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Metaphors and Metonymies in the Teaching of Mathematics
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METAPHORS AND METONYMIES IN THE TEACHING OF MATHEMATICS

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1. Pedantry versus accuracy in presenting mathematics

How to deal with facts that dwell on the borderline between soulless pedantry and reasonable accuracy? This is the eternal question that emerges again and again when preparing mathematical texts for students. In particular one of the trouble spots appears with the communication about division in the domain of natural numbers.

Much has been written about what an equivalence relation is, about the properties of reflexivity, of transitivity and so on. So why then are expressions of the type

\[ 1234 : 27 = 45 \, r \, 19 \]

so widely used and accepted still? Should we tolerate such abuses of well-defined mathematical symbols or should we fight them?

People and fashions in mathematics teaching come and go. But these and some more faults still exist and resist. Are they faults at all? What is it that is actually meant, what do they stand for?

2. Metonymies in natural language

Apparently such instances are like certain facts of life: nobody talks about them too much, everybody does it occasionally, many people are ashamed of it, some are defiant. Though it is possible to eradicate some of such nuisances in a few places, quickly some new ones will emerge elsewhere, just as to mock at our pedantry. Let us look closer at the phenomenon.

The cases arise with the use of a symbol or of a group of symbols. The most striking common feature across the cases is, that there occurs a shift of meaning to the symbols whereas it is absolutely clear, what the actual meaning, what the message is intended to be.
In the example mentioned above the equality sign "=" is also used with a total shift of meaning. In this example it does mean "yields" or "leads to" rather than "is equal to". With this interpretation of the sign there is no trouble with understanding. The only trouble arising is with the legal explaining of the use of the symbols used here.

We very often meet with such cases in every living speech. If one asks: "Have you read Bourbaki?" or "Have you seen my Skemp?" he clearly uses such shift of meaning to refer to a book rather than to a person. We therefore can speak of a shift of reference: the name of a person is used for a copy of a book which the person has written. But this will work only in case we know the partner to whom the message is addressed or at least if we expect to be in tune with this person. It is remarkable also that the choice of that particular symbol is not quite arbitrary though the description through "shift of meaning" might suggest that. The symbol is used just because it was available, and because the use makes the communication short, expedient and effective.

We have a name for such instances in the living speech. Linguists would call it a particular example of figurative speech, and the name of this figure is "metonymy".

Metonymies are formed through contiguity, a sort of abstract relation of closeness. The closeness can come out of the objective world as well as out of the universe of discourse. Anyway it is the contiguity which we have to have in common with our partners in communication as an assumption for metonymies.

Some examples from a natural language may support the understanding of a reader. We understand what is meant by somebody's saying "a voice was singing outside". Of course "a voice" stands for "a person". With "He is the best pen in the country" we can see another shift of reference: "pen" stands for "writing ability". Or "I talked with Warsaw yesterday" said by a visitor over here might mean, he talked with his family at home in Warsaw, and indeed not with the whole family, but with one member of it or with some of them. Furthermore there is no similarity between Warsaw and any member of the family, but there is a relation of contiguity between them, because a person was designated by the general name of the place where this particular person was at the given moment.
Some types of contiguity relations are used frequently for the metonymical shift of reference, for instance the relation between cause and effect ("What's that crowd?" - "It's an accident!"), or the relation between container and content ("Would you like a bottle?"), or the relation between an object and the location of that object ("London calling ...", "... hier spricht Köln"), or the relation between a part and the whole, and many more taken from real situations.

There is also an abstract contiguity relation between that which designates and that which is designated, i.e. between signifier and signified (SAUSSURE). This could be used to produce e.g. the following metonymies:

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\[\text{TITANIC}\]
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or "Forget his name".

Roughly the various types of contiguity relations can be classified into two classes: continguities of real situations and continguities of context. Examples of the second type become evident when we realize that the shift of reference used in a metonymy is not independent of the context given, but is also related to the words or symbols which are adjacent to the figure. We speak of a contiguity of context when the contiguity relation is based on the embedding "contiguous" words, or symbols, or signs. Above all in music and in abstract art this type seems to make the main feature. But also this type of contiguity is present in every metonymical figure of coded thinking, for instance in the relation between the general notion and a specific example ("The lost youth").

It is the contiguity relation described which allows us to speak with a shift of reference and thus to produce metonymies. To say "speak" here is somewhat misleading, because as a matter of fact metonymy is not only a figure of speech but also a figure of thought. Therefore the figure applies to mathematical thinking in particular, because school mathematics and that part of mathematics which is generally accepted as part of our culture can be viewed at as an extension of our natural languages, as English, French, German, Spanish, Russian, Italian etc.

3. Metaphors

In order to complete the picture we have to mention another important figure of speech, the metaphor. In a certain sense metaphors form the dual counterpart to metonymies, as has been described comprehensively by linguists as R. JACOBSON (1963) and M. LE GUERN (1973).
At first it is remarkable that metonymies usually are used and understood in a non-reflected way, like available but unawarely employed routines. Pointing to it in conversation often causes surprise. Metaphors, quite in contrast to this, are nearly always formed deliberately and with intention and sometimes with much intellectual effort. In opposition to the use of metonymies we produce metaphors in cases when we want to evoke a certain understanding, we want to accentuate an aspect, or to lay emphasis on certain properties, and yet we are in lack of common words for it with an established common meaning. The metaphorical figure helps to overcome this difficulty through the use of other words, which are uncommon in the context given, and of which a selected part of their potential meaning can be employed only, thus conveying what we want them to convey decidedly.

In this sense a metaphor is like the square root of 2 expressed within the rationals. Such situations occur very often not only in living speech but also in mathematics teaching, where they seem to build the rule rather than rare exceptions.

4. Some aspects of reasonable accuracy

If at all we want to teach somebody something, then we have to describe the new notions somehow and to expose the new structures in terms of other structures which the student has assimilated or already mastered. By means of a metaphor new meaning can be created, or at least a meaning which is new for the hearer. By means of a metonymy a new local name is created, and very often abandoned later. Metaphors are formed by similarity, or by analogy. Metonymies are formed by contiguity.

In a metaphorical figure characteristically the selected use of parts of the semantic power of the words in the figure goes along with the suppression or forgetting about of the other parts of the potential semantic power. In the context given a metaphor therefore appears as a more or less unusual combination of words, thus creating new contextual situations. This mismatch serves as a hint by which the reader or the listener most often recognizes the figurative use of the words.

In a metonymical figure the whole semantic power but of another word or of another group of words is used. We therefore speak of a shift of reference, or of a total shift of reference, as if one word were disguised or masked by another one.
The word material for a metaphor is selected and chosen because of its meaning, more precise because of a certain part of its meaning. The word material for a metonymy is used because of its availability, with no change of meanings of particular units, but with total shift of local reference, that is using other significations (words) with their unreduced meanings.

Using graphical representation the distinction can be presented as follows:

(METONYMY)

\[ (a \rightarrow s(a)) \quad (b \rightarrow s(b)) \quad (c \rightarrow s(c)) \quad (d \rightarrow s(d)) \quad (e \rightarrow s(e)) \]

(signifier (word or group of words))

(signified (meanings))

(METAPHOR)

\[ (a \rightarrow s(a)) \quad (b \rightarrow s(b)) \quad (c \rightarrow s(c)) \quad (d \rightarrow s(d)) \quad (e \rightarrow s(e)) \]

(signifier/word(s))

\[ s(a) \cap s(b) \quad s(a) \cap s(h) \quad s(a) \cap s(h) \cap s(e) \]

(signified/meanings)

(In the graph the metonymy is produced by using d for a, and the metaphor by using e in the new (reduced) meaning parts of which are described here by \( s(a) \cap s(h) \cap s(e) \).)

The linguistic distinction of metaphors and metonymies mirrors two styles of thinking due to the fact of the deep interaction between language and thought. Mathematics teaching for that purpose has to differentiate between these two styles of thinking, since they regularly require different treatment.

5. School mathematics and figurative speech

A few examples concerning the use of symbolic notation may give support to the insight how commonly the metonymical figure is used in school mathematics:
5.1. An algebraic notation often can be found to be introduced by: We
take a number, say, a, we take another number, say, b, and we form expres-
sions like
\[ a + b, \quad 2a + b, \quad (a + b) \times (c + d), \quad (a + b)(a + b) \quad \text{etc.} \]
In this procedure the metonymical shift of reference is given by taking
letters and pretending to use them just like one would do with numbers.
If both, writer and reader, or speaker and listener, share the same prac-
tice - "are in tune" - then the procedure becomes habitually accepted, and
nobody will ask what or why. It is evident for them then, in which way the
addition of letters should be executed, and that the addition of two letters
will yield another letter of the alphabet.

5.2. Also there is no definition needed when we graciously make up our
mind for a notation as, say,
\begin{align*}
\text{n} & \quad \text{for a natural number} \\
\text{p} & \quad \text{for a prime} \\
\text{r} & \quad \text{for the remainder of a division}
\end{align*}
Clearly we use metonymies here. The decision is taken guided by contiguity
(but not by similarity).

5.3. Also we use E for edges, V for vertices, and F for faces in order to
form Euler's famous theorem.

5.5. Or we just write \( f(x) \) for a function.
When the values of certain functions are more important for us, as is the
case with sequences e.g., then we might prefer another notation, say, \( a_j \).
Do we really have to write pedantically \( a : \mathbb{N} \rightarrow \mathbb{R} \), in place of \( a_n \), no
matter what we want it for?

The most common metonymical figure to notice in a natural language may be
the "pars pro toto" figure. Whereas in mathematical texts perhaps the most
frequent metonymy is taking the value \( f(x) \) for the whole function \( f \). In an
act of condensation we may see much the same, because the reference shifts help
us focus our attention from a set to a number of it, or to a part of it, or from
\( f \) to \( f(x) \), or back from a particular value \( f(x) \) to the entire \( f \), or to some part
of \( f \), or some extension of it as in the "toto-pro-pars" figure.
"... sometimes we would sit for hours in a coffee house. He (Mazur) would write just one symbol or a line like \( y = f(x) \) on a piece of paper or on the top of the marble table. We would both stare at it as various thoughts were suggested and discussed. These symbols in front of us were 'like a crystal ball to help us focus our concentration.'" (S.M. Ulam, 1976, p. 31)

6. **Metaphors and Metonymies as figures of thinking**

6.1. Let us consider a well known story problem: "A brick weighs one kilogramm and half of a brick. How much weighs one brick?"

If \( x \) stands for a brick, then we have a typical metonymy with shift of reference. By mere contiguity one then might write

\[
(1) \quad x = 1 + \frac{1}{2} x. 
\]

As a pure play with symbols this is easy to solve.

But to this effect we have to give up some of the semantic power and to enter a metaphorical mode. We might say, to take \( x \) for a brick is not enough: we have to develop further the idealization of the problem. In fact we mean by \( x \) then something like an "ideal" brick, a mean or average brick and we take a formula \( x = \sqrt[4]{\frac{1}{2}} x \) as a model of the situation. Otherwise we cannot be sure whether we are allowed to take off half of a brick from each side or not. They might be halves from different bricks, and then equilibrium on the scale would be lost, (this comes from actual observation of the classroom).

Analyzing the figures of thinking involved in the whole process of solving the task enables us to describe the process as a sequence of a metonymy, a metaphor, and a metonymy. The first metonymy just leads to writing down the equation (1), with the decided meaning of \( x \) - as pupils often say - "\( x \) is a brick". That is, \( x \) is used like another name for a brick. The second step is characterized by a reduction of the semantic power. The very same expression now is used metaphorically, as a model for an ideal brick and its relations. Finally another metonymy comes into the play by shifting the reference of \( x \) from idealized bricks to numbers, which enables us to calculate and - perhaps, in a way - to solve the problem.
6.2. The fundamental role of metaphors and metonymies is not limited to late stages of school mathematics. The two figures are in function as soon as language is somehow developed. Therefore we add an example from primary mathematics: "There are nine cars at a parking lot. Five cars are red, and five cars are VW's. Are there any red VW's there?"

Representing cars by blocks as usual and with the characteristic shift of reference the pupils can drive "them" into the loops of naive set theory

So they are at least equipped to handle the problem. And they might arrive at a solution by trial and error.

The shift of reference from a car to a block, which stands for a car, is easy and seems to be typical for children.

6.3. For a comparison let us consider a problem with days of the week, similar to the one of 6.2.: "Last week we had five days with some snowing, five days there was some sunshine, and at five days we had pea soup for lunch. Was there at least one day with snowfall, sunshine, and pea soup?"

Days of the week are not so easily representable by blocks as cars are, the "natural" contiguity is lacking. But the sheets from a tear-off calendar might provide for a useful contiguity. Then using the sheets for the days of the week, these seven sheets metonymically are the days of the week.

Moving the sheets into three loops follows the same standard procedure as above with the blocks:
There is no similarity between the sheets and the days, they are chosen by contiguity. Simply the sheets are available and their use is handy. A sheet means a day for the pupil. There is no change of the frame of reference, there is a local shift of reference only, just like borrowing another name with an intact reference for a moment.

Evidently the metonymies used in these examples are figures of thinking rather than figures in mathematics as a language only. And the contiguities involved are a contiguity of context in 6.1., and contiguities of situation in 6.2. and 6.3..

7. Metonymies as effective and essential ingredients

Furthermore there are many examples of metonymical figures in mathematical texts that come directly from the established usage in a given natural language. For instance, the phrase "the radius of this circle is three" seems to be common usage in English language. It is accepted as a kind of idiom with the established meaning of: the measure of the radius in question is three, or has the value three. This would be a fairly banal remark, if such phrases were not to be found right after the definition of the radius legislated to be an interval with the known relation to some circle. Thus in one line the radius appears as an interval, and in another line it is three and can be treated and calculated with like a number, as we do in expressions like $\pi r^2$ or $2\pi r$ and so on. This is only to state that metonymical shifts of reference occur in the best mathematical texts and that they are essential ingredients of effective communication about mathematics.

In some languages, as in Polish for instance, however, phrases like "the radius is three" or "the height of the triangle is three" do not have a comparably idiomatical basis. Nevertheless the metonymies "the height is three" or "the radius is three" are very common in Poland, though many teachers and some textbooks condemn their use as very bad habits. There are also attempts to cure that "bad" state via definitions like "the height of a triangle is a measure, that is a number, which ...". Such attempts are blind to the metonymy of the number-for-measure type and its potential power. And it is funny and convincing together to read just one page later that "the heights of a triangle have a point in common", which aside is an interesting fact about numbers also.
At this point we may state that precise thinking and writing is a good thing, but that a certain grace and neatness of thought is at least of equal importance. The attempt of "debugging" live mathematics from metonyms turns out to be a rather hopeless venture.

A final remark should be given to an interesting kind of contiguity which occurs while practising certain procedures or algorithms. At each stage of the procedure it is by contiguity of situation that the next step or action is taken. But though we - covertly - rely on the existence of these contiguities we very often do not make them explicit in the description of the algorithm. In this way we simplify the description essentially, which characteristically is possible only when we communicate with human beings but not with machines. In this way we arrive at rules of thumb sometimes, abbreviated to the extreme. These rules are not very appealing to those who are able to analyze the performance of a particular algorithm from an often much more sophisticated standpoint. But they are very helpful for the frequent human user.

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