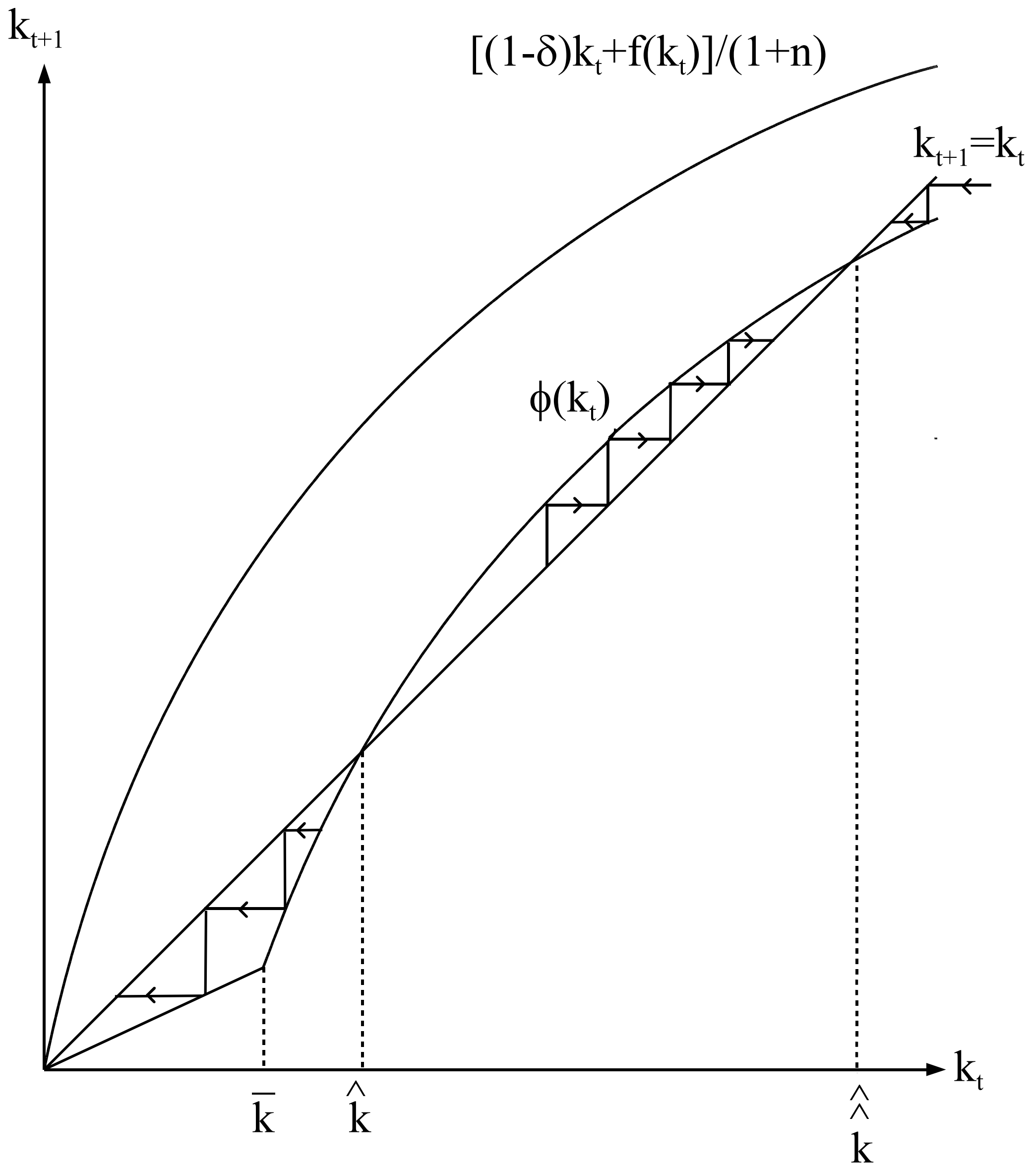
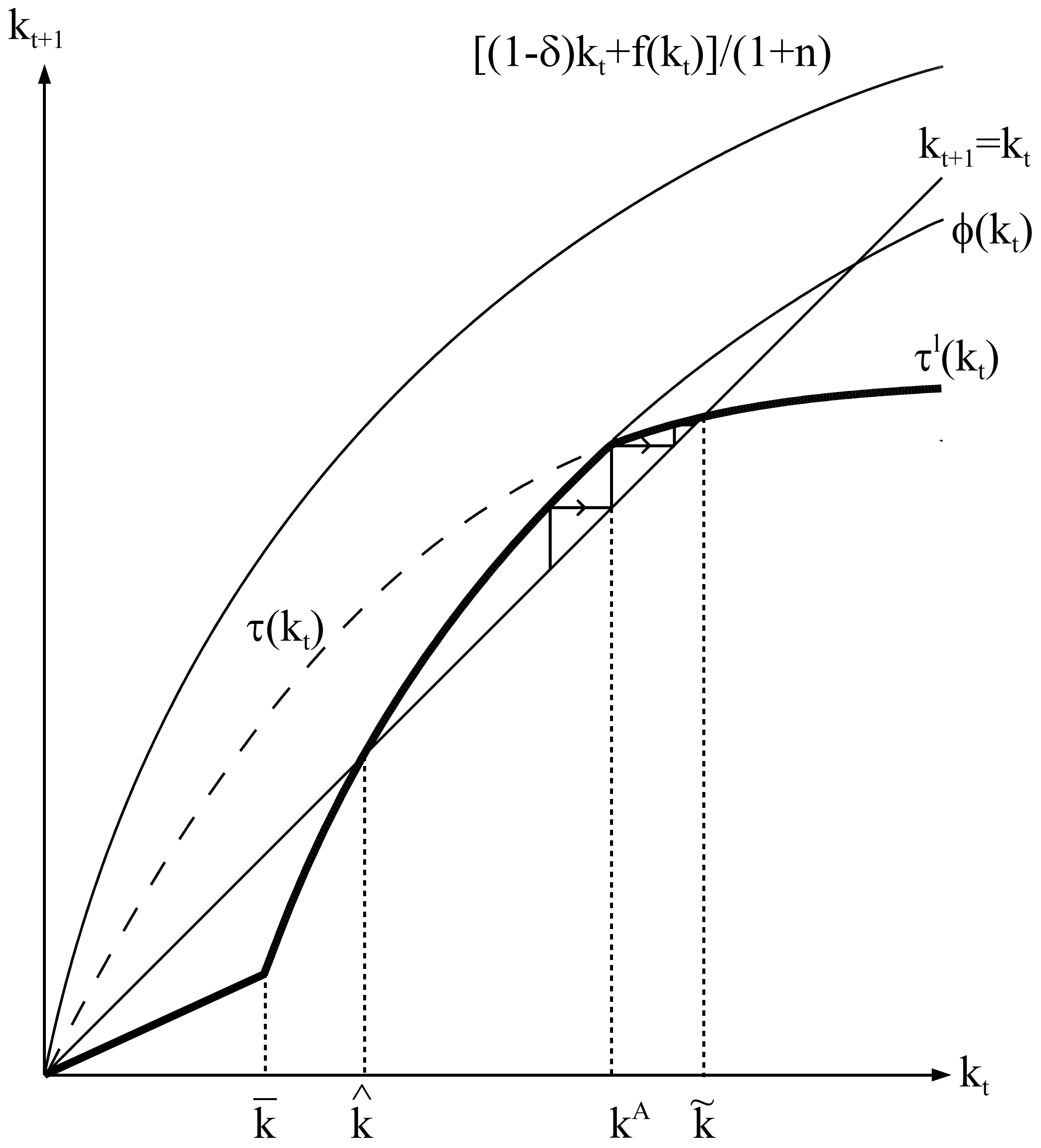
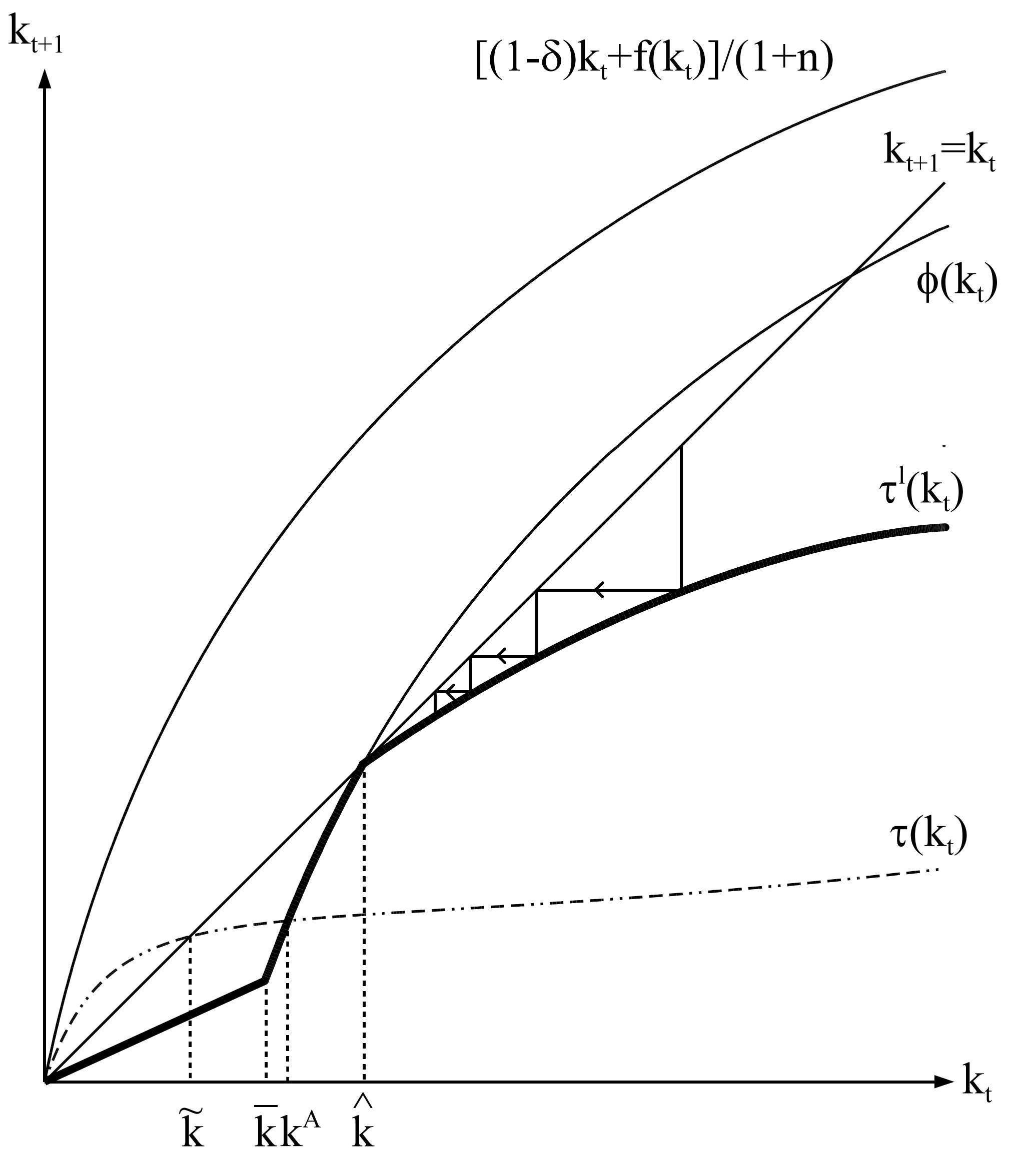


Figure 1



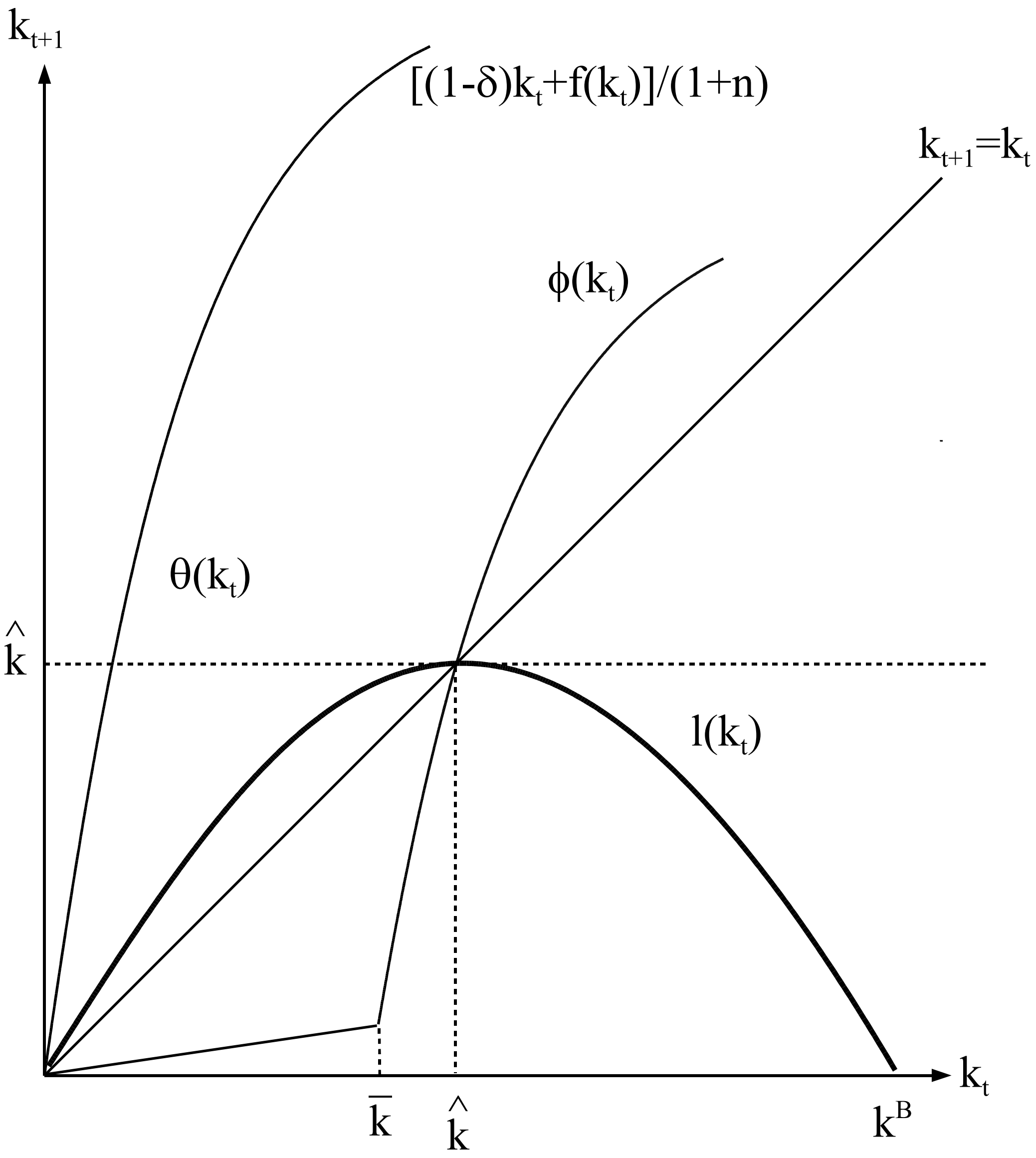
k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\tau^1(k_t)$
$\tau(k_t)$
k_t
$\bar{k}$
$\hat{k}$
k^A
$\tilde{k}$

Figure 2a



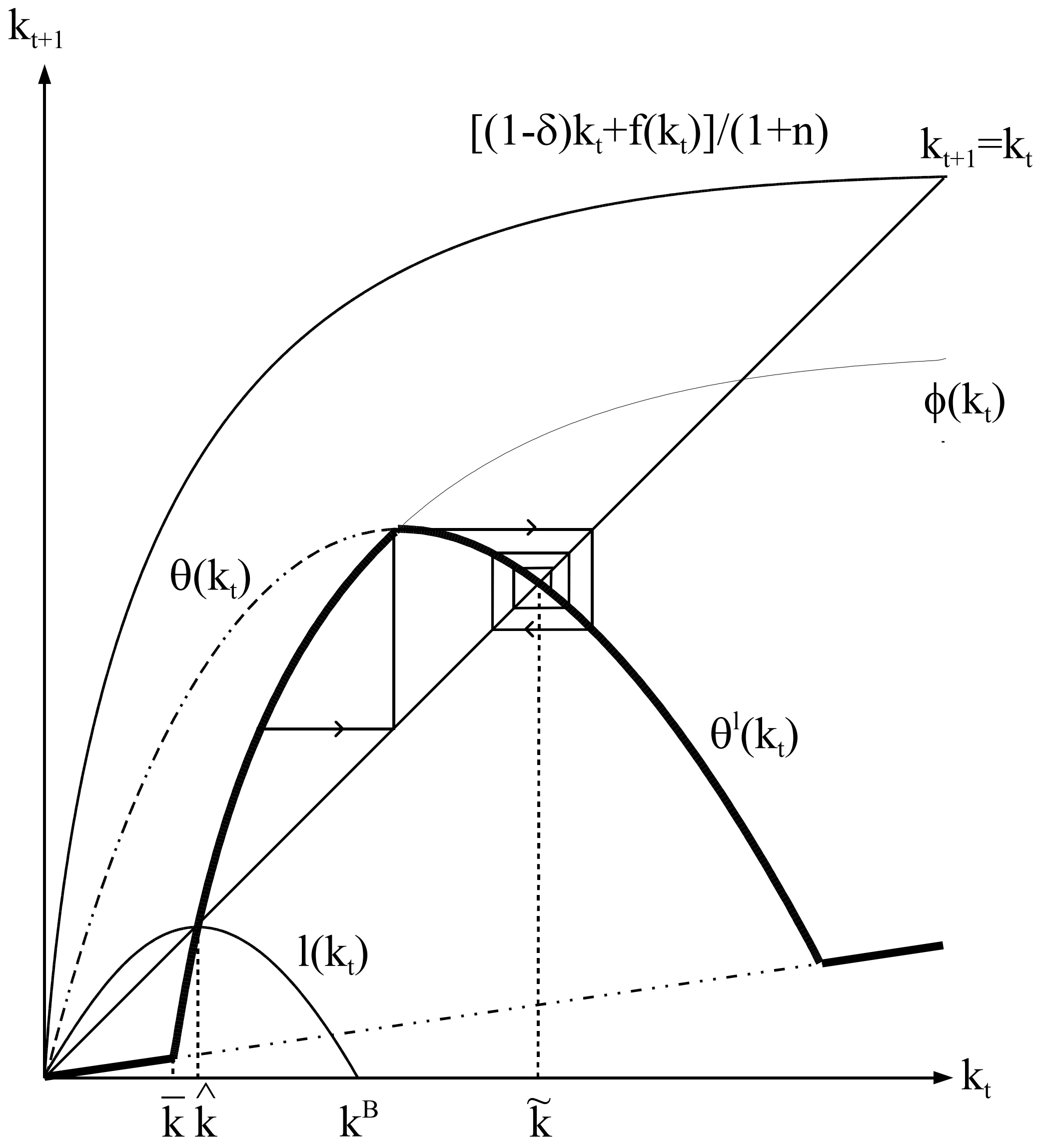
k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\tau^1(k_t)$
$\tau(k_t)$
k_t
$\tilde{k}$
$\bar{k}$
k^A
$\hat{k}$

Figure 2b



k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\theta(k_t)$
$\hat{k}$
$l(k_t)$
k_t
$\bar{k}$
$\hat{k}$
k^B

Figure 3



k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\theta(k_t)$
$\theta^l(k_t)$
$l(k_t)$
k_t
$\bar{k}$
$\hat{k}$
k^B
$\tilde{k}$

Figure 4a

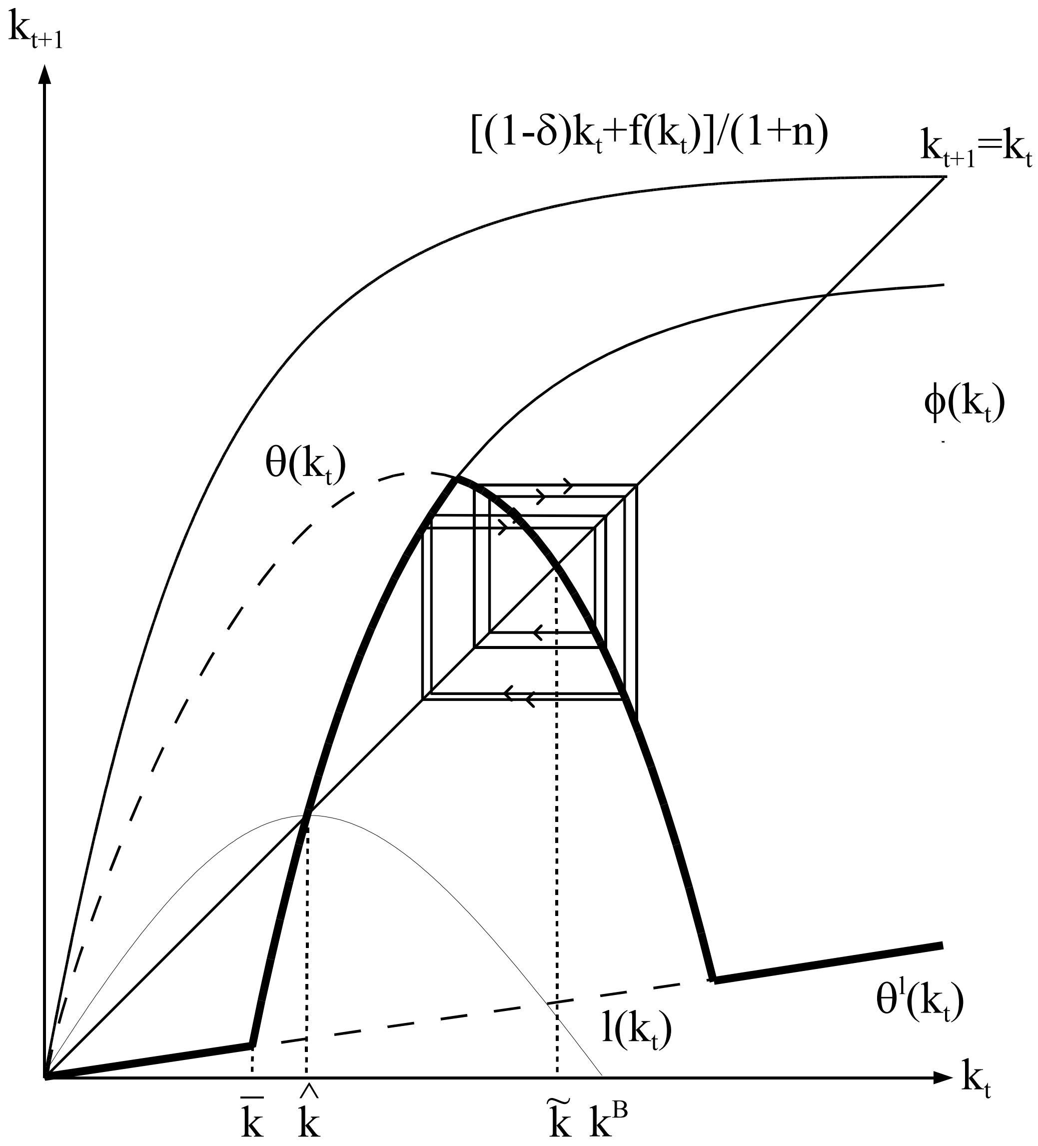

k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\theta(k_t)$
$\theta^l(k_t)$
$l(k_t)$
k_t
$\bar{k}$
$\hat{k}$
$\tilde{k}$
k^B

Figure 4b

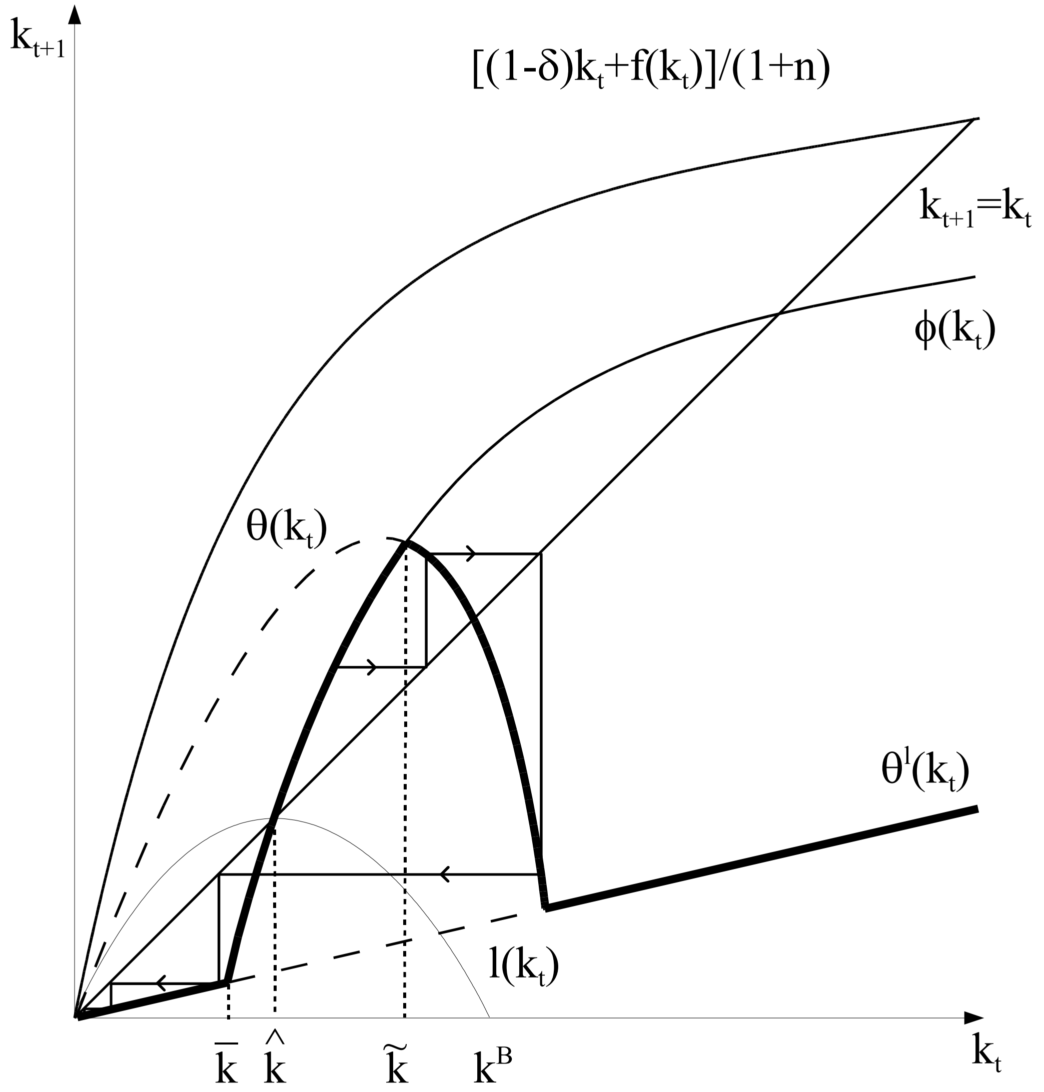
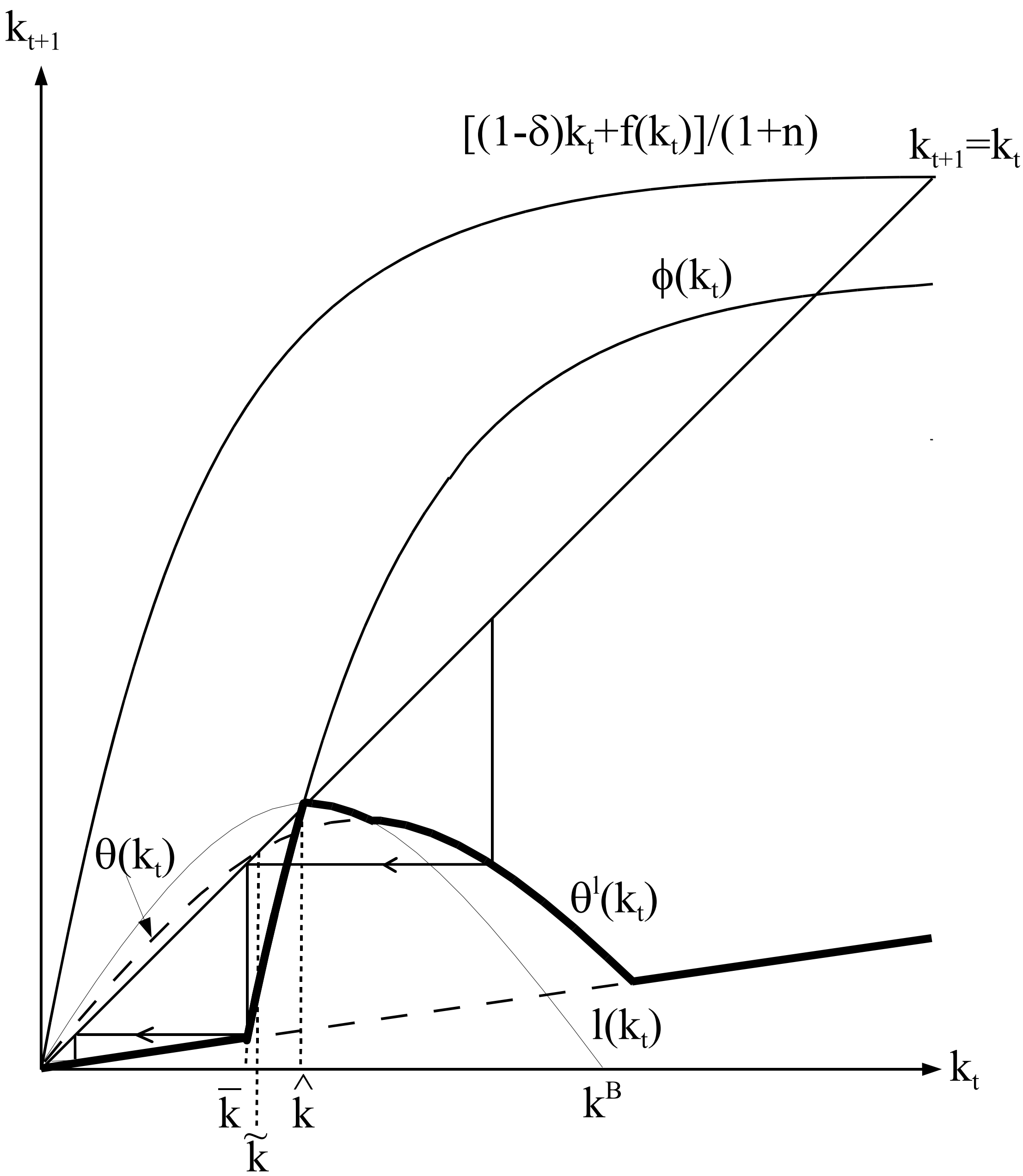

k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\theta(k_t)$
$\theta^l(k_t)$
$l(k_t)$
$\bar{k}$
$\hat{k}$
$\tilde{k}$
k^B
k_t

Figure 4c



k_{t+1}
$[(1-\delta)k_t+f(k_t)]/(1+n)$
$k_{t+1}=k_t$
$\phi(k_t)$
$\theta(k_t)$
$\theta^l(k_t)$
$l(k_t)$
k_t
$\bar{k}$
$\hat{k}$
k^B
$\tilde{k}$

Figure 4d